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INTRINSIC CHARACTERIZATION OF CERTAIN CONFORMALLY FLAT SPACES

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If V_n is a conformally flat hypersurface of a conformally flat space V_{n+1}, then V_n is a quasi-umbilical hypersurface, that is, there exists a non-zero vector field v_i on V_n such that the second fundamental tensor h_{ji} is given by $h_{ji}=\alpha g_{ji}+\beta v_j v_i$ for some functions α, β on V_n, here g_{ji} is the metric tensor on V_n (See [1]). If V_{n+1} is a space of constant curvature k, Chen and Yano showed in [1] that the curvature tensor $K_{kji}{}^h$ is given by

$$K_{kji}{}^h=(k+\alpha^2)(\delta_k^h g_{ji}-\delta_j^h g_{ki})+\alpha\beta[(\delta_k^h v_j-\delta_j^h v_k)v_i+(v_k g_{ji}-v_j g_{ki})v^h]. \tag{0.1}$$

In the present paper, the authors would like to consider an instrinsic characterization of a Riemannian manifold V_n with curvature tensor $K_{kji}{}^h$ given in the form of (0.1). In fact we shall prove the following

THEOREM. *Let V_n be an n-dimensional Riemannian manifold*[1] *with a unit vector field u^h. Then the necessary and sufficient conditions for V_n having the properties*:

(I) *The curvature operator $K_{kji}{}^h v^k w^j$ associated with two vectors v^h and w^h orthogonal to u^h annihilates u^h*:

$$K_{kji}{}^h v^k w^j u^i=0; \tag{0.2}$$

(II) *Sectional curvature with respect to a section containing u^h is a constant*;

(III) *Sectional curvature with respect to a section orthogonal to u^h is a constant is that the Riemann-Christoffel curvature tensor of V_n has the form*

$$K_{kji}{}^h=\lambda(\delta_k^h g_{ji}-\delta_j^h g_{ki})+\mu[(\delta_k^h u_j-\delta_j^h u_k)u_i+(u_k g_{ji}-u_j g_{ki})u^h] \tag{0.3}$$

for some functions λ and μ. In this case, V_n is a conformally flat space for $n>3$.

§ 1. Preliminaries.

Let V_n be an n-dimensional Riemannian space with metric $ds^2=g_{ji}d\eta^j d\eta^i$, $h, i, j, \cdots=1, 2, \cdots, n$, where $\{\eta^h\}$ is a local coordinate system. We denote by $\{{}_j{}^h{}_i\}$

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1) Manifolds, mappings, functions, ⋯ are assumed to be sufficiently differentiable.

the Christoffel symbols formed with g_{ji} and by ∇_j the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$. We denote by $K_{kji}{}^h$ the Riemann-Christoffel curvature tensor of V_n:

$$K_{kji}{}^h=\partial_k\left\{\begin{matrix} h \\ j\ \ i\end{matrix}\right\}-\partial_j\left\{\begin{matrix} h \\ k\ \ i\end{matrix}\right\}+\left\{\begin{matrix} h \\ k\ \ t\end{matrix}\right\}\left\{\begin{matrix} t \\ j\ \ i\end{matrix}\right\}-\left\{\begin{matrix} h \\ j\ \ t\end{matrix}\right\}\left\{\begin{matrix} t \\ k\ \ i\end{matrix}\right\}, \tag{1.1}$$

where $\partial_k=\partial/\partial\eta^k$. Then the Ricci tensor and the scalar curvature are given respectively by

$$K_{ji}=K_{tji}{}^t \tag{1.2}$$

and

$$K=g^{ji}K_{ji}, \tag{1.3}$$

where g^{ji} are contravariant components of the fundamental metric tensor.

We define a tensor field L_{ji} of type $(0,2)$ by

$$L_{ji}=-\frac{K_{ji}}{n-2}+\frac{Kg_{ji}}{2(n-1)(n-2)}. \tag{1.4}$$

The conformal curvature tensor $C_{kji}{}^h$ is then given by

$$C_{kji}{}^h=K_{kji}{}^h+\delta_k^hL_{ji}-\delta_j^hL_{ki}+L_k{}^hg_{ji}-L_j{}^hg_{ki}, \tag{1.5}$$

where δ_k^h are the Kronecker deltas and $L_k{}^h=L_{kt}g^{th}$.

A Riemannian manifold V_n is called a *conformally flat space* if we have

$$C_{kji}{}^h=0 \tag{1.6}$$

and

$$\nabla_kL_{ji}-\nabla_jL_{ki}=0. \tag{1.7}$$

It is well known that (1.6) holds automatically for $n=3$ and (1.7) can be derived from (1.6) for $n>3$.

If there exist, on a conformally flat space, two functions α and β such that α is positive and

$$L_{ji}=-\frac{\alpha^2}{2}g_{ji}+\beta(\nabla_j\alpha)(\nabla_i\alpha), \tag{1.8}$$

then the space V_n is called a *special conformally flat space*.

§ 2. Proof of the theorem.

If the curvature tensor of V_n has the form (0.3), then it is trivial to see that properties (I), (II) and (III) are satisfied.

We now assume that V_n with a unit vector field u^h satisfies (I), (II) and (III).

We take $n-1$ linearly independent vectors $B_b{}^h$, $a, b, c, \cdots 1, 2, \cdots, n-1$, orthogonal to u^h and let $B^a{}_i$, u_i be determined in such a way that

$$(B_b{}^h,\ u^h)^{-1}=(B^a{}_i,\ u_i). \tag{2.1}$$

Then we have

$$B_a{}^h B^a{}_i + u^h u_i = \delta_i^h$$

or

$$B_a{}^h B^a{}_i = \delta_i^h - u_i u^h.$$

The condition (I) is expressed as

$$B_e{}^k B_d{}^j K_{kji}{}^h u^i = 0.$$

Transvecting $B^e{}_m B^d{}_l$ to this we find

$$(\delta_m^k - u_m u^k)(\delta_l^j - u_l u^j) K_{kji}{}^h u^i = 0$$

or

$$K_{mli}{}^h u^i - K_{mji}{}^h u^j u_l u^i - K_{kli}{}^h u^k u_m u^i = 0,$$

from which

$$K_{kji}{}^h u^i = M_k{}^h u_j - M_j{}^h u_k \tag{2.2}$$

where

$$M_k{}^h = K_{kji}{}^h u^j u^i. \tag{2.3}$$

$M_k{}^h$ satisfies

$$M_k{}^h u_h = 0, \qquad M_k{}^h u^k = 0, \qquad M_{ji} = M_{ij}, \tag{2.4}$$

where $M_{ji} = M_j{}^t g_{ti}$. From condition (II) we have

$$K_{kjih} u^k v^j u^i v^h = \text{constant} \tag{2.5}$$

for any unit vector v^h orthogonal to u^h. (2.5) can be written as

$$M_{ji} v^j v^i = \text{constant}. \tag{2.6}$$

Thus we have

$$B_c{}^j B_b{}^i M_{ji} = \lambda g_{cb}.$$

Transvecting $B^c{}_m B^b{}_l$ to this we find

$$(\delta_m^j - u_m u^j)(\delta_l^i - u_l u^i) M_{ji} = \lambda (g_{ml} - u_m u_l),$$

from which using (2.4)

$$M_{ji}=\lambda(g_{ji}-u_ju_i). \tag{2.7}$$

Thus (2.2) becomes

$$K_{kji}{}^h u^i=\lambda(\delta_k^h u_j-\delta_j^h u_k). \tag{2.8}$$

From condition (III) we have

$$K_{kjih}B_d{}^k B_c{}^j B_b{}^i B_a{}^h=k(g_{da}g_{cb}-g_{ca}g_{db}).$$

Transvecting $B^d{}_s B^c{}_r B^b{}_q B^a{}_p$ to this we find

$$\begin{aligned}&K_{kjih}(\delta_s^k-u_su^k)(\delta_r^j-u_ru^j)(\delta_q^i-u_qu^i)(\delta_p^h-u_pu^h)\\&=k[(g_{sp}-u_su_p)(g_{rq}-u_ru_q)-(g_{rp}-u_ru_p)(g_{sq}-u_su_q)],\\&K_{srqp}-K_{srqh}u^hu_p-K_{srip}u^iu_q-K_{sjqp}u^ju_r-K_{krqp}u^ku_s\\&\quad+M_{sp}u_ru_q+M_{rq}u_su_p-M_{rp}u_su_q-M_{sq}u_ru_p\\&=k[(g_{sp}g_{rq}-g_{rp}g_{sq})-(g_{sp}u_r-g_{rp}u_s)u_q-(g_{rq}u_s-q_{sq}u_r)u_p].\end{aligned}$$

Substituting (2.2) into this, we find

$$\begin{aligned}&K_{srqp}+(M_{sq}u_r-M_{rq}u_s)u_p-(M_{sp}u_r-M_{rp}u_s)u_q+(M_{qs}u_p-M_{ps}u_q)u_r\\&\quad-(M_{qr}u_p-M_{pr}u_q)u_s+M_{sp}u_ru_q+M_{rq}u_su_p-M_{rp}u_su_q-M_{sq}u_ru_p\\&=k[(g_{sp}g_{rq}-g_{rp}g_{sq})-(g_{sp}u_r-g_{rp}u_s)u_q-(g_{rq}u_s-g_{sq}u_r)u_p],\end{aligned}$$

from which using (2.7)

$$\begin{aligned}&K_{srqp}+\lambda(g_{sq}u_r-g_{rq}u_s)u_p-\lambda(g_{sp}u_r-g_{rp}u_s)u_q\\&=k[(g_{sp}g_{rq}-g_{rp}g_{sq})-(g_{sp}u_r-g_{rp}u_s)u_q-(g_{rq}u_s-g_{sq}u_r)u_p]\end{aligned}$$

and consequently

$$K_{kji}{}^h=k(\delta_k^hg_{ji}-\delta_j^hg_{ki})+(\lambda-k)[(\delta_k^hu_j-\delta_j^hu_k)u_i+(g_{ji}u_k-g_{ki}u_j)u^h]. \tag{2.9}$$

This is the form of (0.3).

Now we shall show that V_n with $K_{kji}{}^h$ given by (2.9) is conformally flat for $n>3$. From (2.9), we have

$$K_{ji}=K_{hji}{}^h=[(n-2)k+\lambda]g_{ji}+(\lambda-k)(n-2)u_ju_i, \tag{2.10}$$

$$K=g^{ji}K_{ji}=(n-1)[(n-2)k+2\lambda]. \tag{2.11}$$

Substituting (2.10), (2.11) in (1.4), we find

(2.12)
$$L_{ji}=-\frac{1}{2}kg_{ji}-(\lambda-k)u_j u_i,$$

(2.13)
$$L_j{}^h=-\frac{1}{2}k\delta_j^h-(\lambda-k)u_j u^h.$$

Substituting (2.9), (2.12) and (2.13) in (1.5), we find

$$\begin{aligned}C_{kji}{}^h&=K_{kji}{}^h+\delta_k^h L_{ji}-\delta_j^h L_{ki}+L_k{}^h g_{ji}-L_j{}^h g_{ki}\\&=k(\delta_k^h g_{ji}-\delta_j^h g_{ki})+(\lambda-k)(\delta_k^h u_j-\delta_j^h u_k)u_i+(\lambda-k)(g_{ji}u_k-g_{ki}u_j)u^h\\&\quad+\delta_k^h\left[-\frac{k}{2}g_{ji}-(\lambda-k)u_j u_i\right]-\delta_j^h\left[-\frac{k}{2}g_{ki}-(\lambda-k)u_k u_i\right]\\&\quad+\left[-\frac{k}{2}\delta_k^h-(\lambda-k)u_k u^h\right]g_{ji}-\left[-\frac{k}{2}\delta_j^h-(\lambda-k)u_j u^h\right]g_{ki}\\&=0\end{aligned}$$

Hence V_n is conformally flat. This completes the proof of the theorem.

COROLLARY. *Let V_n be a simple connected n-dimensional Riemannian manifold with a unit vector field u^h and satisfies* (I), (II), (III). *Furthermore if the constant k required in* (III) *is $k>0$ and u^h is the vector field given by*

(2.14)
$$u_h=\lambda\nabla_h\phi(k)$$

for some functions λ and ϕ. Then V_n can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.

Proof. If $u_h=\lambda\nabla_h\phi(k)=2\lambda\phi'(k)\sqrt{k}\,\nabla_h\sqrt{k}$ with $k>0$, then L_{ij} takes the form of (1.8):

$$L_{ij}=-\frac{1}{2}kg_{ji}+\beta\nabla_j\sqrt{k}\,\nabla_i\sqrt{k}.$$

Thus V_n is a special conformally flat space. By theorem 1 of [1] V_n can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.

REFERENCES

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