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CONSTRUCTION OF BRANCHING MARKOV PROCESSES WITH AGE AND SIGN*

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It is a familiar fact in the theory of Markov processes that solutions for a wide class of *linear* parabeolic and elliptic equations can be investigated in terms of Markov processes (e.g. Dynkin [3] and Ito-McKean [8]). On the other hand it was known that a class of *semi-linear* parabolic equations plays an important rôle in the theory of branching processes (e.g. Bartlett [2], Harris [5], Moyal [15], and Skorohod [21]). The mathematical structure that reveals the mechanism of how this non-linearity appears in the theory of Markov processes should, therefore, be investigated systematically. Attempts in this direction were recently performed in several articles, especially, in Moyal [13], [14], [15], Skorohod [21], and Ikeda-Nagasawa-Watanabe [6], [7]. A branching Markov process X_t is defined as a strong Markov process on a large state space $\hat{\boldsymbol{S}}=\cup_{n=0}^{\infty} S^n \cup \{\varDelta\}$ having the following branching property[1)]

$$T_t\hat{f}(x)=\widehat{(T_t\hat{f})|_S}(x), \qquad x\in\hat{\boldsymbol{S}}, \tag{1}$$

where T_t is the semi-group of the "large" Markov process on $\hat{\boldsymbol{S}}$ and $\hat{f}$ is a function on $\hat{\boldsymbol{S}}$ defined by

$$\hat{f}(x)=\begin{cases} 1 \quad , & \text{if } x\in S^0, \\ \prod_{i=1}^{n} f(x_i), & \text{if } x=(x_1,x_2,\cdots,x_n)\in S^n, \\ 0 \quad , & \text{if } x=\varDelta, \end{cases}$$

where f is a measurable function on S with $\|f\|=\sup_{x\in S}|f(x)|\leqq 1$. Then

$$u(t,x)=T_t\hat{f}(x), \qquad x\in S,$$

is the (minimal)[2)] solution of a non-linear integral equation:

$$u(t,x)=T_t^0 f(x)+\int_0^t K(x,dsdy)F[y,u(t-s,\cdot)], \qquad x\in S, \tag{2}$$

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1) Definition of the notation appearing in the following will be found in § 1.
2) This is taken to mean that if $f\geqq 0$, $u(t,x)=T_t\hat{f}(x)$ is the minimal solution of (2).

where F is defined for a measurable function u on S by

$$(3)\qquad F[x, u]=\sum_{\substack{n=0\\ n\neq 1}}^{\infty} q_n(x)\int_{S^n} \pi_n(x, d\boldsymbol{y})\hat{u}(\boldsymbol{y}).$$

T_t^0, K, q_n and π_n appearing in the above equation have the following probabilistic meaning: T_t^0 describes the motion of one particle before the instance τ of the first splitting i.e. T_t^0 is a semi-group subordinate to T_t, $K(x, dsdy)$ is the joint distribution of τ and $X_{\tau-}$ for a particle whose starting point is $x\in S$, $q_n(x)\geqq 0$ is the probability that a particle splits into n-particles at $x\in S$, and $\pi_n(x, dy)$ is the distribution of the n-particles which have been produced by splitting at x.

It should be remarked that, in the above treatment, $q_n(x)$ is non-negative, $n=1$ must be excluded in the summation defining F in (3), and $T_t^0 1(x)<1$ at some point $x\in S$. In order to eliminate the above restriction from the probabilistic treatment of the equation (2), a method was proposed by Sirao [20].[3] The crucial point of it is to introduce new parameters "age" and "sign" to describe the state of particles. We shall call a branching Markov process which has the additional parameters a *branching Markov process with age and sign.*

Now, a fundamental problem concerning the general theory of branching Markov process is how to construct a strong Markov process having the branching property (1) by a given system of quantities $\{T_t^0, K, q_n, \pi_n\}$ which describes fundamental nature of branching Markov processes and is given usually as an experimental data in many applications. Several methods were given in [6] to construct branching Markov processes. One is a probabilistic way in which the process is constructed by piecing out path functions of one particle which is originaly given. One of the other two analytic ways is based on Moyal's method given in [13], and another one is through finding a solution of the non-linear integral equation (1). The most fundamental one is, in the author's opinion, the probabilistic construction, because this method is directly based on and reveals probabilistic structure i.e. mechanism of piecing out of path functions of branching Markov processes.

The purpose of this paper is to give a probabilistic construction of branching Markov processes *with age and sign* and to discuss some properties of the processes constructed. In § 1 we will state the main theorem of the paper. A theorem on piecing out of path functions will be stated in § 2. We will construct the *Markov process with age* in § 3 and *branching Markov process with age and sign* in § 4 by means of the theorem of piecing out. In § 5, main properties, especially the branching property, of the processes constructed will be proved. A proof is given in Ikeda-Nagasawa-Watanabe [6], [7]. The proof for the present case is essentially the same as one given in [6]. We will give, however, proof of Lemmas for completeness. In § 6 we shall give some additional comments on probabilistic solutions of the non-linear integral equation (2).

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3) Similar idea was used in [7] to discuss branching transport processes.

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§ 1. Main theorem.

Let D be a compact Hausdorff space with a countable open base, $N=\{0,1,2,3,\cdots\}$, and $S=D\times N$. A point $(x,k)\in D\times N$ describes a state of a particle whose position and age are x and k, respectively. We shall consider the following motion: A particle moving in D and growing older produces $(n-1)$-sons whose ages are zero at a random time τ and on $x\in D$ with probability $q_n^+(x)$ or $q_n^-(x)$, where $q_n^+(x)q_n^-(x)=0$. Then if $q_n^+(x)>0$, the n-particles continue the same but mutually independent movement in $S=D\times N$, say positive world, but if $q_n^-(x)>0$ the n-particles make transition to another world a copy of the original $S=D\times N$ and continue the same but mutually independent movement. Then the n-particles repeat such reproduction. Of course, if a particle is originally in another world, say negative world, and $q_n^-(x)>0$, then n-particles immigrate to the positive world. That is, if $+$ sign apperrs on the shoulder of q_n, then all particles stay in the same world, but if $-$ sign appears, all particles immigrate to the opposite world. Precisely speaking, to carry out a technical procedure satisfactorily we prepare four copies of S, two of them stand for positive world and the remainder negative world.

Accordingly a state of n-particles can be described by a point of the product space of $J=\{0,1,2,3\}$ and the n-fold symmetric topological product of S^n of $S=D\times N$. Therefore an adequate state space of the above mentioned motion is

$$\hat{\boldsymbol{S}}=\left(\bigcup_{n=0}^{\infty} S^n\times J\right)\cup\{\Delta\},\qquad S=D\times N,$$

where Δ is an extra point added by the one-point compactification, and $S^0=\{\partial\}\times N$, where ∂ is another extra point.

The precise meaning of S^n stated above is given as follows: Let $S^{(n)}=(D\times N)^{(n)}$, $(n=1,2,\cdots)$, be the n-fold direct product of $D\times N$. We identify a point $((x_1,k_1),(x_2,k_2),\cdots,(x_n,k_n))$ with another point $((x_{\pi(1)},k_{\pi(1)}),(x_{\pi(2)},k_{\pi(2)}),\cdots,(x_{\pi(n)},k_{\pi(n)}))$, where $\{\pi(1),\pi(2),\cdots,\pi(n)\}$ is a permutation of $\{1,2,\cdots,n\}$. This identification defines a equivalence relation. S^n is the quotient space of $S^{(n)}$ by this relation. An element of S^n will be denoted by $(\boldsymbol{x},\boldsymbol{k})$ where $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ and $\boldsymbol{k}=(k_1,k_2,\cdots,k_n)$, and a point of $\hat{\boldsymbol{S}}$ will be denoted by $(\boldsymbol{x},\boldsymbol{k},j)$.[4]

We will construct a "large" Markov process on the "large" state space $\hat{\boldsymbol{S}}$ by a system of given quantities: (i) a conservative strong Markov process on D which has right continuous path functions with left limits, which we shall call the *basic*

4) A point $(\boldsymbol{x},\boldsymbol{k},j)\in S^n\times J$ stands for the following state of particles: A k_i-year old particle is distributed at x_i, $i=1,2,\cdots,n$ and if $j=0$ or 1 the n-particles are in a positive world and if $j=2$ or 3 they are in a negative world. Our definition of state space $\hat{S}$ is slightly different from the one given in [20], where $((x_1,k_1),\cdots,(x_n,k_n))$ and $((x_1',k_1'),\cdots,(x_n',k_n'))$ are identified if $(x_1,\cdots,x_n)$ is a permutation of $(x_1',\cdots,x_n')$ and $\sum_{i=1}^n k_i=\sum_{i=1}^n k_i'$.

Markov process; (ii) Bounded non-negative measurable function $c(x)$ on D, which will be called the *killing rate*; (iii) A sequence $\{q_n(x); n=0,1,2,\cdots\}$ of measurable functions on D satisfying

$$\sum_{n=0}^{\infty}|q_n|(x)=1, \qquad x\in D,$$

where $|q_n|=q_n^+ + q_n^-$, $q_n^+=q_n\vee 0$, and $q_n^-=(-q_n)\vee 0$; (iv) A sequence $\{\pi_n(x,d\boldsymbol{y}); n=1,2,3,\cdots\}$ of probability kernels defined on $D\times D^{(n)}$, where $D^{(n)}$ is the n-fold topological product of D.

Our main theorem is stated as follows:

THEOREM 1.1. *Assume there are given a basic Markov process on D and a system of quantities* $\{c, q_n, \pi_n\}$ *stated above. Then there exists a strong Markov process* Z_t *on* $\hat{\boldsymbol{S}}=(\cup_{n=0}^{\infty} S^n\times J)\cup\{\Delta\}$ *which has right continuous path functions with left limits before the terminal time* ζ *such that*:

(i) *Let* U_t *be the semi-group of the process* Z_t,[5] f *a bounded measurable function on D and* λ *a positive constant. Then* U_t *satisfies the following extended "branching property"*

$$U_t\widetilde{f\cdot\lambda}=\widetilde{(U_t\widetilde{f\cdot\lambda})|_D\cdot\lambda},\text{[6]} \tag{1.1}$$

where

$$\widetilde{f\cdot\lambda}(\boldsymbol{x},\boldsymbol{k},j)=(-1)^{[j/2]}\widehat{f\cdot\lambda}(\boldsymbol{x},\boldsymbol{k}), \tag{1.2}$$

where [ ] *is Gauss's bracket i.e.* [0/2], [1/2]=0, [2/2], [3/2]=1, *and*

$$\widehat{f\cdot\lambda}(\boldsymbol{z})=\begin{cases}\lambda^{|\boldsymbol{k}|}\prod_{i=1}^{n}f(x_i), & \text{if } \boldsymbol{z}=(\boldsymbol{x},\boldsymbol{k})\in S^n, \quad \text{and} \quad |\boldsymbol{k}|=\sum_{i=1}^{n}k_i,\\ \lambda^k, & \text{if } \boldsymbol{z}=(\partial,k)\in S^0,\\ 0, & \text{if } \boldsymbol{z}=\Delta.\end{cases} \tag{1.3}$$

(ii) *Put*

$$u(t,x)=U_t\widetilde{f\cdot\lambda}(x,0,0), \qquad x\in D, \tag{1.4}$$

then it satisfies the following non-linear integral equation

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5) The semi-group U_t of Z_t is usually defined for a bounded measurable function g on $\hat{\mathbf{S}}$ with $g(\Delta)=0$ by

$$U_tg(\cdot)=E.[g(Z_t)], \qquad \cdot\in\hat{\mathbf{S}},$$

where $E.$ denotes the integration with respect to the probability measure $P.$ of the process. However we define U_tg by the above equation even when g is unbounded but $g(Z_t)$ is integrable with respect to $P.$. Therefore it happens that $U_tg(\cdot)$ blows up for some $t>0$.

6) For a function g defined on $\hat{\mathbf{S}}$, we write as $(g)|_D(x)=g(x,0,0)$, $x\in D$.

(1. 5) $$u(t,x)=T_t^{(\lambda)}f(x)+\int_0^t ds\,T_s^{(\lambda)}(cF[\cdot,u(t-s,\cdot)])(x),\qquad x\in D,$$

where $T_t^{(\lambda)}$ *is the semi-group defined in terms of the given basic Markov process* x_t *on D by*

(1. 6) $$T_t^{(\lambda)}f(x)=E_x[f(x_t)\exp((\lambda-2)\varphi_t)],$$

where

(1. 7) $$\varphi_t(w)=\int_0^t c(x_s(w))ds,$$

and F is defined by

(1. 8) $$F[x,u]=q_0(x)+\sum_{n=1}^{\infty}q_n(x)\int_{D^{(n)}}\pi_n(x,d\boldsymbol{y})\hat{u}(\boldsymbol{y}),\qquad x\in D,$$[7]

where u is a measurable function on D and

$$\hat{u}(\boldsymbol{y})=\prod_{i=1}^{n}u(y_i),\qquad \text{if}\quad \boldsymbol{y}=(y_1,y_2,\cdots,y_n)\in D^{(n)}.$$

The equalities in (1. 1) *and* (1. 5) *are taken to mean that if the left hand side has a definite value then both sides are equal.*

Moreover when $||f||\leqq r<1$, *there exists* $\varepsilon>0$ *depending on r and* $||f||$ *such that* $\widetilde{f\cdot\lambda}(Z_t)$, $t\in[0,\varepsilon)$, *is integrable with respect to the measure* $P_{(x,k,j)}$ *and hence* $u(t,x)=U_t\widetilde{f\cdot\lambda}(x,0,0)$, *where* $t\in[0,\varepsilon)$ *and* $x\in D$, *has a definite value and is a unique local solution of* (1. 5) *with the initial data f.*

DEFINITION. We shall call the Markov process Z_t which is stated in the above Theorem a *branching Markov process with age and sign.* The constant λ appearing in (1. 2) will be called a *weight of age.*

REMARK 1. For applications, it is useful to observe the fact that the existence of a solution $u(t,x)$ of (1. 5) is reduced to the integrability of $\widetilde{f\cdot\lambda}(Z_t)$, an unbounded function when $\lambda>1$, with respect to the measure $P_{(x,k,j)}$ of the branching Markov process with age and sign. When the initial data f and the weight λ satisfy $||f||<1$ and $0<\lambda\leqq 1$, respectively, there is no diffieulty. In fact, under the condition, $\widetilde{f\cdot\lambda}$ is a bounded function on $\hat{\boldsymbol{S}}$ and hence $u(t,x)=U_t\widetilde{f\cdot\lambda}(x,0,0)$ is well-defined for all $t\geqq 0$. Moreover we have $|u(t,x)|<1$ and therefore $u(t,x)$ is the unique global solution of (1. 5). The uniqueness follows from the following inequality: For any r satisfying $0<r<1$, there exists a constant $c(r)>0$ such that

(1. 9) $$||F[\cdot,u]-F[\cdot,v]||\leqq c(r)||u-v||,$$

7) $D^{(n)}=\underbrace{D\times D\times\cdots\times D}_{n}.$

provided $||u||$, $||v||\leqq r$, (cf. Lemma 1. 3 of [6]).

REMARK 2. Theorem 1. 1 is still valid when we take a non-negative continuous additive functional φ_t instead of the one defined in (1. 7). In this case (1. 5) becomes

$$u(t,x)=T_t^{(\lambda)}f(x)+\int_0^t\int_S K((x,0),dsd(y,k))\lambda^k F[y,u(t-s,\cdot)],\qquad x\in D, \tag{1. 5'}$$

where K is a kernel defined on $S\times S$ by

$$K((x,0),dsd(y,k))=P_{(x,0)}[\zeta\in ds, X_{\zeta-}\in d(y,k)],\qquad x\in D,$$

in terms of a *Markov process with age* that will be constructed in § 3.

REMARK 3. We assumed in Theorem 1. 1 that the basic Markov process is conservative i.e. the life time $\zeta=\infty$, P_x-a.e., but it is useful in applications to distinguish the following case: the state space D contains a point ∂ as a terminal point and $x_{\zeta-}=\partial$, P_x-a.e., where ζ is the first hitting time of ∂. For example, x_t is the part of the d-dimensional Brownian motion on a bounded domain U (cf. [3]) and $D=U\cup\{\partial\}$ is the one-point compactification of U, then this is the case stated above. We will give a remark about this point when we construct a branching Markov process with age and sign in § 3 and § 4.

REMARK 4. If we take

$$\pi_n(x,d\boldsymbol{y})=\delta_{\underbrace{(x,x,\cdots,x)}_{n}}(d\boldsymbol{y}),$$

then

$$F[x,u]=\sum_{n=0}^{\infty}q_n(x)u(x)^n, \tag{1. 10}$$

and if we take 2 as a weight of age, then

$$T_t^{(2)}f(x)=E_x[f(x_t)]=T_tf(x),$$

where T_t is the semi-gromp of the basic Markov process. This is the typical case appearing in many applications.

REMARK 5. We give a simple example. Let D be the d-dimensional Euclidean space with one point compactification, the basic Markov process on D be the d-dimensional standard Brownian motion, and let F be given in (1. 10) with a condition $|c(x)\sum_{n=0}^{\infty}q_n(x)-c(y)\sum_{n=0}^{\infty}q_n(y)|\leqq\gamma|x-y|$, where γ is a positive constant and $|x-y|$ is a distance of D. Moreover we assume that $c, q_n(x)$ $(n=0,1,2,3,\cdots)$, and f are bounded continuous function on D. Then we can obtain from (1. 5) that $u(t,x)=U_t\widetilde{f\cdot 2}(x,0,0)$, $x\in D$, $t\in[0,\varepsilon)$, $||f||\leqq r<1$, satisfies

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+c(x)F[u],\qquad u(0,x)=f(x), \tag{1. 11}$$

where Δ is d-dimensional Laplacian operator and $\varepsilon>0$ depends on r, (cf. Remark 1 above, and Theorem 1 of [10]). In the case of simple branching Brownian motion (i.e. without age and sign), $u(t,x)=E_x[\hat{f}(X_t)]$, $x\in R^d$ satisfies

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+c(x)\cdot\{F[u]-u\}, \qquad u(0,x)=f(x), \tag{1.12}$$

where $\|f\|<1$, $t\in[0,\infty)$ and

$$F[u]=\sum_{\substack{n=0\\ n\neq 1}}^{\infty} q_n(x)u(x)^n, \qquad q_n(x)\geqq 0,$$

(cf. [6]). The difference between two cases brought by introducing age, sign, and weight of age should be remarked.

§ 2. Preliminaries.

Since the construction of the branching Markov process with age and sign Z_t that will be given in the following sections heavily leans upon a theorem of piecing out paths of Markov process [6], we will give a brief description of the theorem to make this paper self-contained.

Let E be a locally compact Hausdorff space with a countable base and $\hat{E}=E\cup\{\Delta\}$ be the one-point compactification of E (if E is compact, Δ is attached as an extra point). Let $\{W, N_t, P_x, x_t, \zeta, \theta_t\}$ be a Markov process on $\hat{E}=E\cup\{\Delta\}$ with Δ as a terminal point, where W is the path space, N_t is the smallest Borel field on W with respect to which $x_s (s\leqq t)$ are measurable and ζ is the life time of x_t i.e. the first hitting time to Δ,[8] and θ_t is the shift operator of path i.e. $\theta_t w(s)=w(t+s)$, (e.g. [3], [8]). We assume in addition that all path functions are right continuous and with left limits, and the process satisfies the *strong Morkov property*: For every N_{t+}-Markov time T and for any Borel set A of E, it holds that

$$P_x[x_{T+t}\in A, T<\infty | N_{T+}]=I_{\{T<\infty\}}P_{x_T}[x_t\in A], \qquad P_x\text{-a.e.},$$

where $N_{T+}=\{B;\ B\in N_\infty$ and for every $t>0$ $B\cap\{w;\ T(w)\leqq t\}\in N_{t+}\}$ and T is said to be *N_{t+}-Markov time* if $\{w;\ T(w)\leqq t\}\in N_{t+}$.[8]

Let $\mu(w,dx)$ be a probability kernel defined on $W\times\hat{E}$, i.e. $\mu(w,\cdot)$ is a probability measure on $\hat{E}$ and $\mu(\cdot,A)$ is a N_∞-measurable function for any Borel subset A of $\hat{E}$. The kernel $\mu(w,dx)$ will be called an *instantaneous distribution*, if it has the properties: (i) For $w\in W$ such that $\zeta(w)=0$ or $\zeta(w)=\infty$, $\mu(w,dy)=\delta_\Delta(dy)$, and (ii) for any N_{t+}-Markov time $T(w)$,

$$P_x[\mu(w,dy)=\mu(\theta_T w,dy), T<\zeta]=P_x[T<\zeta], \qquad x\in E, \tag{2.1}$$

where ζ is the life time of x_t.

Let $\tilde{\Omega}$ be the infinite product of $\Omega=W\times\hat{E}$, i.e.

8) We assume that $x_t(w)=\Delta$ for $t\geqq\zeta(w)$. $N_{t+}=\bigcap_{\varepsilon>0}N_{t+\varepsilon}$.

$$\tilde{\Omega} = \prod_{j=1}^{\infty} \Omega_j, \qquad (\Omega_j = \Omega), \tag{2.2}$$

with the product Borel field

$$\widetilde{\mathscr{B}} = \prod_{j=1}^{\infty} \mathscr{F}_j, \qquad (\mathscr{F}_j \equiv \mathscr{F} = N_\infty \otimes \mathscr{B}(\hat{E})), \tag{2.3}$$

where $\mathscr{B}(\hat{E})$ is the Borel field of $\hat{E}$, and define a probability kernel $Q(x, d\omega)$ on $\hat{E} \times \Omega$ by

$$\int_A Q(x, d\omega) = \int_A \int P_x[dw] \mu(w, dy), \qquad A \in \mathscr{F}. \tag{2.4}$$

Then there exists a unique system of probability measures $\{\tilde{P}_x, x \in \hat{E}\}$ on $(\tilde{\Omega}, \widetilde{\mathscr{B}})$ such that for any bounded measurable function $F(\omega^1, \omega^2, \cdots, \omega^n)$ on $(\Pi_{j=1}^n \Omega_j, \Pi_{j=1}^n \mathscr{F}_j)$, $(n = 1, 2, 3, \cdots)$,

$$\tilde{P}_x[F(\omega^1, \cdots, \omega^n)] = \int \cdots \int Q(x, d\omega^1) Q(x_1, d\omega^2) Q(x_2, d\omega^3) \cdots Q(x_{n-1}, d\omega^n) \cdot F(\omega^1, \omega^2, \cdots, \omega^n), \tag{2.5}$$

where $\omega^j = (w_j, x_j)$. The existence of such measure $\tilde{P}_x$ is guaranteed by Ionescu-Tulcea's Theorem (e.g. Loéve [11], p. 137).

Now we shall define a new process X_t on $(\tilde{\Omega}, \widetilde{\mathscr{B}}, \tilde{P}_x)$ by piecing out the paths of the given Markov process x_t as follows: First of all we put for $\omega = (w, y) \in \Omega$

$$\dot{x}_t(\omega) = \begin{cases} x_t(w), & \text{if } t < \zeta(w) \\ y, & \text{if } t \geqq \zeta(w). \end{cases} \tag{2.6}$$

$\{\dot{x}_t, Q_x\}$ is generally not Markov process, however we can obtain a strong Markov process by piecing out $\dot{x}_t$ infinitely many times. Putting for $\tilde{\omega} = (\omega^1, \omega^2, \cdots) \in \tilde{\Omega}$, $\omega^i = (w_i, x_i)$

$$N(\tilde{\omega}) = \begin{cases} \inf \{j; \zeta(w_j) = 0\} \\ \infty, \quad \text{if such } j \text{ does not exist,} \end{cases}$$

we define $X_t(\tilde{\omega})$ on $(\tilde{\Omega}, \widetilde{\mathscr{B}})$ by

$$X_t(\tilde{\omega}) = \begin{cases} \dot{x}_t(\omega^1), & \text{if } 0 \leqq t \leqq \zeta(w_1) \\ \dot{x}_{t-\zeta(w_1)}(\omega^2), & \text{if } \zeta(w_1) < t \leqq \zeta(w_1) + \zeta(w_2) \\ \cdots\cdots & \\ \dot{x}_{t-(\zeta(w_1)+\cdots+\zeta(w_n))}(\omega^{n+1}), & \text{if } \sum_{j=1}^{n} \zeta(w_j) < t \leqq \sum_{j=1}^{n+1} \zeta(w_j) \\ \cdots\cdots & \\ \Delta, \quad \text{if } t \geqq \sum_{j=1}^{N(\tilde{\omega})} \zeta(w_j), & \end{cases} \tag{2.7}$$

and define the life time of X_t by

(2.8) $$\tilde{\zeta}(\tilde{\omega})=\sum_{j=1}^{N(\tilde{\omega})}\zeta(w_j),$$

and introduce a sequence of random times

$$\tau_0(\tilde{\omega})=0, \qquad \tau(\tilde{\omega})=\tau_1(\tilde{\omega})=\zeta(w_1), \qquad \tau_n(\tilde{\omega})=\left\{\sum_{j=1}^{n}\zeta(w_j)\right\}\wedge\tilde{\zeta}(\tilde{\omega}).$$

It is easily seen that $\tilde{P}_x[\tilde{\Omega}_0]=1$, where $\tilde{\Omega}_0=\{\tilde{\omega};\, X_t(\tilde{\omega})$ is right continuous with respect to $t\geqq 0\}$. The shift operator θ_t of $\tilde{\omega}\in\tilde{\Omega}_0$ is defined by

(2.9) $$\theta_t\tilde{\omega}=((\theta_{t-\tau_k(\tilde{\omega})}w_{k+1}, x_{k+1}), \omega^{k+2}, \cdots), \qquad \text{if} \quad \tau_k(\tilde{\omega})\leqq t<\tau_{k+1}(\tilde{\omega}).$$

Let $T(\tilde{\omega})$ be a random variable on $\tilde{\Omega}_0$ taking values in $[0,\infty]$. We shall call $\tilde{\omega}$ and $\tilde{\omega}'\in\tilde{\Omega}_0$ R_T-equivalent and denote as $\tilde{\omega}\sim\tilde{\omega}'$ (R_T) if: (a) $T(\tilde{\omega})=T(\tilde{\omega}')$, (b) $X_s(\tilde{\omega})=X_s(\tilde{\omega}')$, for $s\leqq T(\tilde{\omega})$, and (c) if $\tau_k(\tilde{\omega})\leqq T(\tilde{\omega})<\tau_{k+1}(\tilde{\omega})\leqq\tilde{\zeta}(\tilde{\omega})$, then $\tau_k(\tilde{\omega}')\leqq T(\tilde{\omega}')<\tau_{k+1}(\tilde{\omega}')\leqq\tilde{\zeta}(\tilde{\omega}')$ and $\tau_j(\tilde{\omega})=\tau_j(\tilde{\omega}')$ for every $j\leqq k$, while if $T(\tilde{\omega})\geqq\tilde{\zeta}(\tilde{\omega})$, then $T(\tilde{\omega}')\geqq\tilde{\zeta}(\tilde{\omega}')$ and $\tau_j(\tilde{\omega})=\tau_j(\tilde{\omega}')$ for every $j\geqq 0$. Put

$$\tilde{\mathscr{B}}_T=\{A;\ \text{(i)}\ A\in\tilde{\mathscr{B}}\cap\tilde{\Omega}_0 \text{ and (ii) if } \tilde{\omega}\in A \text{ and } \tilde{\omega}\sim\tilde{\omega}'(R_T) \text{ then } \tilde{\omega}'\in A\}.$$

Then $\tilde{\mathscr{B}}_T$ is a Borel field of $\tilde{\Omega}_0$ and $\{\tilde{\mathscr{B}}_t : t\geqq 0\}$ is an increasing family and $\tilde{N}_t\subset\tilde{\mathscr{B}}_t$, where $\tilde{N}_t$ is the Borel field generated by X_s, $(s\leq t)$.

On the basis of the above notation our theorem reads as follows:

THEOREM 2.1 ([6]). *Let* $\{W, N_t, P_x, x_t, \zeta, \theta_t\}$ *be a strong Markov process on* $E=E\cup\{\Delta\}$ *which has right continuous path functions with left limit with* Δ *as a terminal point, and* $\mu(w,dy)$ *be an instantaneous distribution. Then the system* $\{\tilde{\Omega}_0, \tilde{\mathscr{B}}_{t+}, P_x, X_t, \tilde{\zeta}, \theta_t\}$[9] *defined above is a strong Markov process on* $\hat{E}$ *which has right continuous path functions with left limits before* $\tilde{\zeta}$ *with* Δ *as a terminal point having the properties* (i) *The sub-process* $\{X_t, \tilde{P}_x, t<\tau\}$ *of* $\{X_t\, \tilde{P}_x\}$ *is equivalent to the Markov process* $\{x_t, P_x, t<\zeta\}$, *and* (ii) *for* $\Gamma\in\mathscr{B}(\hat{E})$

(2.10) $$\tilde{P}_x[X_\tau\in\Gamma, \{\tilde{\omega};\, w_1\in B\}]=\int_B P_x[dw]\mu(w,\Gamma),$$

where B belongs to the Borel field generated by x_s, $0\leqq s<\zeta$, *and we write* $\tilde{\omega}=(\omega^1,\omega^2,\cdots)$ *and* $\omega^i=(w_i, x_i)$, $i=1,2,\cdots$. *Moreover if* $x_t(w)$ *is quasi-left-continuous and* ζ *is non-accessible* (*i.e. totally inaccessible in ihe strong sense* [12] p. 130), *then* $X_t(\tilde{\omega})$ *is quasi-left-continuous before* $\tilde{\zeta}(\tilde{\omega})$.

§ 3. Markov process with age.

We will construct a Markov process with a new parameter *age* from a given

9) $\tilde{\mathscr{B}}_{t+}=\bigcap_{\varepsilon>0}\tilde{\mathscr{B}}_{t+\varepsilon}$. We may take, if necessary, a completion of $\tilde{\mathscr{B}}_{t+}$ instead of $\tilde{\mathscr{B}}_{t+}$ by the standard argument (e.g. [3]).

Markov process and a killing rate $c(x)$. Let D be a compact Hausdorff space with a countable open base. Let $\{W, N_t, P_x, x_t, \zeta, \theta_t\}$ be a conservative strong Markov process on D which has right continuous path functions with left limit. In the following we shall call the process x_t the *basic Morkov process*.

Taking a measurable function c on D, we put

$$\text{(3. 1)} \qquad \begin{cases} \varphi_t(w)=\int_0^t |c|(x_s(w))ds, \\ \varphi_t^+(w)=\int_0^t c^+(x_s(w))ds, \\ \varphi_t^-(w)=\int_0^t c^-(x_s(w))ds, \end{cases}$$

where we write $c^+=c\vee 0$, $c^-=(-c)\vee 0$, and $|c|=c^++c^-$.

Now let $\{\bar{W}, \bar{N}_t, \bar{P}_x, \bar{x}_t, \bar{\zeta}, \bar{\theta}_t\}$ be a sub-process[10] of x_t that is obtained from x_t by curtailing the life time with killing rate $|c|(x)$, such that

$$\text{(3. 2)} \qquad \bar{E}_x[f(\bar{x}_t)]=E_x[f(x_t)m_t],$$

where

$$\text{(3. 3)} \qquad m_t(w)=\exp(-\varphi_t(w)).$$

First of all we have

LEMMA 3. 1. *Let $g(x, s)$ be a bounded measurable function on $D\times[0,\infty)$ and $\bar{x}_{\bar{\zeta}-}=\lim_{t\uparrow\bar{\zeta}}\bar{x}_t$, then*

$$\text{(3. 4)} \qquad \bar{E}_x[g(\bar{x}_{\bar{\zeta}-}, \bar{\zeta}), \bar{\zeta}\leqq t]=E_x\left[\int_0^t g(x_s, s)e^{-\varphi_s}|c|(x_s)ds\right].$$

Proof. ([16]) It is sufficient to prove (3. 4) for $g(x, t)=f(x)e^{-\lambda t}$, bounded continuous f on D and $\lambda>0$. In this case we have

$$\begin{aligned} &E_x\left[\int_0^t f(x_s)e^{-\lambda s}e^{-\varphi_s}|c|(x_s)ds\right] \\ &=E_x\left[\int_0^t f(x_{s-})e^{-\lambda s}d(-e^{-\varphi_s})\right] \\ &=E_x\left[\lim_{h\downarrow 0}\sum_{ih<t} f(x_{ih})e^{-\lambda ih}(e^{-\varphi_{ih}}-e^{-\varphi_{(i+1)h}})\right] \\ &=\lim_{h\downarrow 0}\sum_{ih<t}\bar{E}_x[e^{-\lambda ih}f(\bar{x}_{ih}); ih<\bar{\zeta}\leqq(i+1)h] \\ &=\bar{E}_x[e^{-\lambda\bar{\zeta}}f(\bar{x}_{\bar{\zeta}-}); \bar{\zeta}\leqq t]. \end{aligned}$$

Therefore the condition (c. 2) of [6] is automatically satisfied i.e.

10) Cf. Dynkin [3] or Ito-McKean [8].

(3. 5) $$\bar{P}_x[\bar{\zeta}=t]=0 \qquad \text{for any fixed } t\in[0,\infty).$$

Now let $x_t^{\bullet}$ be an infinite collection of $\bar{x}_t$, precisely speaking, we define $x_t^{\bullet}$ as follows: put

(3. 6)
$$\hat{S}=(D\times N)\cup\{\varDelta\}, \qquad N=\{0,1,2,3,\cdots\},$$
$$W^{\bullet}=(\overline{W}\times N)\cup\{w_{\varDelta}\},$$

where $w_{\varDelta}$ is an extra point, and

$$x_t^{\bullet}(w^{\bullet})=\begin{cases}(\bar{x}_t(\overline{w}),k), & \text{if } w^{\bullet}=(\overline{w},k), \text{ and } t<\zeta(\overline{w})\\ \varDelta, & \text{if } w^{\bullet}=w_{\varDelta} \text{ or } t\geqq\bar{\zeta}(\overline{w}),\end{cases}$$

$$\zeta^{\bullet}(w^{\bullet})=\begin{cases}\bar{\zeta}(\overline{w}), & \text{if } w^{\bullet}\neq w_{\varDelta}\\ 0, & \text{if } w^{\bullet}=w_{\varDelta},\end{cases}$$

(3. 7)
$$\theta_t^{\bullet}(w,k)=\begin{cases}(\bar{\theta}_t\overline{w},k), & \text{if } t<\bar{\zeta}(\overline{w})\\ w_{\varDelta}, & \text{if } t\geqq\bar{\zeta}(\overline{w}),\end{cases}$$

$$P_{(x,k)}^{\bullet}[A]=\bar{P}_x[\{\overline{w};(\overline{w},k)\in A\}], \qquad A\in N_{\infty}^{\bullet},$$

$$P_{\varDelta}^{\bullet}[A]=\bar{P}_{\varDelta}[A\cap\overline{W}],$$

where $N_t^{\bullet}$ is the Borel field generated by $x_s^{\bullet}(s\leq t)$, $N_{\infty}^{\bullet}=\vee_{t>0}N_t^{\bullet}$, and we write as $x_t^{\bullet}=(\bar{x}_t,k_t)$.

Clearly $\{W^{\bullet},N_t^{\bullet},P_{(x,k)}^{\bullet},x_t^{\bullet},\zeta^{\bullet},\theta_t^{\bullet}\}$ is a strong Markov process on $\hat{S}=(D\times N)\cup\{\varDelta\}$ which has right continuous path functions with left limits with $\varDelta$ as a terminal point.

Next we define a probability kernel $\pi(w^{\bullet},dy^{\bullet})$ on $W^{\bullet}\times\hat{S}$ by

(3. 8) $$\pi(w^{\bullet},dy^{\bullet})=\begin{cases}\pi'(x^{\bullet}_{\zeta^{\bullet}-}(w^{\bullet}),dy^{\bullet}) & \text{if } 0<\zeta^{\bullet}(w^{\bullet})<+\infty,\\ \delta_{\varDelta}(dy^{\bullet}), & \text{if } \zeta^{\bullet}(w^{\bullet})=0, \text{ or } +\infty,\end{cases}$$

where

(3. 9)
$$\pi'((x,k),dy^{\bullet})=\begin{cases}\delta_x(dy)\delta_{k+1,k'} & \text{if } c^{+}(x)\geqq0, \text{ and } y^{\bullet}=(y,k'),^{11)}\\ \delta_{\varDelta}(dy^{\bullet}), & \text{if } c^{-}(x)>0,\end{cases}$$
$$\pi'(\varDelta,dy^{\bullet})=\delta_{\varDelta}(dy^{\bullet}).$$

11) When D contains a point δ as a terminal point of the basic Markov process and $x_{\zeta-}=\delta$, where ζ is the first hitting time to δ, we put $\pi'((\delta,k),dy^{\bullet})=\delta_{\delta}(dy)\delta_{k,k'}$, where $y^{\bullet}=(y,k')$, and assume $c(\delta)=0$.

Then it is easily seen that $\pi(w^\cdot, dy^\cdot)$ is an instantaneous distribution of $x_t^\cdot$. We can get, therefore, a strong Markov process $\{\Omega, N_t, P_{(x,k)}, (x,k)\in\hat{S}, X_t=(x_t, k_t), \zeta, \theta_t\}$ on $\hat{S}$ which has right continuous path functions with left limit with Δ as a terminal point by means of theorem of piecing out stated in § 2. We shall call this process $X_t\equiv(x,k_t)$ on $\hat{S}$ *Markov process with age.* We give a picture of this process in Figure 1, where

$$
\begin{aligned}
&\sigma(\omega)=\inf\{t; X_t\notin D\times k, \text{ when } k_0(\omega)=k\},\\
(3.10)\qquad &\quad=\infty, \quad \text{if } \{\ \} \text{ is void},\\
&\sigma_0(\omega)\equiv 0, \qquad \sigma_1(\omega)=\sigma(\omega), \qquad \sigma_n(\omega)=\sigma_{n-1}(\omega)+\theta_{\sigma_{n-1}}\sigma(\omega), \qquad (n\geqq 2).
\end{aligned}
$$ [12]

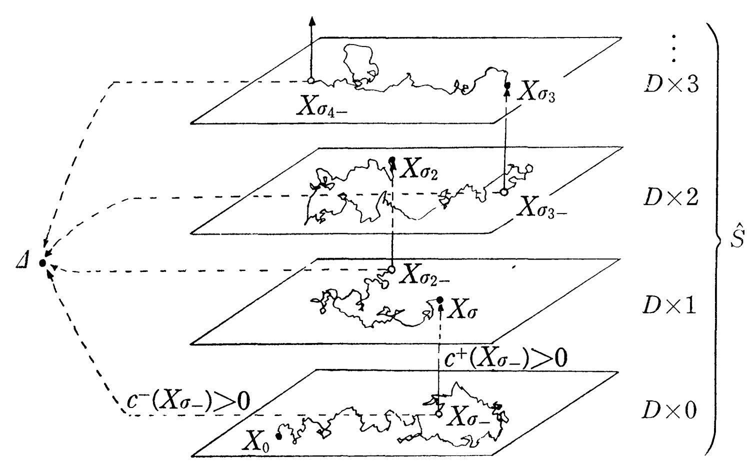

Figure 1.

LEMMA 3.2. *Let f be a non-negative measurable function on D, λ a positive constant, α a non-negative constant, and*

$$b_t(w)=\int_0^t b(x_s(w))ds, \qquad b_t(\omega)=\int_0^t b(x_s(\omega))ds,$$

where b is a non-negative measurable function on D. Then we have[13]

$$(3.11)\qquad E_{(x,k)}[f\cdot\lambda(X_t)e^{-\alpha b_t}; \sigma_n\leq t<\sigma_{n+1}]=E_x\left[f(x_t)e^{-\alpha b_t-\varphi_t}\frac{(\lambda\varphi_t^+)^n}{n!}\right]\lambda^k,$$

where the expectation E_x of the right hand side is that of the basic process, and $f\cdot\lambda$ is a function on $\hat{S}$ defined by

12) σ_n $(n=0,1,2,\cdots)$ is N_{t+}-Markov time, and $P_{(x,k)}[\sup_n \sigma_n<\infty\,;\ \forall_n, \sigma_n<\zeta]=0$.

13) We will use the same notation b_t, but it may produce no confusion.

$$f\cdot\lambda(x,k)=f(x)\lambda^k, \qquad if \quad (x,k)\in D\times N,\text{[14)]} \tag{3.12}$$
$$f\cdot\lambda(\Delta)=0.$$

Proof. By the way of construction of X_t, we have $E_{(x,k)}[f\cdot\lambda(X_t)e^{-\alpha b_t}; t<\sigma] = E_x[f(x_t)\exp(-\alpha b_t-\varphi_t)]\lambda^k$. Thus (3. 11) is true for $n=0$. Assume that (3. 11) is valid for $n-1$. Then we have

$$\begin{aligned}
&E_{(x,k)}[f\cdot\lambda(X_t)e^{-\alpha b_t}; \sigma_n\leqq t<\sigma_{n+1}] \\
&=E_{(x,k)}[f\cdot\lambda(X_t)e^{-\alpha b_t}; \sigma_n\leqq t<\sigma_{n+1}, c^-(x_{\sigma-})=0] \\
&=E_{(x,k)}[e^{-\alpha b_\sigma}E_{(x_\sigma,k_\sigma)}[\lambda^{k_{t-s}}f(x_{t-s})e^{-\alpha b_{t-s}}; \sigma_{n-1}\leqq t-s<\sigma_n]|_{s=\sigma}; \sigma\leqq t, c^-(x_{\sigma-})=0], \\
&=E_{(x,k)}[e^{-\alpha b_\sigma}E_{(x_{\sigma-},k_\sigma)}[\lambda^{k_{t-s}}f(x_{t-s})e^{-\alpha b_{t-s}}; \sigma_{n-1}\leqq t-s<\sigma_n]|_{s=\sigma}; \sigma\leqq t, c^-(x_{\sigma-})=0],
\end{aligned}$$

noting (3. 4) and $k_\sigma=k_0+1$, we can continue the above, using the induction hypothesis and Lemma 3. 1, as

$$\begin{aligned}
&=E_x\left[\int_0^t d\varphi_s^+ e^{-\varphi_s-\alpha b_s}E_{x_s}\left[f(x_{t-s})e^{-\varphi_{t-s}-\alpha b_{t-s}}\frac{(\lambda\varphi_{t-s}^+)^{n-1}}{(n-1)!}\right]; c^-(x_{s-})=0\right]\lambda^{k+1} \\
&=E_x\left[\int_0^t d\varphi_s^+ e^{-\varphi_s-\alpha b_s}f(x_t)e^{-\varphi_{t-s}(\theta_s w)-\alpha b_{t-s}(\theta_s w)}\frac{(\lambda\varphi_{t-s}^+(\theta_s w))^{n-1}}{(n-1)!}\right]\lambda^{k+1} \\
&=E_x\left[e^{-\varphi_t-\alpha b_t}f(x_t)\int_0^t \lambda d\varphi_s^+\frac{(\lambda(\varphi_t^+-\varphi_s^+))^{n-1}}{(n-1)!}\right]\lambda^k \\
&=E_x\left[f(x_t)e^{-\varphi_t-\alpha b_t}\frac{(\lambda\varphi_t^+)^n}{n!}\right]\lambda^k.
\end{aligned}$$

Thus (3. 11) is true for n.

COROLLARY 3. 3. *For a non-negative measurable function f on D and a positive constant λ*

$$E_{(x,k)}[f\cdot\lambda(X_t)]=E_x[f(x_t)e^{-\varphi_t+\lambda\varphi_t^+}]\lambda^k, \tag{3.13}$$

$$E_{(x,k)}[f\cdot\lambda(X_t)e^{-\varphi_t}]=E_x[f(x_t)e^{-2\varphi_t+\lambda\varphi_t^+}]\lambda^k. \tag{3.14}$$

REMARK 1. From (3. 11) we have

$$E_{(x,k)}[f\cdot\lambda(X_t); \sigma_n\leqq t<\sigma_{n+1}]=E_{(x,0)}[f\cdot\lambda(X_t); \sigma_n\leqq t<\sigma_{n+1}]\lambda^k. \tag{3.15}$$

REMARK 2. If we take $\lambda=2$ and put

$$T_t'f(x)=E_{(x,0)}[f\cdot 2(X_t)],$$

14) When $\lambda>1$ ($\lambda<1$), $\lambda^{+\infty}=+\infty$ $(+0)$ and when $\lambda=1$, $\lambda^{+\infty}=1$.

then

$$T'_t f(x)=E_x\Big[f(x)\exp\Big(\int_0^t c(x_s)ds\Big)\Big]. \tag{3.16}$$

Therefore, if $E_x[\exp(\int_0^t c(x_s)ds)]\leqq e^{c_0 t}$ with some constant $c_0>0$, it defines a semi-group on $\boldsymbol{B}(D)$, the space of bounded measurable functions on D, which may describe the creation of mass.

§ 4. Branching Markov process with age and sign.

We shall construct a *branching Markov process with age and sign* which has the Markov process with age as a "non-branching part" (this terminology comes from [6]). Let $\{\Omega, N_t, P_{(x,k)}, (x,k)\in D\times N, X_t\equiv(x_t, k_s), \theta_t\}$ be a Markov process with age on $\hat{S}=(D\times N)\cup\{\varDelta\}$ constructed in the previous section, but here we assume that $c(x)$ which appeared in the first paragraph of § 3, is non-negative. Taking the same functioh c, we put

$$\varphi_t(\omega)=\int_0^t c(x_s(\omega))ds, \tag{4.1}$$

and

$$m_t(\omega)=\exp(-\varphi_t(\omega)). \tag{4.2}$$

Let $\{\bar{\Omega}, \bar{N}_t, \bar{P}_{(x,k)}, \bar{X}_t, \bar{\zeta}, \bar{\theta}_t\}$ be the m_t-sub-process of the Markov process with age X_t. Next we make the *n-fold symmetric direct product* $\{\bar{\Omega}^n, \bar{N}^n_t, \bar{P}^n_{(x,k)}, \bar{X}^n_t, \bar{\zeta}^n, \bar{\theta}^n_t\}$ of $\bar{X}_t$ which stands for n-particles moving on and growing older independently. Here

$$\begin{aligned}
&\bar{\Omega}^n=\bar{\Omega}\times\bar{\Omega}\times\cdots\times\bar{\Omega},\\
&\bar{\zeta}^n(\omega)=\min\{\bar{\zeta}(\omega^k); k=1,2,\cdots,n\},\\
&\bar{X}^n_t(\omega)=\begin{cases}\rho(\bar{X}_t(\omega^1),\cdots,\bar{X}_t(\omega^n)), & \text{if } t<\bar{\zeta}^n(\omega),\\ \varDelta, & \text{if } t\geqq\bar{\zeta}^n(\omega),\end{cases}\\
&\bar{\theta}^n_t\omega=(\bar{\theta}_t\omega^1, \bar{\theta}_t\omega^2,\cdots,\bar{\theta}_t\omega^n),
\end{aligned} \tag{4.3}$$

where $\omega=(\omega^1,\omega^2,\cdots,\omega^n)\in\bar{\Omega}^n$ and ρ is the natural mapping from the direct product $S^{(n)}$ onto the quotient space S^n. Let $\bar{N}^n_t$ be the smallest Borel field with respect to which $\bar{X}^n_s$ $(s\leqq t)$ is measurable and put for $A\in\bar{N}^n_\infty$,

$$\bar{P}^n_z[A]=\begin{cases}\bar{P}_{(x_1,k_1)}\times\bar{P}_{(x_2,k_2)}\times\cdots\times\bar{P}_{(x_n,k_n)}[A], & \text{if } z=(\boldsymbol{x},\boldsymbol{k})\in S^n,\\ \bar{P}_\varDelta\times\cdots\times\bar{P}_\varDelta[A], & \text{if } z=\varDelta.\end{cases} \tag{4.4}$$

PROPOSITION 4. 1.[15] *The n-fold symmetric direct product* $\{\bar{\Omega}^n, \bar{N}_t^n, \bar{P}_{(x,k)}^n, \bar{X}_t^n, \bar{\zeta}^n, \bar{\theta}_t^n\}$ *defined above is a strong Markov process on* $S^n \cup \{\Delta\}$ *which has right continuous path functions with left limits, with* Δ *as a terminal point.*

Now we prepare four copies of the direct sum of all $\bar{X}_t^n$, $(n=1, 2, 3, \cdots)$. Strictly speaking we construct a large Markov process Z_t^0 on $\hat{\boldsymbol{S}}$ in the following way: We put

$$\hat{\boldsymbol{S}}=\left\{\left(\bigcup_{n=0}^{\infty} S^n\right)\times J\right\}\cup\{\Delta\}. \tag{4. 5}$$

where $S=D\times N$, $N=\{0, 1, 2, 3, \cdots\}$, $J=\{0, 1, 2, 3\}$, $S^0=\{\partial\}\times N$, ∂ is an extra point, and S^n, $(n\geq 1)$ is the symmetric n-fold direct product of S. Moreover we put

$$\Omega^0=\bigcup_{n=0}^{\infty} \bar{\Omega}^n\times J, \qquad \bar{\Omega}^0\times J=\{\omega_{\partial kj};\, k\in N, j\in J\} \tag{4. 6}$$

where $\{\omega_{\partial kj}\}$ are extra points, and

$$\zeta^0(\omega^0)=\begin{cases}\bar{\zeta}^n(\omega), & \text{if } \omega^0=(\omega, j),\ \ \omega\in\bar{\Omega}^n,\\ +\infty, & \text{if } \omega^0\in\bar{\Omega}^0\times J,\end{cases}$$

$$Z_t^0(\omega^0)=\begin{cases}(\bar{X}_t^n(\omega), j), & \text{if } \omega^0=(\omega, j)\in\bar{\Omega}^n\times J,\ \ t<\zeta^0(\omega^0),\\ \Delta, & \text{if } \omega^0\in\bar{\Omega}^n\times J,\ \ t\geqq\zeta^0(\omega^0),\\ (\partial, k, j), & \text{if } \omega^0=\omega_{\partial kj}\in\bar{\Omega}^0\times J,\end{cases} \tag{4. 7}$$

$$\theta_t^0\omega^0=\begin{cases}(\bar{\theta}_t^n\omega, j), & \text{if } \omega^0=(\omega, j)\in\bar{\Omega}^n\times J,\\ \omega_{\partial kj}, & \text{if } \omega^0=\omega_{\partial kj}.\end{cases}$$

Let N_t^0 be the smallest Borel field with respect to which Z_s^0, $(s\leq t)$ is measurable, and put for $A\in N_\infty^0$

$$\begin{cases}P_{(x,k,j)}^0[A]=\bar{P}_{(x,k)}^n[\{\omega;\, \omega^0=(\omega, j)\in A,\, \omega\in\bar{\Omega}^n\}], \qquad (x, k)\in S^n,\\ P_{(\partial,k,j)}^0[A]=\delta_{\{\omega_{\partial kj}\}}(A),\\ P_\Delta^0 \text{ is any probability measure on } (\Omega^0, N_\infty^0) \text{ such that}\\ P_\Delta^0[Z_t(\omega^0)=\Delta \text{ for all } t\in[0, \infty)]=1.\end{cases} \tag{4. 8}$$

PROPOSITION 4. 2. *The system* $\{\Omega^0, N_t^0, P_{(x,k,j)}^0, Z_t^0, \zeta^0, \theta_t^0\}$ *defined above is a*

15) A proof is found in [6].

strong Markov process on $\hat{\boldsymbol{S}}=\{(\cup_{n=0}^{\infty} S^n)\times J\}\cup\{\Delta\}$ *which has right continuous path functions with left limits with* Δ *as a terminal point.*

The process Z_t^0 is a mathematical model of motion of arbitrary number of particles moving on and growing older independently in four countries (the same copies).

We will write in the following as $Z_t^0=(X_t^0, J_t^0)$.[16]

The next step we have to do is to define a law that governs the situation of splitting of particles and that decides a country where those particles will imigrate.

Let $\{q_n; n=0,1,2,\cdots\}$ be a a sequence of bounded measurable functions on D satisfying

$$\sum_{n=0}^{\infty} |q_n|(x)=1, \qquad x\in D, \tag{4. 9}$$

where $|q_n|=q_n^+ + q_n^-$, $q_n^+=q_n\vee 0$, $q_n^-=(-q_n)\vee 0$, and take a sequence $\{\pi_n(x, d\boldsymbol{y}); n=1,2,3,\cdots\}$ of probability kernels defined on $D\times D^{(n)}$, $n=1,2,3,\cdots$, where $D^{(n)}$ is the n-fold direct product of D.[17]

First of all we define a probability kernel on $S\times S^n$, $n=0,1,2,3,\cdots$ by

$$\begin{cases} \pi_0((x,l), d(\boldsymbol{y},\boldsymbol{k}))=\delta_{(\partial,l)}(d(\boldsymbol{y},\boldsymbol{k})), \\ \pi_n((x,l), d(\boldsymbol{y},\boldsymbol{k}))=\pi_n'((x,l), \rho^{-1}(d(\boldsymbol{y},\boldsymbol{k}))), \\ \pi_n'((x,l), d((y_1,k_1),\cdots,(y_n,k_n)))=\pi_n(x, d\boldsymbol{y})\delta_{(l,\underbrace{0,\cdots,0}_{n-1})}(d\boldsymbol{k}), n\geqq 1, \end{cases} \tag{4. 10}$$

where $(\boldsymbol{y},\boldsymbol{k})=((y_1,k_1),\cdots,(y_n,k_n))\in S^n$ and ρ is the natural mapping from the n-fold product $S^{(n)}$ to the n-fold symmetric product S^n, and put

$$\begin{cases} \pi^+((x,l), d(\boldsymbol{y},\boldsymbol{k}))=\sum_{n=0}^{\infty} q_n^+(x)\pi_n((x,l), d(\boldsymbol{y},\boldsymbol{k})\cap S^n), \\ \pi^-((x,l), d(\boldsymbol{y},\boldsymbol{k}))=\sum_{n=0}^{\infty} q_n^-(x)\pi_n((x,l), d(\boldsymbol{y},\boldsymbol{k})\cap S^n). \end{cases} \tag{4. 11}$$

π^+ and π^- are kernels defined on $S\times(\cup_{n=0}^{\infty} S^n)$ and $\pi^+ + \pi^-$ is a probability kernel on $S\times(\cup_{n=0}^{\infty} S^n)$.

Next we define kernels μ_0^+ and μ_0^- on $(\bar{\Omega}^n\cap\{\omega; 0<\zeta^n(\omega)<\infty\})\times(\cup_{k=0}^{\infty} S^k)^{(n)}$, $n=1,2,3,\cdots$ by

$$\begin{aligned} &\overset{(-)}{\mu_0^+}(\omega, d(\boldsymbol{x}_1,\boldsymbol{k}_1), d(\boldsymbol{x}_2,\boldsymbol{k}_2),\cdots, d(\boldsymbol{x}_n,\boldsymbol{k}_n)) \\ &=\sum_{i=1}^{n} I_{[\zeta^n(\omega)=\zeta(\omega^i)]}(\omega)\cdot \overset{(-)}{\pi^+}(\bar{X}_{\zeta(\omega^i)-}(\omega^i), d(\boldsymbol{x}_i,\boldsymbol{k}_i))\cdot \prod_{\substack{j=1\\ j\neq i}}^{n} \delta_{(\bar{X}_{\zeta^n(\omega)-}(\omega^j))}(d(\boldsymbol{x}_j,\boldsymbol{k}_j)), \end{aligned} \tag{4. 12}$$

16) X_t^0 takes values in $\cup_{n=0}^{\infty} S^n$, and J_t^0 in J.

17) When D contains a point δ as a terminal point of the basic Markov process and $x_{\zeta-}=\delta$, we put $|q_1|(\delta)=1$ (therefore $|q_n|(x)=0$, $n\neq 1$) and $\pi_1(\delta, dy)=\delta_\delta(dy)$.

where we write $\omega=(\omega^1, \omega^2, \cdots, \omega^n)\in\bar{\Omega}^n$. Then we put

$$\overset{(-)}{\mu^{+}}(\omega, d(\boldsymbol{x}, \boldsymbol{k}))=\overset{(-)}{\mu_0^{+}}(\omega, \gamma^{-1}(d(\boldsymbol{x}, \boldsymbol{k}))), \tag{4.13}$$

where γ is the natural mapping from $(\cup_{k=0}^{\infty} S^k)^{(n)}$ to $\cup_{k=0}^{\infty} S^k$ defined below:

$$\begin{aligned}&\gamma((\boldsymbol{x}_1, \boldsymbol{k}_1), \cdots, (\boldsymbol{x}_n, \boldsymbol{k}_n))\\ &=\begin{cases}(\partial, k_1+k_2+\cdots k_n), & \text{if} \quad (\boldsymbol{x}_i, \boldsymbol{k}_i)=(\partial, k_i) \quad \text{for all} \quad i=1,2,\cdots,n,\\ \rho((x_1^1, k_1^1), (x_1^2, k_1^2), \cdots, (x_1^{m_1}, k_1^{m_1}), (x_2^1, k_2^1), \cdots, (x_n^1, k_n^1), \cdots, (x_n^{m_n}, k_n^{m_n})), & \text{otherwise,}\end{cases}\end{aligned} \tag{4.14}$$

where we take all $(\boldsymbol{x}_i, \boldsymbol{k}_i)=((x_i^1, k_i^1), \cdots, (x_i^{m_i}, k_i^{m_i}))$ which differ from (∂, k), and ρ is the natural mapping from $\cup_{n=0}^{\infty} S^{(n)}$ onto $\cup_{n=0}^{\infty} S^n$.

Finally we define a probability kernel $\mu(\omega^0, d(\boldsymbol{x}, \boldsymbol{k}, j))$ on $\Omega^0\times\hat{\boldsymbol{S}}$ by

$$\mu(\omega^0, d(\boldsymbol{x}, \boldsymbol{k}, j'))=\begin{cases}\mu^{+}(\omega, d(\boldsymbol{x}, \boldsymbol{k}))\delta_{j_1, j'}+\mu^{-}(\omega, d(\boldsymbol{x}, \boldsymbol{k}))\delta_{j_2, j'}, & \\ \qquad \text{if} \quad \omega^0=(\omega, j)\in\bigcup_{n=1}^{\infty}\bar{\Omega}^n\times J \quad \text{and} \quad 0<\zeta^0(\omega^0)<\infty, & \\ \delta_{\Delta}(d(\boldsymbol{x}, \boldsymbol{k}, j')), \quad \text{if} \quad \omega^0\in\bigcup_{n=1}^{\infty}\bar{\Omega}^n\times J \quad \text{and} \quad \zeta^0(\omega^0)=0 \quad \text{or} \quad +\infty, & \\ \delta_{(\partial, k, j)}(d(\boldsymbol{x}, \boldsymbol{k}, j')), \quad \text{if} \quad \omega^0=\omega_{\partial k j}\in\bar{\Omega}^0\times J, & \end{cases} \tag{4.15}$$

where j_1 and j_2 are the functions of j (of $\omega^0=(\omega, j)$) defined by the table:

j	j_1	j_2
0	1	2
1	0	3
2	3	0
3	2	1

Table 1.

A picture given in Figure 2 describes the mechanism of transition of particles governed by the kernel $\mu(\omega^0, d(\boldsymbol{x}, \boldsymbol{k}, j'))$ defined above (but only for q_2).

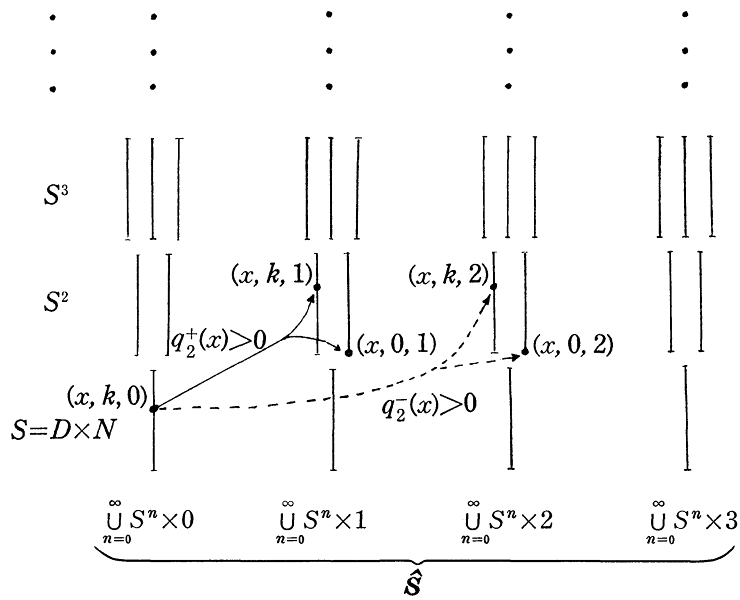

Figure 2.

We may write the transition mechanism more simply and symbolically as shown in Figure 3, because the crucial point is the change of the variable j (of (x, k, j)) depending on the sign.

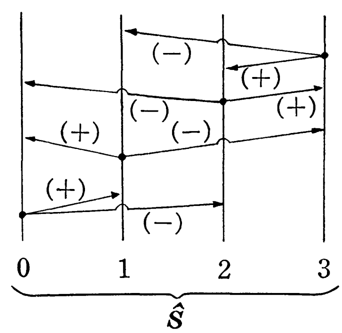

Figure 3.

The symbol (+) and (−) in Figure 3 correspond to the sign appearing on the shoulder of $q_n^{(\cdot)}$ i.e. to the case $q_n^+(x)>0$ and $q_n^-(x)>0$, $(n=0, 1, 2, \cdots)$, respectively.

It is clear from the definition that the kernel $\mu(\omega^0, d(x, k, j'))$ is an instantaneous distribution for Z_t^0. Consequently we can apply Theorem 2. 1 to the Markov process Z_t^0 and the instantaneous distribution μ. Hence we have obtained a strong Markov process on $\hat{S}$ which has right continuous path functions with left limits before the life time ζ. We will denote hence-forth the process as $\{\Omega, N_t, Z_t=(X_t, K_t, J_t), P_{(x,k,j)}, (x, k, j)\in\hat{S}, \zeta, \theta_t\}$ and call it a *branching Markov process with age and sign* (corresponding to the basic Markov process x_t and the system of quantities $\{c, q_n, \pi_n\}$). The components X_t, K_t, and J_t of Z_t stand for positions, ages and sign of particles at instant t, respectively, taking values in $\cup_{n=0}^{\infty} D^{(n)}$, $\cup_{n=0}^{\infty} N^{(n)}$, and J.

Theorem 2. 1 implies that this process has the following properties: (i) The sub-process $\{Z_t, P_{(x,k,j)}, t<\tau\}$ is equivalent to $\{Z_t^0, P_{(x,k,j)}^0, t<\zeta^0\}$, where τ is the "*first branching time* (*splitting time*)" of Z_t, i.e.

$$\tau(\omega)=\inf\{t; J_t(\omega)\neq J_0(\omega)\},^{18)} \tag{4. 16}$$

and (ii) as a particular case of (2. 10)

$$\begin{aligned}&P_{(y,0,0)}[Z_\tau\in d(x', k', j')|Z_{\tau-}]\\ &=\sum_{n=0}^{\infty}\pi_n((x,l), d(x', k'))\{q_n^+(x)\delta_{1,j'}+q_n^-(x)\delta_{2,j'}\}, \qquad y\in D,\end{aligned} \tag{4. 17}$$

where $Z_{\tau-}=(x, l, 0)$.

Thus we have obtained a Markov process which was expected to exist in Theorem 1. 1. We will prove in the next section the remaining properties of the process that are stated in Theorem 1. 1.

§ 5. Some properties of $Z_t=(X_t, K_t, J_t)$.

In this section we shall prove that the semi-group U_t of the process Z_t has the branching property (1. 1) and provides a solution of non-linear integral equation (1. 5). Proof of the branching property of Z_t can be done in the same way given in Ikeda-Nagasawa-Watanabe [6][19] with minor notational change caused by new components K_t and J_t of Z_t, which will be dealt with the following lemma and Lemma 5. 2.

LEMMA 5. 1. *Let f be a measurable function on $\hat{S}$ which is independent of k and j. Then*

$$E_{(x,k,j)}[(-1)^{[J_\tau/2]}\lambda^{|K_\tau|}f(X_\tau)]=(-1)^{[j/2]}\lambda^{|k|}E_{(x,0,0)}[(-1)^{[J_\tau/2]}\lambda^{|K_\tau|}f(X_\tau)] \tag{5. 1}$$

where $[\cdot]$ *denotes Gauss's bracket and* $|k|=\sum_{i=1}^n k_i$ *if* $k=(k_1, k_2, \cdots, k_n)$. *The equality in* (5. 1) *is taken to mean that if the left hand side has definite value then both sides are equal.*

---

18) τ is N_{t+}-Markov time.

19) Cf. Proof of the equivalence theorem, especially, Property B. III$\rightarrow$Branching property.

Proof. According to the way of construction of the measure $P_{(k,k,j)}$,[20] the left hand side of (5. 1) is equal to

$$\iint P^0_{(x,k,j)}[d\omega^0]\mu(\omega^0, d(x', k', j'))(-1)^{[j'/2]}\lambda^{|k'|}f(x')=I, \quad \text{say.}$$

Now remembering that $P^0_{(x,k,j)}$ is the direct product of the sub-process of the Markov process with age,[21] and noting the following

$$\mu(\omega^0, d(x', k', j'))(-1)^{[j'/2]}\lambda^{|k'|}f(x')=\lambda^{|K_{\zeta^0}-|}\mu(\omega^0, d(x', k', j'))(-1)^{[j'/2]}f(x')$$

which follows from the definition of the kernel μ, we have, using Corollary 3. 3,

$$I=\iint \lambda^{|k|}P^0_{(x,0,j)}[d\omega^0]\mu(\omega^0, d(x', k', j'))(-1)^{[j'/2]}\lambda^{|k'|}f(x').$$

On the other hand if we take account of the function relation of j_1 and j_2, given in the Table 1 and the form $(-1)^{[j'/2]}$ of the integrand, we have

$$\begin{aligned} I&=(-1)^{[j/2]}\lambda^{|k|}\iint P^0_{(x,0,0)}[d\omega^0]\mu(\omega^0, d(x', k', j'))(-1)^{[j'/2]}\lambda^{|k'|}f(x') \\ &=(-1)^{[j/2]}\lambda^{|k|}E_{(x,0,0)}[(-1)^{[J_\tau/2]}\lambda^{|K_\tau|}f(X_\tau)], \end{aligned}$$

which is the right hand side of (5. 1).

Let us introduce some notation to state fundamental properties of Z_t. For a bounded measurable fuction f on D, we put

$$U^{(r)}_t[\widetilde{f\cdot\lambda}](x, k, j)=E_{(x,k,j)}[\widetilde{f\cdot\lambda}(Z_t); \tau_r\leqq t<\tau_{r+1}], \qquad r\geq 0, \tag{5. 2}$$

when $\widetilde{f\cdot\lambda}(Z_t)$ is integrable on $\{\tau_r\leqq t<\tau_{r+1}\}$ with respect to $P_{(x,k,j)}$, where τ_r is defined by

$$\begin{cases}\tau_0(\omega)\equiv 0, \qquad \tau_1(\omega)=\tau(\omega), \qquad \text{and} \\ \tau_r(\omega)=\tau_{r-1}(\omega)+\theta_{\tau_{r-1}}\tau(\omega), \qquad r\geqq 2.\end{cases} \tag{5. 3}$$

We write also U^0_t instead of $U^{(0)}_t$. Let ϕ be the joint distribution of τ and Z_τ, i.e.

$$\phi((x, k, j), dsd(x', k', j'))=P_{(x,k,j)}[\tau\in ds, Z_\tau\in d(x', k', j')]. \tag{5. 4}$$

Then we have the following property that corresponds to the property B. III. of [6]. We will, therefore, call it also property B. III.

20) Cf. § 2, (2. 4).

21) Cf. (4. 8) and (4. 4).

[*Property* B. III] *Let $f(x)$ and $f(x,v)$ be bounded measurable functions on D and $D\times[0,\infty)$, respectively. Then the following two properties hold*:

(i) $$U_t^0\widetilde{f\cdot\lambda}(\boldsymbol{x},\boldsymbol{k},j)=\widehat{(U_t^0\widetilde{f\cdot\lambda})|_D\cdot\lambda}(\boldsymbol{x},\boldsymbol{k},j),$$

(ii) $$\int_0^t\int_{\hat{S}}\psi((\boldsymbol{x},\boldsymbol{0},0),dsd(x'\,\boldsymbol{k}',j'))\widetilde{f(\cdot,s)\cdot\lambda}(x',\boldsymbol{k}',j')$$
$$=\sum_{i=1}^n\int_0^t\int_{\hat{S}}\psi((x_i,0,0),dsd(x',\boldsymbol{k}',j'))\widetilde{f(\cdot,s)\cdot\lambda}(x',\boldsymbol{k}',j')\cdot\prod_{\substack{p=1\\p\neq i}}^n U_s^0[\widetilde{f(\cdot,s)\cdot\lambda}](x_p,0,0),\text{[22]}$$

where $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$, $\boldsymbol{k}=(k_1,k_2,\cdots,k_n)$, and $\widetilde{f\cdot\lambda}$ is defined in (1. 2).

Proof. (i) is verified as follows: For $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\boldsymbol{k}=(k_1,\cdots,k_n)$,

$$\begin{aligned}U_t^0\widetilde{f\cdot\lambda}(\boldsymbol{x},\boldsymbol{k},j)&=E_{(\boldsymbol{x},\boldsymbol{k},j)}[\widetilde{f\cdot\lambda}(Z_t);t<\tau]\\&=(-1)^{[j/2]}E^0_{(\boldsymbol{x},\boldsymbol{k},0)}[\widetilde{f\cdot\lambda}(Z_t^0);t<\zeta^0],\end{aligned}$$

but since $P^0_{(\boldsymbol{x},\boldsymbol{k},0)}$ is equivalent to the direct product measure $\bar{P}^n_{(\boldsymbol{x},\boldsymbol{k})}$ of the subprocess of the Markov process with age,[23] the above line is equal to, by Corollary 3. 3,

$$\begin{aligned}&(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}\bar{E}^n_{(\boldsymbol{x},0)}[\widehat{f\cdot\lambda}(\bar{X}_t^n);t<\bar{\zeta}^n]\\&=(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}\prod_{i=1}^n E_{(x_i,0,0)}[\widetilde{f\cdot\lambda}(Z_t);t<\tau]\\&=\widehat{(U_t^0\widetilde{f\cdot\lambda})|_D\cdot\lambda}(\boldsymbol{x},\boldsymbol{k},j).\end{aligned}$$

(ii) is verified as follows:

$$\begin{aligned}&E_{(\boldsymbol{x},\boldsymbol{0},0)}[\widetilde{f(\cdot,\tau)\cdot\lambda}(Z_\tau);\tau\leqq t]\\&=\sum_{j'=0}^3E_{(\boldsymbol{x},\boldsymbol{0},0)}[\widetilde{f(\cdot,\tau)\cdot\lambda}(Z_\tau);J_\tau=j',\tau\leqq t]\end{aligned}$$

but since when $\omega=(\omega^1,\omega^2,\cdots,\omega^n)$, $\tau(\omega)=\inf(\tau(\omega^i),i=1,2,\cdots,n)$, we can continue as follows:

22) When $x\in D$, $U_s^0[\widetilde{f(\cdot,s)\cdot\lambda}](x,0,0)=U_s^0[f(\cdot,s)\cdot\lambda](x,0,0)$, because $Z_t\in S$, $t<\tau$, $P_{(x,0,0)}$-a.e. and $\widetilde{f(\cdot,s)\cdot\lambda}(y,k)=f(\cdot,s)\cdot\lambda(y,k)$, $(y,k)\in S$.

23) Cf. (4. 8) and (4. 4).

$$=\sum_{j'=0}^{3}\sum_{i=1}^{n}\int_0^t(-1)^{[j'/2]}E_{(x,\mathbf{0},0)}[\widehat{f(\cdot,\tau)\cdot\lambda(X_\tau,K_\tau)};J_\tau=j',\tau=\tau(\omega^i)\in ds,s<\tau(\omega^p),\quad\text{for } p\neq i]$$

$$=\sum_{j'=0}^{3}\sum_{i=1}^{n}\int_0^t(-1)^{[j'/2]}E^0_{(x,\mathbf{0},0)}\Big[\Big\{\int_{\hat{S}}\mu(\omega^i,d(\boldsymbol{y},\boldsymbol{k},j))\delta_{j,j'}\widehat{f(\cdot,s)\cdot\lambda(\boldsymbol{y},\boldsymbol{k})}\Big\}$$

$$\cdot\prod_{p\neq i}f(\cdot,s)\cdot\lambda(\bar{X}_s(\omega^p));\zeta^0(\omega^i)\in ds,s<\zeta^0(\omega^p),p\neq i\Big].^{24)}$$

$$=\sum_{j'=0}^{3}\sum_{i=1}^{n}\int_0^t(-1)^{[j'/2]}E^0_{(x_i,0,0)}\Big[\int_{\hat{S}}\mu(\omega,d(\boldsymbol{y},\boldsymbol{k},j))\delta_{j,j'}\widehat{f(\cdot,s)\cdot\lambda(\boldsymbol{y},\boldsymbol{k})};\zeta^0\in ds\Big]$$

$$\cdot\prod_{p\neq i}\bar{E}_{(x_p,0)}[f(\cdot,s)\cdot\lambda(\bar{X}_s);s<\bar{\zeta}]$$

$$=\sum_{i=1}^{n}\int_0^t E_{(x_i,0,0)}[\widetilde{f(\cdot,\tau)\cdot\lambda(Z_\tau)};\tau\in ds]\cdot\prod_{p\neq i}E_{(x_p,0,0)}[f(\cdot,s)\cdot\lambda(Z_s);s<\tau],$$

completing the proof.

LEMMA 5. 2. *For a bounded measurable function f on D and positive λ, it holds that*

$$U_t^{(r)}\widetilde{f\cdot\lambda}(x,\boldsymbol{k},j)=(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}U_t^{(r)}\widetilde{f\cdot\lambda}(x,\mathbf{0},0),\tag{5. 5}$$

$$U_t\widetilde{f\cdot\lambda}(x,\boldsymbol{k},j)=(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}U_t\widetilde{f\cdot\lambda}(x,\mathbf{0},0).\tag{5. 6}$$

These are taken to mean that if the left hand side is definite then both sides are equal.

Proof. We prove (5. 5) by induction. When $r=0$, (5. 5) is nothing but (i) of Property B. III. Let us assume that (5. 5) is valid for $r\geqq 0$.

$$\begin{aligned}&U_t^{(r+1)}\widetilde{f\cdot\lambda}(x,\boldsymbol{k},j)\\&=E_{(x,\boldsymbol{k},j)}[\widetilde{f\cdot\lambda}(Z_t);\tau_{r+1}\leqq t<\tau_{r+2}]\\&=E_{(x,\boldsymbol{k},j)}[E_{(X_\tau,K_\tau,J_\tau)}[\widetilde{f\cdot\lambda}(Z_{t-s});\tau_r\leqq t-s<\tau_{r+1}]|_{s=\tau};\tau\leqq t]\\&=E_{(x,\boldsymbol{k},j)}[(-1)^{[J_\tau/2]}\lambda^{|K_\tau|}E_{(X_\tau,0,0)}[\widetilde{f\cdot\lambda}(Z_{t-s});\tau_r\leqq t-s<\tau_{r+1}]|_{s=\tau};\tau\leqq t],\end{aligned}$$

where we used (5. 5) for r. Then Lemma 5. 1 implies the last line is equal to

$$\begin{aligned}&(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}E_{(x,\mathbf{0},0)}[E_{(X_\tau,0,0)}[\widetilde{f\cdot\lambda}(Z_{t-s});\tau_r\leqq t-s<\tau_{r+1}]|_{s=\tau};\tau\leqq t]\\&=(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}E_{(x,\mathbf{0},0)}[\widetilde{f\cdot\lambda}(Z_t);\tau_{r+1}\leqq t<\tau_{r+2}],\end{aligned}$$

24) Cf. (4. 7) and (4. 3).

which proves (5. 5) for $r+1$. (5. 6) follows from (5. 5) and

$$U_t\widetilde{f\cdot\lambda}=\sum_{r=0}^{\infty}U_t^{(r)}\widetilde{f\cdot\lambda},$$

which is verified by $P_{(x,k,j)}[\lim_{n\to\infty}\tau_n=\zeta]=1$ (cf. (2. 7)), completing the proof.

We shall state several Lemmas which correspond to Lemma 1. 2~Lemma 1. 6 of [6]. Proof of our Lemmas here can be performed by the same way as [6] making slight modification of notation, which we can dispose of mainly by Lemma 5. 2. All equalities that will appear in the following lemmas are taken to mean that the left hand side is definite if and only if the right hand side is, and if this is the case both sides are equal.

LEMMA 5. 3. *For a bounded measurable function f on D and positive λ,*

$$U_s^0(U_{t-s}^{(m)}\widetilde{f\cdot\lambda})(x,\boldsymbol{k},j)=\int_s^t\int_{\hat{S}}\psi((x,\boldsymbol{k},j),dvd(x',\boldsymbol{k}',j'))U_{t-v}^{(m-1)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j'). \tag{5. 7}$$

$$U_t^{(m+1)}\widetilde{f\cdot\lambda}(x,\boldsymbol{k},j)=\int_0^t\int_{\hat{S}}\psi((x,\boldsymbol{k},j),dsd(x',\boldsymbol{k}',j'))U_{t-s}^{(m)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j'). \tag{5. 8}$$

Proof. The above equalities are direct consequence of the strong Markov property of Z_t. In fact,

$$\begin{aligned}
&U_t^{(m+1)}\widetilde{f\cdot\lambda}(x,\boldsymbol{k},j)\\
&=E_{(x,\boldsymbol{k},j)}[\widetilde{f\cdot\lambda}(Z_t);\ \tau_{m+1}\leqq t<\tau_{m+2}]\\
&=E_{(x,\boldsymbol{k},j)}[E_{Z_\tau}[\widetilde{f\cdot\lambda}(Z_{t-s});\ \tau_m\leqq t-s<\tau_{m+1}]|_{s=\tau};\ \tau\leqq t]\\
&=\int_0^t\int_{\hat{S}}P_{(x,\boldsymbol{k},j)}[\tau\in ds,\ Z_t\in d(x',\boldsymbol{k}',j)]\,U_t^{(m)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j'),
\end{aligned}$$

which proves (5. 8). Next we prove (5. 7).

$$\begin{aligned}
&U_s^0(U_{t-s}^{(m)}\widetilde{f\cdot\lambda})(x,\boldsymbol{k},j)\\
&=E_{(x,\boldsymbol{k},j)}[E_{Z_s}[E_{Z_\tau}[\widetilde{f\cdot\lambda}(Z_{t-s-r});\ \tau_{m-1}\leqq t-s-r<\tau_m]|_{r=\tau};\ \tau\leqq t-s];\ s<\tau]\\
&=E_{(x,\boldsymbol{k},j)}[E_{Z_{\tau(\theta_s\omega)}(\theta_s\omega)}[\widetilde{f\cdot\lambda}(Z_{t-s-r});\ \tau_{m-1}\leqq t-s-r<\tau_m]|_{r=\tau(\theta_s\omega)};\ s<\tau,\ \tau(\theta_s\omega)\leqq t-s],
\end{aligned}$$

taking account of $\{\omega;\ \tau(\omega)=s+\tau(\theta_s\omega),\ s<\tau(\omega)\}=\{\omega;\ s<\tau(\omega)\}$, this is equal to

$$=E_{(x,k,j)}[E_{Z_\tau}[\widetilde{f\cdot\lambda}(Z_{t-r});\ \tau_{m-1}\leqq t-r<\tau_m]|_{r=\tau};\ s<\tau\leqq t]$$

$$=\int_s^t\int_{\hat{S}}P_{(x,k,j)}[\tau\in dr,\ Z_\tau\in d(x',k',j')]\,U_{t-r}^{(m-1)}\widetilde{f\cdot\lambda}(x',k',j').$$

which proves (5.7).

LEMMA 5.4. *For $(x,0)\in S^m$, a positive constant λ, and a bounded measurable function $f(x,s)$ defined on $D\times[0,\infty)$, we have*

$$\int_0^t\int_{S^m\times J}\psi((x,0,0),dsd(x',k',j'))\widetilde{f(\cdot,s)\cdot\lambda}(x',k',j')$$

$$=\sum_{i=1}^n\int_0^t\int_{S^{m-(n-1)}\times J}\psi((x_i,0,0),dsd(x',k',j'))\widetilde{f(\cdot,s)\cdot\lambda}(x',k',j') \tag{5.9}$$

$$\cdot\prod_{p\neq i}E_{(x_p,0,0)}[f(\cdot,s)\cdot\lambda(X_s,K_s);\ s<\tau].$$

Proof. If we put af, where a is a non-negative constant, in (ii) of Property B.III instead of f, and compare the coefficients of a^m of both sides of (ii), then we can obtain (5.9).

LEMMA 5.5. *Let $f_i(x,t)$ $(i=1,2,\cdots)$ be bounded measurable functions on $D\times[0,\infty)$ and $\lambda>0$. Then for $m\geqq n-1$, $m\neq n$, and $(x,0)\in S^n$, we have*

$$\int_0^t\int_{S^m\times J}\psi((x,0,0),dvd(x',k',j'))\left\{\frac{1}{m!}\sum_\pi\prod_{i=1}^m f_{\pi(i)}(x_i',v)\lambda^{k_i'}\cdot(-1)^{[j'/2]}\right\}$$

$$=\int_0^t\sum_{i=1}^n\frac{1}{\binom{m}{n-1}}\sum_{(m-n+1)}\frac{1}{(n-1)!}\sum{}^{**}\int_{S^{m-n+1}\times J}\psi((x_i,0,0),dvd(x',k',j')) \tag{5.10}$$

$$\cdot\left\{\frac{1}{(m-n+1)!}\sum{}^*\prod_{h=1}^{m-n+1}f_*(x_h',v)\lambda^{k_h'}(-1)^{[j'/2]}\right\}\prod_{p\neq i}E_{(x_p,0,0)}[f_{**}(\cdot,v)\cdot\lambda(X_v,K_v);\ v<\tau]$$

where $\sum_\pi$ denotes the sum over all permutations π on $(1,2,\cdots,m)$, $\sum_{(m-n+1)}$ the sum over all choices $(q_1,q_2,\cdots,q_{m-n+1})$ from $(1,2,\cdots,m)$, $\sum^*$ the sum over all permutations π on $(q_1,\cdots,q_{m-n+1})$ and $\sum^{**}$ the sum over all permutations π on $(l_1,\cdots,l_{n-1})$ which is the remainder of $(1,2,\cdots,m)$ excluding $(q_1,\cdots,q_{m-n+1})$.

Proof. If all f_i are the same ones, (5.10) reduces to (5.9). In order to deduce (5.10) from (5.9), we have to use a formula of permanent: Let (a_{ij}) be a $m\times m$-matrix, then we have

$$\sum_{\pi}\prod_{j=1}^{m} a_{\pi(i),j}=\prod_{j=1}^{m}\Big(\sum_{k=1}^{m} a_{k,j}\Big)-\sum_{(k_1,\cdots,k_{m-1})}\prod_{j=1}^{m}\Big(\sum_{q=1}^{m-1} a_{k_q,j}\Big)+ \tag{5.11}$$
$$+\sum_{(k_1,\cdots,k_{m-2})}\prod_{j=1}^{m}\Big(\sum_{q=1}^{m-2} a_{k_q,j}\Big)-\cdots+(-1)^{m-1}\sum_{k}\prod_{j=1}^{m} a_{k,j},$$

where $\sum_{(k_1,\cdots,k_r)}$ denotes the sum over all choices $(k_1,\cdots,k_r)$ from $(1,\cdots,m)$. By means of this formula the integrand of the left hand side of (5.10) can be expressd as a sum of functions of the form $\widetilde{g\cdot\lambda}$ therefore we can apply (5.9) to each term and verify (5.10) as follows. The left hand side of (5.10) is equal to, putting $(x',k',j')=z$,

$$\int_0^t\int_{S^m\times J}\phi((x,0,0),dv\,dz)\frac{1}{m!}\Big\{\widetilde{\Big(\sum_{q=1}^{m} f_q(\cdot,v)\Big)\cdot\lambda}(z)$$
$$-\sum_{(k_1,\cdots,k_{m-1})}\widetilde{\Big(\sum_{q=1}^{m-1} f_{k_q}(\cdot,v)\Big)\cdot\lambda}(z)+\cdots+(-1)^m\sum_{k=1}^{m}\widetilde{f_k(\cdot,v)\cdot\lambda}(z)\Big\},$$

and by (5.9) in the previous lemma, this is equal to

$$\sum_{i=1}^{n}\int_0^t\int_{S^{m-(n-1)}}\phi((x_i,0,0),dv\,dz')\frac{1}{m!}\Big\{\widetilde{\Big(\sum_{q=1}^{m} f_q(\cdot,v)\Big)\cdot\lambda}(z')\cdot\prod_{p\neq i}U_v^0\Big[\Big(\sum_{q=1}^{m} f_q(\cdot,v)\Big)\cdot\lambda\Big](x_p,0,0)$$
$$-\sum_{(k_1,\cdots,k_{m-1})}\widetilde{\Big(\sum_{q=1}^{m-1} f_{k_q}(\cdot,v)\Big)\cdot\lambda}(z')\cdot\prod_{p\neq i}U_v^0\Big[\Big(\sum_{q=1}^{m-1} f_{k_q}(\cdot,v)\Big)\cdot\lambda\Big](x_p,0,0)+\cdots\Big\}.$$

Applying again the formula (5.11) of permanent to the integrand { } above, we can express the above line as

$$\sum_{i=1}^{n}\int_0^t\int_{S^{m-(n-1)}}\phi((x_i,0,0),dv\,d(x',k',j'))\Big\{\sum_{\pi}\prod_{k=1}^{m-n+1} f_{\pi(h)}(x'_h,v)\cdot\lambda^{k'_h}\cdot(-1)^{[j'/2]}\Big\}$$
$$\cdot\prod_{p\neq i}U_v^0[f_{\pi(h_p)}(\cdot,v)\cdot\lambda](x_p,0,0),$$

where $h=1,2,\cdots,m-n+1$, and $\{h_p;1\leq p\leq n,p\neq i\}=\{m-n+2,\cdots,m\}$. Since $m!=\binom{m}{n-1}(n-1)!\,(m-n+1)!$ and $\sum_\pi=\sum_{(m-n+1)}\sum^{**}\sum^{*}$, this equal to the right hand side of (5.10).

Lemma 5.6. *For $(x,0)\in S^n$ and a bounded measurable function f on D, we have*

$$\sum_{r_1+\cdots+r_n=r}\int_0^t\sum_{i=1}^n\int_{\hat{S}}\psi((x_i,0,0),dv\,d(x',\boldsymbol{k}',j'))U_{t-v}^{(r_i)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j')$$

$$(5.12)\qquad\cdot\prod_{p\neq i}U_v^0[U_{t-v}^{(r_p)}\widetilde{f\cdot\lambda}](x_p,0,0)$$

$$=\sum_{r_1+\cdots+r_n=r+1}\prod_{i=1}^n U_t^{(r_i)}\widetilde{f\cdot\lambda}(x_i,0,0),$$

where $\sum_{r_1+\cdots+r_n=r}$ *denotes the sum over all combinations* $(r_1\cdots r_n)$ *of non-negative integers such that* $r_r+\cdots+r_n=r$, *permitting* $r_i=r_j$.

Proof. If we put $g^{(r_i)}(v)=U_v^0U_{t-v}^{(r_i)}\widetilde{f\cdot\lambda}(x_i,0,0)$, then the left hand side of (5.12) is equal to, using (5.7) in Lemma 5.3,

$$\sum_{r_1+\cdots+r_n=r}\int_0^t\sum_{i=1}^n d(-g^{(r_i+1)}(v))\prod_{p\neq i}g^{(r_p)}(v).$$

Replacing r_i+1 by r_i and noting $dg^{(0)}(v)\equiv 0$, we can express the above as

$$\sum_{r_1+\cdots+r_n=r+1}\int_0^t\sum_{i=1}^n d(-g^{(r_i)}(v))\prod_{p\neq i}g^{(r_p)}(v)$$

$$=\sum_{r_1+\cdots+r_n=r+1}\prod_{i=1}^n g^{(r_i)}(0),$$

which is equal to the right hand side of (5.12).

LEMMA 5.7. *For* $(x,\boldsymbol{k})\in S^n$, *we have*

$$(5.13)\qquad U_t^{(r)}\widetilde{f\cdot\lambda}(x,\boldsymbol{k},j)=(-1)^{[j/2]}\sum_{r_1+\cdots+r_n=r}\prod_{i=1}^n U_t^{(r_i)}\widetilde{f\cdot\lambda}(x_i,k_i,0).$$

Proof. When $r=0$, (5.13) is just (i) of property B. III. Let us assume (5.13) is true for $1,2,3,\cdots,r$. Since Lemma 5.2 implies

$$U_t^{(r+1)}\widetilde{f\cdot\lambda}(x,\boldsymbol{k},j)=(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}U_t^{(r+1)}\widetilde{f\cdot\lambda}(x,\mathbf{0},0),$$

it is sufficient to treat $U_t^{(r+1)}\widetilde{f\cdot\lambda}(x,\mathbf{0},0)$. However by Lemma 5.3

$$I\equiv U_t^{(r+1)}\widetilde{f\cdot\lambda}(x,\mathbf{0},0)=\int_0^t\int_{\hat{S}}\psi((x,\mathbf{0},0),ds\,d(x',\boldsymbol{k}',j'))\cdot U_{t-s}^{(r)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j'),$$

and hence we can apply (5.13) to the integrand $U_{t-s}^{(r)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j')$ by the induction hypothesis. Then applying Lemma 5.5, we have

$$I=\sum_{m=n-1}^{\infty}\sum_{r_1+r_2+\cdots+r_m=r}\int_0^t\sum_{i=1}^{n}\frac{1}{\binom{m}{n-1}}\sum_{(m-n+1)}\frac{1}{(n-1)!}\sum^{**}\int_{S^{m-n+1}\times J}\phi((x_i,0,0),ds\,d(x',\boldsymbol{k}',j'))$$

$$\cdot\left\{\frac{1}{(m-n+1)!}\sum^{*}\prod_{h=1}^{m-n+1}U_{t-s}^{(r_*)}\widetilde{f\cdot\lambda}(x_h',0,0)\lambda^{k_h'}(-1)^{[j'/2]}\right\}\cdot\prod_{p\neq i}U_s^0[U_{t-s}^{(r_{**})}\widetilde{f\cdot\lambda}](x_p',0,0),$$

where $r_*=r_{q_{\pi(h)}}$, $r_{**}=r_{l_{\pi(p)}}$, and $\sum^*$ and $\sum^{**}$ are given in Lemma 5. 5. Noting that

$$\sum_{r_1+\cdots+r_m=r}\sum_{(m-n+1)}=\sum_{(m-n+1)}\sum_{r_{l_1}+\cdots+r_{l_{n-1}}+r'=r}\sum_{r_{q_1}+\cdots+r_{q_{m-n+1}}=r'},$$

$$\sum_{r_{q_1}+\cdots+r_{q_{m-n+1}}=r'}\frac{1}{(m-n+1)!}\sum^{*}=\sum_{r_{q_1}+\cdots+r_{q_{m-n+1}}=r'}$$

we can express I as

$$I=\sum_{m=n-1}^{\infty}\sum_{r_{l_1}+\cdots+r_{l_{n-1}}+r'=r}\int_0^t\sum_{i=1}^{n}\frac{1}{\binom{m}{n-1}}\sum_{(m-n+1)}\frac{1}{(n-1)!}\sum^{**}$$

$$\cdot\int_{S^{m-n+1}\times J}\phi((x_i,0,0),ds\,d(x',\boldsymbol{k}',j'))$$

$$\cdot\left\{\sum_{r_{q_1}+\cdots+r_{q_{m-n+1}}=r'}\prod_{h=1}^{m-n+1}U_{t-s}^{(r_*)}\widetilde{f\cdot\lambda}(x_h',0,0)\lambda^{k_h'}(-1)^{[j'/2]}\right\}\cdot\prod_{p\neq i}U_s^0[U_{t-s}^{(r_{**})}\widetilde{f\cdot\lambda}](x_p',0,0)$$

$$=\sum_{i=1}^{n}\int_0^t\sum_{m=n-1}^{\infty}\frac{1}{\binom{m}{n-1}}\sum_{(m-n+1)}\sum_{r_1+\cdots+r_n=r}\int_{S^{m-n+1}\times J}\phi((x_i,0,0),ds\,d(x',\boldsymbol{k}',j'))$$

$$\cdot U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j')\cdot\prod_{p\neq i}U_s^0[U_{t-s}^{(r_p)}\widetilde{f\cdot\lambda}](x_p,0,0),$$

where we used the induction hypothesis for $r'\leq r$ and wrote $(r_1,r_2,\cdots,r_n)$ instead of $(r_{l_1},r_{l_2},\cdots,r_{l_{n-1}},r')$. Hence we have

$$I=\sum_{r_1+\cdots+r_n=r}\int_0^t\sum_{i=1}^{n}\int_{\hat{S}}\phi((x_i,0,0),ds\,d(x',\boldsymbol{k}',j'))U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}(x',\boldsymbol{k}',j')\cdot\prod_{p\neq i}U_s^0[U_{t-s}^{(r_p)}\widetilde{f\cdot\lambda}](x_p,0,0)$$

$$=\sum_{r_1+\cdots+r_n=r+1}\prod_{i=1}^{n}U_t^{(r_i)}\widetilde{f\cdot\lambda}(x_i,0,0),$$

by Lemma 5. 6. Thus we have obtained (5. 13) **for** $r+1$.

We are now ready to prove the branching property of U_t.

PROPOSITION 5. 8. *The semi-group U_t of Z_t has the branching property*:

$$U_t\widetilde{f\cdot\lambda}(x, \boldsymbol{k}, j)=\widetilde{(U_t\widetilde{f\cdot\lambda})|_D\cdot\lambda}(x, \boldsymbol{k}, j).^{25)}$$

Proof. First of all we note $P_{(x,\boldsymbol{k},j)}[\lim_{n\to\infty}\tau_n=\zeta]=1$. This follows from the construction of Z_t (cf. § 2, especially (2. 7)). Therefore by Lemma 5. 7 we have, taking $(x, \boldsymbol{k})\in S^n$,

$$
\begin{aligned}
U_t\widetilde{f\cdot\lambda}(x, \boldsymbol{k}, j) &= \sum_{r=0}^{\infty} U_t^{(r)}\widetilde{f\cdot\lambda}(x, \boldsymbol{k}, j)\\
&=(-1)^{[j/2]}\sum_{r=0}^{\infty}\sum_{r_1+\cdots+r_n=r}\prod_{i=1}^{n} U_t^{(r_i)}\widetilde{f\cdot\lambda}(x_i, k_i, 0)\\
&=(-1)^{[j/2]}\sum_{r_1=0}^{\infty}\cdots\sum_{r_n=1}^{\infty}\prod_{i=1}^{n} U_t^{(r_i)}\widetilde{f\cdot\lambda}(x_i, k_i, 0)\\
&=(-1)^{[j/2]}\prod_{i=1}^{n}\sum_{r=0}^{\infty} U_t^{(r)}\widetilde{f\cdot\lambda}(x_i, k_i, 0)\\
&=(-1)^{[j/2]}\lambda^{|\boldsymbol{k}|}\prod_{i=1}^{n} U_t\widetilde{f\cdot\lambda}(x_i, 0, 0)\\
&=\widetilde{(U_t\widetilde{f\cdot\lambda})|_D\cdot\lambda}(x, \boldsymbol{k}, j).
\end{aligned}
\tag{5. 14}
$$

On the other hand since (∂, k, j) and Δ are traps, we have

$$
\begin{aligned}
&U_t\widetilde{f\cdot\lambda}(\partial, k, j)=(-1)^{[j/2]}=\widetilde{(U_t\widetilde{f\cdot\lambda})|_D\cdot\lambda}(\partial, k, j),\\
&U_t\widetilde{f\cdot\lambda}(\Delta)=0=\widetilde{(U_t\widetilde{f\cdot\lambda})|_D\cdot\lambda}(\Delta).
\end{aligned}
\tag{5. 15}
$$

(5. 14) and (5. 15) prove the branching property of U_t.

Now we shall prove (1. 5) of Theorem 1. 1. At first we define a kernel π on $(S\times\{0\})\times\hat{\boldsymbol{S}}$ by

$$\pi((x, l, 0), d(x', \boldsymbol{k}', j'))=\pi^+((x, l), d(x', \boldsymbol{k}'))\delta_{1,j'}+\pi^-((x, l), d(x', \boldsymbol{k}'))\delta_{2,j'}, \tag{5. 16}$$

where π^+ and π^- are defined in (4. 11).

LEMMA 5. 9. *Let B_1, B_2, and B_3 be Borel subsets of* $[0, \infty]$, $D\times N\times J$, *and* $\hat{\boldsymbol{S}}$,

25) We define $U_t\widetilde{f\cdot\lambda}(x, k, j)$ when $\widetilde{f\cdot\lambda}(Z_t)$ is integrable with respect to $P_{(x,k,j)}$.

respectively. Then we have

(5. 17) $$E_{(x,0,0)}[I_{B_1}(\tau)I_{B_2}(Z_{\tau-})I_{B_3}(Z_\tau)]=E_{(x,0,0)}[I_{B_1}(\tau)I_{B_2}(Z_{\tau-})\pi(Z_{\tau-}, B_3)].$$

Proof. This follows directly from the way of construction of $P_{(x,0,0)}$:

$$\begin{aligned}
&E_{(x,0,0)}[I_{B_1}(\tau)I_{B_2}(Z_{\tau-})I_{B_3}(Z_\tau)]\\
&=E^0_{(x,0,0)}\left[I_{B_1}(\zeta^0)\,I_{B_2}(Z^0_{\zeta^0-})\int\mu(\omega^0, d(x', \boldsymbol{k}', j'))I_{B_3}((x', \boldsymbol{k}', j'))\right]\\
&=E^0_{(x,0,0)}[I_{B_1}(\zeta^0)I_{B_2}(Z^0_{\zeta^0-})\pi(z^0_{\zeta^0-}, B_3)]\\
&=E_{(x,0,0)}[I_{B_1}(\tau)I_{B_2}(Z_{\tau-})\pi(Z_{\tau-}, B_3)].
\end{aligned}$$

In order to write down the integral equation which is satisfied by $U_t\widetilde{f\cdot\lambda}(x, 0, 0)$, we have to introduce some notation:

(5. 18) $$\begin{cases} F[x, u]=q_0(x)+\sum_{n=1}^{\infty} q_n(x)\int_{D^n}\pi_n(x, d\boldsymbol{y})\hat{u}(\boldsymbol{y}),\\ K(x, dt\, d(y, k))=P_{(x,0,0)}[\tau\in dt, Z_{\tau-}\in d(y, k, 0)],\\ u(t, x)=U_t\widetilde{f\cdot\lambda}(x, 0, 0),\ x\in D, \end{cases}$$

where f is a bounded measurable function on D.

In the following we shall assume that $u(t, x)$ has definite value in $[0, T)$, $T\in(0, \infty]$, that is, $\widetilde{f\cdot\lambda}(Z_t)$ is integrable with respect to $P_{(x,0,0)}$. Then we have by the strong Markov property of Z_t

$$u(t, x)=E_{(x,0,0)}[\widetilde{f\cdot\lambda}(Z_t); t<\tau]+E_{(x,0,0)}[U_{t-\tau}\widetilde{f\cdot\lambda}(Z_\tau); t\geqq\tau].$$

By Lemma 5. 9, the second term of the right hand side is equal to

$$\begin{aligned}
I=&\int_0^t E_{(x,0,0)}\left[\tau\in ds, Z_{\tau-}\in d(y, k, 0); \int\pi((y, k, 0), d(\boldsymbol{y}', \boldsymbol{k}', j'))U_{t-s}\widetilde{f\cdot\lambda}(\boldsymbol{y}', \boldsymbol{k}', j')\right]\\
=&\int_0^t\int_S K(x, ds\, d(y, k))\left\{\int_{\hat{S}}\pi^+((y, k), d(\boldsymbol{y}', \boldsymbol{k}'))U_{t-s}\widetilde{f\cdot\lambda}(\boldsymbol{y}', \boldsymbol{k}', 1)\right.\\
&\left.+\int_{\hat{S}}\pi^-((y, k), d(\boldsymbol{y}', \boldsymbol{k}'))U_{t-s}\widetilde{f\cdot\lambda}(\boldsymbol{y}', \boldsymbol{k}', 2)\right\},
\end{aligned}$$

but by the branching property of U_t and (5. 6) in Lemma 5. 2, we have

$$U_{t-s}\widetilde{f\cdot\lambda}(\boldsymbol{y}', \boldsymbol{k}', 1)=\lambda^{|\boldsymbol{k}'|}(\widehat{U_{t-s}\widetilde{f\cdot\lambda}})|_D(\boldsymbol{y}'),$$

$$U_{t-s}\widetilde{f\cdot\lambda}(\boldsymbol{y}', \boldsymbol{k}', 2)=-\lambda^{|\boldsymbol{k}'|}(\widehat{U_{t-s}\widetilde{f\cdot\lambda}})|_D(\boldsymbol{y}'),$$

and hence

$$I=\int_0^t\int_S K(x, ds\,d(y, k))\lambda^k\Big\{(q_0^+(y)-q_0^-(y))+\sum_{n=1}^{\infty}(q_n^+(y)-q_n^-(y))\int_{D^n}\pi_n(y, d\boldsymbol{y}')\widehat{u(t-s, \cdot)}(\boldsymbol{y}')\Big\}$$

$$=\int_0^t\int_S K(x, ds\,d(y, k))\lambda^k F[y, u(t-s, \cdot)].$$

Thus we have proved that $u(t, x)$ satisfies the following non-linear integral equation.

PROPOSITION 5. 10. *If $\widetilde{f\cdot\lambda}(Z_t)$ is integrable with respect to $P_{(x,0,0)}$ for $t\in[0, T)$, then $u(t, x)=E_{(x,0,0)}[\widetilde{f\cdot\lambda}(Z_t)]$ satisfies*

$$u(t, x)=U_t^0\widetilde{f\cdot\lambda}(x, 0, 0)+\int_0^t\int_{D\times N}K(x, ds\,d(y, k))\lambda^k F[y, u(t-s, \cdot)], \tag{5.19}$$

where f is a bounded measurable function on D ane λ is a positive constant.

LEMMA 5. 11. *For a non-negative measurable function f on D, we have*

$$\int_{D\times N}K(x, ds\,d(y, k))\lambda^k f(y)=T_s^{(\lambda)}(c\cdot f)(x)ds, \tag{5.20}$$

$$U_t^0\widetilde{f\cdot\lambda}(x, 0, 0)=T_t^{(\lambda)}f(x), \tag{5.21}$$

where $T_t^{(\lambda)}$ is the semi-group defined in terms of the basic Markov process x_t on D by $T_t^{(\lambda)}f(x)=E_x[f(x_t)e^{(\lambda-2)\varphi_t}]$.

Proof.

$$\int_{D\times N}K(x, ds\,d(y, k))\lambda^k f(y)$$

$$=E_{(x,0,0)}[\tau\in ds; \widehat{f\cdot\lambda}(X_{\tau-}, K_{\tau-})]$$

$$=\bar{E}_{(x,0)}[\zeta\in ds; f\cdot\lambda(\bar{X}_{\zeta-})]^{26)}$$

$$=E_{(x,0)}[f\cdot\lambda(X_s)e^{-\varphi_s}d\varphi_s]^{27)}$$

$$=I, \text{ say.}$$

26) $\{\bar{X}_t, \bar{P}_{(x,k)}\}$ is the $\exp(-\varphi_t)$-subprocess of the Markov process with age where $\varphi_t=\int_0^t c(x_s)ds$, cf. the first paragraph of § 4.

27) $\{X_t, P_{(x,k)}\}$ is the Markov process with age.

Since $d\varphi_s=c(x_s)ds$, we have

$$I=E_{(x,0)}[(c\cdot f)\cdot\lambda(X_s)e^{-\varphi_s}]ds.$$

By Corollary 3. 3, we have

$$E_{(x,0)}[(c\cdot f)\cdot\lambda(X_s)\exp(-\varphi_s)]=E_x[c(x_s)f(x_s)e^{(\lambda-2)\varphi_t}]. \tag{5.22}$$

Thus we have proved (5. 20).

Next we show (5. 21);

$$\begin{aligned}&U_t^0\widetilde{f\cdot\lambda}(x,0,0)\\&=E_{(x,0,0)}[\widetilde{f\cdot\lambda}(Z_t);t<\tau]\\&=\bar{E}_{(x,0)}[f\cdot\lambda(X_t,K_t)]\\&=E_{(x,0)}[f\cdot\lambda(X_t)e^{-\varphi_t}]\\&=E_x[f(x_t)e^{(\lambda-2)\varphi_t}]\\&=T_t^{(\lambda)}f(x),\end{aligned}$$

where we used Corollary 3. 3.

Combining proposition 5. 10 and Lemma 5. 11, we have

PROPOSITION 5. 12. *If $\widetilde{f\cdot\lambda}(Z_t)$ is integrable with respect to $P_{(x,0,0)}$, then $u(t,x)=E_{(x,0,0)}[\widetilde{f\cdot\lambda}(Z_t)]$ satisfies*

$$u(t,x)=T_t^{(\lambda)}f(x)+\int_0^t dsT_s^{(\lambda)}(cF[\cdot,u(t-s)])(x), \tag{5.23}$$

where F is given in (5. 18).

Now, in the following we will prove the last statement of Theorem 1. 1. First of all we need the following lemma.

LEMMA 5. 13. *For an initial data f satisfying $\|f\|\leqq r<1$, there exists $\varepsilon>0$ depending on r and $\|f\|$ such that there exists the unique local solution $v(t,x)$, $t\in[0,\varepsilon)$, of the integral equation* (5. 23).

Proof. For any r satisfying $0<r<1$, there exists a constant $c(r)$ such that for bounded measurable functions u and v on D,

$$\sup_n\sup_{x\in D^n}|\hat{u}(x)-\hat{v}(x)|\leqq c(r)\|u-v\|, \tag{5.24}$$

provided $\|u\|$, $\|v\|\leqq r$, (cf. Lemma 0. 1 of [6]). On the other hand we have

(5. 25) $$|F[t, u]-F[x, v]|\leq \sum_{n=1}^{\infty}|q_n(x)| \sup_n \sup_{x\in D^n}|\hat{u}(x)-\hat{v}(x)|.$$

Therefore it holds for u and v satisfying $||u||$, $||v||\leqq r$ that

(5. 26) $$||F[\cdot, u]-F[\cdot, v]||\leqq c(r)||u-v||.$$

Hence we are able to apply a Theorem due to Segal ([19] Theorem 1. 1, cf. also Kato [9]) to the present case, that is, if we put

$$u_0(t, x)=T_t^{(\lambda)}f(x),$$

$$u_n(t, x)=T_t^{(\lambda)}f(x)+\int_0^t ds\, T_s^{(\lambda)}(cF[\cdot, u_{n-1}(t-s)])(x), \qquad n\geqq 1,$$

then we can find $\varepsilon>0$ such that $u_n(t, x)$ converges for $t\in[0, \varepsilon)$ and it supplies a unique local solution of (5. 23), and hence we obtain the assertion of the lemma.

Next we borrow the following lemma from [6],[28] which is easily proved by induction for n.

LEMMA 5. 14. *If $v_t(x)$ is a solution of* (5. 23), *then it satisfies*

(5. 27) $$\prod_{i=1}^{n} T_s^{(\lambda)}v_{t-s}(x_i)=\prod_{i=1}^{n} T_t^{(\lambda)}f(x_i)+\int_s^t dr\sum_{i=1}^{n} T_r^{(\lambda)}(cF[\cdot, v_{t-r}])(x_i)\prod_{p\neq i} T_r^{(\lambda)}v_{t-r}(x_p).$$

PROPOSITION 5. 15.[29] *If $v_t(x)$ is a solution of* (5. 23) *and if we put*

(5. 28) $$\begin{cases} \tilde{v}_t(x, \boldsymbol{k}, j)=(-1)^{[j/2]}\prod_{i=1}^{n} v_t(x_i)\lambda^{k_i}, & if \quad (x, \boldsymbol{k})\in S^n, \\ \tilde{v}_t(\partial, k, j)=(-1)^{[j/2]}\lambda^k, & \\ \quad \tilde{v}_t(\Delta)=0, & \end{cases}$$

then $\tilde{v}_t$ satisfies the following integral equation of renewal type[30]

(5. 29) $$\tilde{v}_t(x, \boldsymbol{k}, j)=U_t^0\widetilde{f\cdot\lambda}(x, \boldsymbol{k}, j)+\int_0^t\int_S \psi((x, \boldsymbol{k}, j), ds\, d(x', \boldsymbol{k}', j'))$$
$$\cdot\tilde{v}_{t-s}(x', \boldsymbol{k}, j'), (x, \boldsymbol{k}, j)\in\hat{\boldsymbol{S}},$$

28) Cf. Lemma 4. 4 in [6].

29) This proposition is stated here in its own interest. We shall not use the proposition to prove the last statement of Theorem 1. 1, but we use a version of it in the following proposition.

30) This is called *M-equation* in [6].

where

$$\psi((x, \boldsymbol{k}, j), ds\, d(x', \boldsymbol{k}', j')) = P_{(x,\boldsymbol{k},j)}[\tau \in ds, Z_\tau \in d(x', \boldsymbol{k}', j')].$$

Proof. Since $\tilde{v}_t(x, \boldsymbol{k}, j)$ has clearly the following property

$$\tilde{v}_t(x, \boldsymbol{k}, j) = \widetilde{\tilde{v}_t(\cdot)|_D \cdot \lambda}(x, \boldsymbol{k}, j),$$

the equation (5. 29) can be written in the form

$$\tag{5. 30} \begin{aligned} \tilde{v}_t(x, \boldsymbol{k}, j) &= (-1)^{[j/2]} \prod_{j=1}^{n} T_t^{(\lambda)} f(x_j) \lambda^{k_i} \\ &+ \int_0^t ds \sum_{i=1}^{n} (-1)^{[j/2]} \lambda^{k_i} T_s^{(\lambda)} (cF[\cdot, v_{t-s}])(x_i) \cdot \prod_{p \neq i} T_s^{(\lambda)} v_{t-s}(x_p) \lambda^{k_p}, \qquad (x, \boldsymbol{k}) \in S^n, \end{aligned}$$

by the Property B. III and Lemma 5. 11. However (5. 30) follows directly from (5. 27).

PROPOSITION 5. 16. *For any initial data f satisfying $\|f\| \leq r < 1$, there exists $\varepsilon > 0$ depending on r and $\|f\|$ such that $\widetilde{f \cdot \lambda}(Z_t)$, $t \in [0, \varepsilon)$, is integrable with respect to the measure $P_{(x,\boldsymbol{k},j)}$ of the branching Markov process with age and sign, and hence $u(t, x) = U_t \widetilde{f \cdot \lambda}(x, 0, 0)$, where $t \in [0, \varepsilon)$ and $x \in D$, is a unique local solution of* (5. 23).

Proof. Let f be a non-negative[31] measurable function on D such that $\|f\| \leq r < 1$. First of all we remark that Lemma 5. 13 is valid when we replace F in (5. 23) by

$$\tag{5. 31} |F|[x, u] = |q_0(x)| + \sum_{n=1}^{\infty} |q_n(x)| \int_{D^n} \pi_n(x, d\boldsymbol{y}) \hat{u}(\boldsymbol{y}).$$

Let $v_t(x)$, $t \in [0, \varepsilon)$, be the solution of (5. 23) with $|F|$ instead of F, the existence of which is assured in Lemma 5. 13. Then $v_t(x)$ satisfies (5. 27), where F must be replaced by $|F|$. Put

$$\tag{5. 32} \begin{cases} \hat{v}_t(x, \boldsymbol{k}, j) = \prod_{i=1}^{n} v_t(x_i) \lambda^{k_i}, & \text{if} \quad (x, \boldsymbol{k}) \in S^n \\ \hat{v}_t(\partial, k, j) = \lambda^k, & \\ \hat{v}_t(\Delta) = 0. & \end{cases}$$

Then it is easy to see that $\hat{v}_t$ satisfies

31) $|\widetilde{f \cdot \lambda}| = \widetilde{|f| \cdot \lambda}|$. Therefore it is sufficieent to consider a non-negative f.

$$\hat{v}_t(x, \boldsymbol{k}, j) = U_t^0 |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) + \int_0^t \int_{\hat{S}} \psi((x, \boldsymbol{k}, j), ds\, d(x', \boldsymbol{k}', j')) \cdot \hat{v}_{t-s}(x', \boldsymbol{k}', j'), \qquad (x, \boldsymbol{k}, j) \in \hat{\boldsymbol{S}}. \tag{5. 33}$$

In fact, (5. 33) is equal to

$$\hat{v}_t(x, \boldsymbol{k}, j) = \prod_{i=1}^n T_t^{(\lambda)} f(x_i) \lambda^{k_i} + \int_0^t ds \sum_{i=1}^n \lambda^{k_i} T_s^{(\lambda)} (c|F|[\cdot, v_{t-s}])(x_i) \prod_{p \neq i} T_s^{(\lambda)} v_{t-s}(x_p) \lambda^p, \tag{5. 34}$$

by a version of the Property B. III for $\widehat{f\cdot\lambda}(x, \boldsymbol{k}, j) = \widehat{f\cdot\lambda}(x, \boldsymbol{k})$. (5. 34) follows from (5. 27) with $|F|$.

Now, it is clear from (5. 33) that

$$U_t^0 |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) \leqq \hat{v}_t(x, \boldsymbol{k}, j).$$

Assume that

$$\sum_{n=0}^N U_t^{(n)} |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) \leqq \hat{v}_t(x, \boldsymbol{k}, j).$$

By the definition of $U_t^{(n)}$ and the strong Markov property of Z_t, we have

$$\sum_{n=0}^{N+1} U_t^{(n)} |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) = U_t^0 |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) + \int_0^t \int_{\hat{S}} \psi((x, \boldsymbol{k}, j), ds\, d(x', \boldsymbol{k}', j')) \left(\sum_{n=0}^N U_{t-s}^{(n)} |\widetilde{f\cdot\lambda}|(x', \boldsymbol{k}', j') \right).$$

By the induction hypothesis, the second term of the right hand side of the above equality is less than and equal to

$$\int_0^t \int_{\hat{S}} \psi((x, \boldsymbol{k}, j), ds\, d(x', \boldsymbol{k}', j')) \hat{v}_{t-s}(x', \boldsymbol{k}', j').$$

Therefore we have

$$\sum_{n=0}^{M+1} U_t^{(n)} |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) \leq \hat{v}_t(x, \boldsymbol{k}, j) < \infty, \qquad t \in [0, \varepsilon).$$

and hence

$$U_t |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) = \lim_{N\to\infty} \sum_{n=0}^N U_t^{(n)} |\widetilde{f\cdot\lambda}|(x, \boldsymbol{k}, j) \leq \hat{v}_t(x, \boldsymbol{k}, j) < \infty, \qquad t \in [0, \varepsilon), \tag{5. 35}$$

which proves the integrability of $\widetilde{f\cdot\lambda}(Z_t)$, $t \in [0, \varepsilon)$, with respect to the measure

$P_{(x,k,j)}$. Accordingly $u(t, x)=U_t\widetilde{f\cdot\lambda}(x, 0, 0)$, $t\in[0, \varepsilon)$, has a definite value and hence it is a solution of (5. 23) by (ii) of Theorem 1. 1, which has been proved already. The uniqueness part of Lemma 5. 13 implies $v(t, x)=u(t, x)$, where $v(t, x)$ is the solution guaranteed in Lemma 5. 13.

Thus we have completely proved Theorem 1. 1.

REMARK. When $F[x, u]$ has the following form

$$F[x, u]=q_0(x)+\sum_{n=1}^{N} q_n(x)\int_{D^n}\pi_n(x, d\boldsymbol{y})\hat{u}(\boldsymbol{y}), \tag{5. 36}$$

where $N<+\infty$, the previous proposition is true for any bounded initial data f on D. In fact, when F is given by (5. 36), it satisfies

$$|F[x, u]-F[x, v]|\leqq c(r)\|u-v\|, \tag{5. 37}$$

provided $\|u\|$, $\|v\|\leqq r$, where r is any positive number. Therefore Lemma 5. 13 is true for any $r>0$, and all arguments following Lemma 5. 13 are of course applicable for this case.

§ 6. Some comments.

(A) When all $q_n(x)$ are non-negative, we may use $\bigcup_{n=0}^{\infty} S^n\times\{0, 1\}$ as the state space instead of $\bigcup_{n=0}^{\infty} S^n\times\{0, 1, 2, 3\}$. In fact particles starting from $\bigcup_{n=0}^{\infty} S^n\times\{0, 1\}$ do not make transition into $\bigcup_{n=0}^{\infty} S^n\times\{2, 3\}$ and vice versa. In this case we shall call the process Z_t on $\bigcup_{n=0}^{\infty} S^n\times\{0, 1\}$ *branching Markov process with age.*

Let Z_t be a branching Markov process with age on $\bigcup_{n=0}^{\infty} S^n\times\{0, 1\}$. We have already proved that when $u(t, x)=E_{(x,0,0)}[\widehat{f\cdot\lambda}(Z_t)]$ has a definite value it is a solution of the non-linear integral equation:

$$u(t, x)=T_t^{(\lambda)}f(x)+\int_0^t ds T_s^{(\lambda)}(cF[u(t-s, \cdot)])(x), \qquad x\in D. \tag{6. 1}$$

We shall remark that $u(t, x)=E_{(x,0)}[f\cdot\lambda(Z_t)]$ is the *minimal solution* of (6. 1) when f is non-negative.

LEMMA 6. 1. *Let f be a bounded non-negative measurable function on D, then*

$$u_t(x, \boldsymbol{k}, j)=E_{(x,\boldsymbol{k},j)}[\widehat{f\cdot\lambda}(Z_t)], \qquad (x, \boldsymbol{k}, j)\in\bigcup_{n=0}^{\infty} S^n\times\{0, 1\}$$

is the minimal solution of the following integral equation of renewal type

$$u_t(x, \boldsymbol{k}, j) = U_t^0 \widehat{f\cdot\lambda}(x, \boldsymbol{k}, j)$$

(6. 2)

$$+\int_0^t\int_{\boldsymbol{S}} \psi((x, \boldsymbol{k}, j), ds\, d(x', \boldsymbol{k}', j'))u_{t-s}(x', \boldsymbol{k}', j'), \qquad (x, \boldsymbol{k}, j)\in \boldsymbol{S}$$

where $\boldsymbol{S}=\cup_{n=0}^{\infty} S^n\times\{0, 1\}$ *and*

$$\psi((x, \boldsymbol{k}, j), ds\, d(x', \boldsymbol{k}', j'))=P_{(x,\boldsymbol{k},j)}[\tau\in ds,\ Z_\tau\in d(x', \boldsymbol{k}', j')].$$

Proof. If $v_t(x, \boldsymbol{k}, j)$ is a solution of (6. 2), then clearly

$$v_t(x, \boldsymbol{k}, j)\geq U_t^0\widehat{f\cdot\lambda}(x, \boldsymbol{k}, j).$$

Assume $v_t(x, \boldsymbol{k}, j)\geqq\sum_{n=0}^{N} U_t^{(n)}\widehat{f\cdot\lambda}(x, \boldsymbol{k}, j)$, $N\geq 0$, then

$$v_t(x, \boldsymbol{k}, j)\geq U_t^0\widehat{f\cdot\lambda}(x, \boldsymbol{k}, j)+\int_0^t\int_{\boldsymbol{S}} \psi((x, \boldsymbol{k}, j),\ ds\, d(x', \boldsymbol{k}', j'))\sum_{n=0}^{N} U_{t-s}^{(n)}\widehat{f\cdot\lambda}(x', \boldsymbol{k}', j')$$

$$=\sum_{n=0}^{N+1} U_t^{(n)}\widehat{f\cdot\lambda}(x, \boldsymbol{k}, j).$$

Therefore

$$v_t(x, \boldsymbol{k}, j)\geq u_t(x, \boldsymbol{k}, j)=\sum_{n=0}^{\infty} U_t^{(n)}\widehat{f\cdot\lambda}(x, \boldsymbol{k}, j),$$

completing the proof.

Let $v_t(x)$ be a solution of (6. 1) for non-negative f, then

$$\hat{v}_t(x, \boldsymbol{k}, j)=\prod_{i=1}^{n} v_t(x_i)\lambda^{k_i}, \qquad (x, \boldsymbol{k})\in S^n,$$

is a solution of (6. 2), which is already proved in the previous section. Suppose that $v_t(x)<u_t(x)=E_{(x,0,0)}[\widehat{f\cdot\lambda}(Z_t)]$, then we have $\hat{v}_t(x, \boldsymbol{k}, j)<u_t(x, \boldsymbol{k}, j)$. This contradicts the fact that $u_t(x, \boldsymbol{k}, j)$ is the minimal solution of (6. 2). This implies that $u_t(x)$ is the minimal solution of (6. 1).

(**B**) When $q_1(x)\leqq 0$, (but the other $q_n(x)$ are arbitrary), we need not use $\cup_{n=0}^{\infty} S^n\times\{0, 1, 2, 3\}$. Instead, we may use $\cup_{n=0}^{\infty} S^n\times\{0, 1\}$ as the state space. In this case, we have to replace $\widetilde{f\cdot\lambda}$ in (1. 2) by

$$\widetilde{f\cdot\lambda}(x, \boldsymbol{k}, j)=(-1)^j\widehat{f\cdot\lambda}(x, \boldsymbol{k}), \qquad j=0, 1, \tag{1. 2'}$$

and the table 1 by Table 1′ below to define the instantaneous distribution in (4. 15).

j	j_1	j_2
0	0	1
1	1	0

Table 1′.

Figure 3 is simplified as follows:

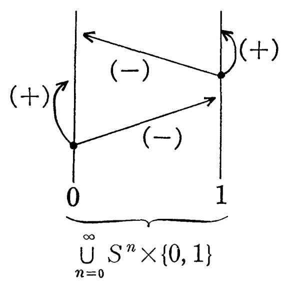

Figure 3′.

(C) When all $q_n(x)$ are non-negative and in addition if $q_1(x)\equiv 0$, then we do not need any discussions given in § 4 and § 5, because we are able to use the branching Markov process X_t defined on $\bigcup_{n=0}^{\infty} S^n$, $S=D\times N$, developed in [6], taking the Markov process with age constructed in § 3, as a non-branching part. We shall call this Markov process on $\bigcup_{n=0}^{\infty} S^n \cup \{\varDelta\}$ *branching Markov process with age*, too.

As an application, we can treat the problem of the blowing up of solutions for the following non-linear integral equation

$$u(t, x)=T_t f(x)+\int_0^t ds T_s(c\cdot u(t-s, \cdot)^{\beta})(x), \qquad \beta=2, 3, \cdots,$$

in terms of branching Markov process with age. Here T_t is the semi-group of d-dimensional symmetric stable process with index α, $(0<\alpha\leqq 2)$. The behaviour of blowing up of the solution depends on the dimension d, the index α, and the power β, (cf. [4] and [17]).

(D) Let E be a compact Hausdorff space with a countable open base. Assume there are given conservative strong Markov processes that have right continuous path functions with left limits, $\{W^1, N_t^1, P_x^1, \zeta^1, \theta_t^1\}$ and $\{W^2, N_t^2, P_x^2, \zeta^2, \theta_t^2\}$ on $E\times\{1\}$ and $E\times\{2\}$, respectively. If we put

$$\begin{cases} D=E\times\{1,2\}, \\ W=W^1\cup W^2, \end{cases}$$

$$(6.3)\qquad \begin{cases} x_t(w)=x_t^i(w), & \text{if} \quad w\in w^i, \quad i=1,2, \\ N_t=N_t^1\vee N_t^2, \\ P_{(x,i)}=P_x^i, & i=1,2, \\ \zeta(w)=\zeta^i(w), & \text{if} \quad w\in W^i, \quad i=1,2, \\ \theta_t=\theta_t^i & \text{on} \quad W^i, \quad i=1,2, \end{cases}$$

then $\{W, N_t, x_t, P_{(x,i)}, \zeta, \theta_t\}$ is clearly a strong Markov process on $D=E\times\{1,2\}$, which has right continuous path functions with left limits.

Now we take this process as a basic Markov process. Let $\{q_n^i(x); n=0,1,2,3,\cdots\}$, $i=1,2$, be sequences of measurable functions on E satisfying

$$(6.4)\qquad \sum_{n=0}^{\infty}|q_n^i(x)|=1, \qquad i=1,2,$$

and taking sequences $\{a_n^{ij}(x); n=0,1,2,3,\cdots\}$, $(i,j=1,2)$, of non-negative measurable functions on E such that

$$(6.5)\qquad a_n^{i1}(x)+a_n^{i2}(x)=1, \qquad i=1,2, \quad \text{and} \quad n=0,1,2,\cdots,$$

we put

$$(6.6)\qquad \begin{aligned} \pi_n((x,i), d(x',\boldsymbol{i}'))=&a_n^{i1}(x)\delta_{(\underbrace{(x,1),(x,1),\cdots,(x,1)}_{n})}(d(x',\boldsymbol{i}')) \\ &+a_n^{i2}(x)\delta_{(\underbrace{(x,2),(x,2),\cdots,(x,2)}_{n})}(d(x',\boldsymbol{i}')). \end{aligned}$$

Then π_n is a probability kernel defined on $D\times D^{(n)}$.[32] Finally let $c^i(x)$, $i=1,2$, be non-negative measurable functions on E, and put

$$\boldsymbol{c}(x,i)=c^i(x).$$

We can apply Theorem 1.1 to the system of quantities: the Markov process $\{W, N_t, P_{(x,i)}, x_t, \zeta, \theta_t\}$, $\boldsymbol{c}(x,i)$, $\{q_n^i\}$, and $\{\pi_n((x,i), d(x',\boldsymbol{i}'))\}$ defined above. Therefore we have a branching Markov process with age and sign

$$\{\Omega, N_t, P_{((x,\boldsymbol{i}),\boldsymbol{k},j)}, Z_t, \zeta, \theta_t\} \qquad \text{on} \qquad \hat{\boldsymbol{S}}=\left(\bigcup_{n=0}^{\infty} S^n\times J\right)\cup\{\Delta\},$$

32) $D^{(n)}=\underbrace{D\times D\times\cdots\times D}_{n}$.

where $S=E\times\{1,2\}\times N$.

Now we define a function $\boldsymbol{f}$ on $D=E\times\{1,2\}$ by

$$\boldsymbol{f}(x,i)=f(x), \qquad i=1,2, \tag{6.7}$$

where f is a bounded measurable function on E. Taking 2 as a weight of age, we put

$$u_i(t,x)=E_{((x,i)0,0)}[\widetilde{\boldsymbol{f}\cdot 2}(Z_t)], \qquad i=1,2;\ x\in E. \tag{6.8}$$

Then $u_i(t,x)$ $i=1,2$ satisfy, when they have a definite value i.e. $\widetilde{f\cdot 2}(Z_t)$ is integrable, the following system of non-linear integral equation:

$$\begin{cases} u_1(t,x)=T_t^1f(x)+\displaystyle\int_0^t dsT_s^1\Big\{c^1\sum_{n=0}^{\infty}q_n^1(a_n^{11}u_1(t-s,\cdot)^n+a_n^{12}u_2(t-s,\cdot)^n)\Big\}(x), \\ u_2(t,x)=T_t^2f(x)+\displaystyle\int_0^t dsT_s^2\Big\{c^2\sum_{n=0}^{\infty}q_n^2(a_n^{21}u_1(t-s,\cdot)^n+a_n^{22}u_2(t-s,\cdot)^n)\Big\}(x), \qquad x\in E, \end{cases} \tag{6.9}$$

where T_t^1 and T_t^2 are the semi-group of x_t^1 and x_t^2 defined by

$$\begin{cases} T_t^1f(x)=E_{(x,1)}[f(x_t^1)], & x\in E, \\ T_t^2f(x)=E_{(x,2)}[f(x_t^2)], & x\in E. \end{cases} \tag{6.10}$$

Thus Theorem 1.1 is of use to treat a system of non-linear integral equations in terms of Markov processes.

For example, the equation of Nagumo [18] for an active pulse transmission line simulating nerve axon

$$\begin{cases} \dfrac{\partial u_1}{\partial t}=ch\dfrac{\partial^2 u_1}{\partial x^2}+c\Big(u_1-\dfrac{1}{3}u_1^3\Big)+cu_2, \\ \dfrac{\partial u_2}{\partial t}=\dfrac{1}{c}(a-u_1)-\dfrac{b}{c}u_2 \end{cases} \tag{6.11}$$

can be treated as a special case of (6.9), (cf. also [1] and [22]).

Added in proof. As for a limit theorem similar to the one for nerve axon, cf. M. Nagasawa, A limit theorem of a pulse-like wave form for a Markov process. Proc. Japan Acad. **44** (1968), 491–494.

REFERENCES

[1] ARIMA, R., AND Y. HASEGAWA, On global solutions for mixed problem of a semi-linear differential equation. Proc. Japan Acad. **39** (1963), 721–725.

[2] BARTLETT, M. S., An introduction to stochastic processes with special reference to methods and application. Cambridge Univ. Press (1955).

[3] DYNKIN, E. B., Markov processes. Springer (1965).

[4] FUJITA, H., On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. Journ. Fac. Sci. Univ. Tokyo, Sec. 1, **13** Part 2 (1966), 109–124.

[5] HARRIS, T. E., The theory of branching processes. Springer (1963).

[6] IKEDA, N., M. NAGASAWA, AND S. WATANABE, Branching Markov processes (to appear in J. Math. Kyoto Univ.). Abstracts: Proc. Japan Acad. **41** (1965), 816–821, **42** (1966), 252–257, 370–375, 380–384, 719–724, 1016–1021, 1022–1026.

[7] IKEDA, N., M. NAGASAWA, AND S. WATANABE, Foundation of Branching Markov processes. Seminar on Probability **23** (1966). (Japanese).

[8] ITO, K., AND H. P. MCKEAN. JR., Diffusion processes and their sample paths. Springer (1965).

[9] KATO, T., Nonlinear evolution equations in Banach spaces. Proc. Symp. in Appl. Math. **17** (1965), 50–67.

[10] KOLMOGOROV, A. N., I. PETROVSKY, AND N. PISCOUNOFF, Etude de l'équation de la diffusion avec croisance de la quantité de matière et son application à un problèm biologigue. Bull. State Univ. Moscow. Sect. A. Math. **1**, Fasc. 6, (1937), 1–25.

[11] LOÉVE, M., Probability theory, third edition. D. van Nostrand (1963).

[12] MEYER, P. A., Probability and Potentials. Blaisdell Pub. Co. (1966).

[13] MOYAL, J. E., Discontinuous Markov processes. Acta Math. **98** (1957), 221–264.

[14] MOYAL, J. E., The general theory of stochastic population processes. Acta Math. **108** (1962), 1–31.

[15] MOYAL, J. E., Multiplicative population processes. J. Appl. Prob. **1** (1964), 267–283.

[16] NAGASAWA, M., AND K. SATO, Some theorems on time change and killing of Markov processes. Kōdai Math. Sem. Rep. **15** (1963), 195–219.

[17] NAGASAWA, M., AND T. SIRAO, Probabilistic treatment of blowing up of solutions for a non-linear integral equation. To appear in Trans. Amer. Math. Soc.

[18] NAGUMO, J., S. ARIMOTO, AND S. YOSIZAWA, An active pulse transmission lines simulating nerve axon. Proceeding of the IRE **50** (1962), 2061–2070.

[19] SEGAL, I., Non-linear semi-groups. Ann. of Math. **78** (1963), 339–364.

[20] SIRAO, T., On signed Branching Markov Processes with age. (To appear in Nagoya Math. J.). Abstract: Proc. Japan Acad. **42** (1966), 885–890.

[21] SKOROHOD, A. V., Branching diffusion processes. Theory Prob. Appl. **9** (1964), 492–497.

[22] YAMAGUCHI, M., The asymptotic behaviour of the solution of a semi-linear partial differential equation related on an active pulse transmission line. Proc. Japan Acad. **39** (1963), 726–730.

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