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ON THE SOLUTION OF THE FUNCTIONAL EQUATION $f\circ g(z)=F(z)$, IV

BY MITSURU OZAWA

In our previous paper [2] we discussed the transcendental unsolvability of the functional equation $f\circ g(z)=F(z)$ with three entire functions f, g and F. In this note we shall consider the functional equation $f\circ g(z)=F(z)$ in a case that F is a given transcendental meromorphic function, f an unknown transcendental mermorphic function and g an unknown transcendental entire function. Our main interest is concerned with the unsolvability criteria of the functional equation in the above sense, which are based upon an elegant theorem due to Edrei [1] and which are extensions of our earlier results in [2].

Edrei's theorem may be stated in the following manner:

Let $f(z)$ be an entire function. If there exists an unbounded sequence $\{a_n\}$ such that almost all equations $f(z)=a_n$, $n=1, 2, \cdots$ have their roots on a single straight line, then $f(z)$ is a polynomial of degree at most two.

Here and in the sequel the term "almost all" means "all but excepting a finite number". We shall use the notations ρ_f and $\hat{\rho}_f$ as follows:

$$\rho_f=\overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log r} \qquad \text{and} \qquad \hat{\rho}_f=\overline{\lim_{r\to\infty}}\frac{\log\log T(r,f)}{\log r}.$$

THEOREM 1. *Let $F(z)$ be a meromorphic function of finite order. Assume that there are two values A and B such that almost all the roots of $F(z)=A$ are lying on a straight line l_A and almost all the roots of $F(z)=B$ are lying on a straight line l_B. Then the functional equation $f\circ g(z)=F(z)$ is transcendentally unsolvable, that is, any transcendental meromorphic function f and any transcendental entire function g do not satisfy the functional equation $f\circ g(z)=F(z)$.*

Proof. If $f(w)=A$ has an infinite number of roots $\{w_n\}$, then $g(z)=w_n$, $n=1, 2, \cdots$ have roots on the straight line l_A with the exception of a finite number of indices. Hence we can use Edrei's theorem. Then $g(z)$ must be a polynomial. This contradicts the transcendency of $g(z)$. The same holds for $f(w)=B$. Hence two equations $f(w)=A$ and $f(w)=B$ have only a finite number of roots, respectively. Therefore we have

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$$\frac{f(w)-A}{f(w)-B}=R(w)e^{L(w)}$$

with a rational function $R(w)$ and an entire function $L(w)$. Therefore

$$\frac{F(z)-A}{F(z)-B}=R\circ g(z)\cdot e^{L\circ g(z)}.$$

Here the order of $R\circ g(z)$ is equal to that of g, but the order of $\exp(L\circ g(z))$ must be infinite by Pólya's theorem. Hence the order of F must be infinite, which is a contradiction.

This theorem 1 contains the following corollary.

COROLLARY 1. *Let $F(z)$ be a meromorphic function of finite order with two Picard exceptional values. Then there is no pair of transcendental solutions f and g of the functional equation $f\circ g(z)=F(z)$.*

THEOREM 2. *Let $F(z)$ be a meromorphic function with a finite $\hat{\rho}_F$. Assume that there are two constants A and B such that almost all the roots of two equations $F(z)=A$ and $F(z)=B$ lie on two straight lines l_A and l_B, respectively, and the maximum of orders of $N(r;\ A, F)$ and $N(r;\ B, F)$ is greater than $\hat{\rho}_F$. Then the functional equation $f\circ g(z)=F(z)$ is transcendentally unsolvable.*

Proof. By the proof of theorem 1 we may assume that each of two equations $F(z)=A$ and $F(z)=B$ has only a finite number of roots. Then

$$\frac{F(z)-A}{F(z)-B}=R\circ g(z)\cdot e^{L\circ g(z)}$$

with a rational function $R(w)$ and an entire function $L(w)$. By our assumption and by an elementary consideration we have

$$\hat{\rho}_F<\max(\rho_{N(r;A,F)},\quad \rho_{N(r;B,F)})\leqq\rho_g.$$

However by the form of F we have that $\hat{\rho}_F\geqq\rho_g$, which is really a contradiction.

THEOREM 3. *Let $F(z)$ be a meromorphic function of finite order. Assume that almost all the zeros of $F'(z)$ lie on a straight line and almost all the poles of $F'(z)$ lie on a straight line. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. We consider the derived functional equation $f'\circ g(z)\cdot g'(z)=F'(z)$. If $f'(w)=\infty$ has an infinite number of solutions $\{w_n\}$, almost all equations $g(z)=w_n$ has roots lying on a straight line. Indeed at any such root $F'(z)=\infty$, which is shown in the following manner: Denote $g(z)=w_n+c(z-z_{nj})^{p_j}+\cdots$ around z_{nj} and

$$f'(w)=\frac{A}{(w-w_n)^{q_n}}+\cdots$$

around w_n. Then

$$F'(z)=f'\circ g(z)\cdot g'(z)=\frac{B}{(z-z_{nj})^{q_n p_j-p_j+1}}(1+O(z-z_{nj}))$$

around z_{nj}. By Edrei's theorem $g(z)$ must be a polynomial, which contradicts the transcendency of $g(z)$. The same holds for $f(w)=0$. Hence we have

$$f'(w)=R(w)e^{L(w)}$$

with a rational function $R(w)$ and an entire function $L(w)$. Therefore

$$F'(z)=R\circ g(z)e^{L\circ g(z)}\cdot g'(z).$$

By this form the order of $F'(z)$ and hence that of $F(z)$ must be infinite, which contradicts the assumption.

THEOREM 4. *Let $F'(z)$ be a meromorphic function of finite hyper-order $\hat{\rho}_F'$. Assume further that almost all the zeros of $F(z)$ lie on a straight line and almost all the poles of $F(z)$ lie on a straight line and the maximum of the orders of $N(r;0,F')$ and $N(r;\infty,F')$ is greater than $\hat{\rho}_{F'}$. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

We shall omit the proof.

We shall give another extension of corollary 1.

THEOREM 5. *Let $F(z)$ be a meromorphic function. Assume that $F(z)$ has two Picard exceptional values A and B both of which are taken at least once by $F(z)$. Then the functional equational $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. If $f(w)=A$ has at least two solutions w_1 and w_2, at least one of $g(z)=w_1$ and $g(z)=w_2$ has an infinite number of roots, since $g(z)$ is transcendental entire function. This is a contradiction. Further, if $f(w)=A$ and $f(w)=B$ have two roots w_1, w_2 altogether, $g(z)=w_1$ or $g(z)=w_2$ has an infinite number of roots, which is again a contradiction.

THEOREM 6. *Let $F(z)$ be a meromorphic function with $\hat{\rho}_{F'}<1$. Assume, further that $0,\infty$ are two Picard exceptional values of $F'(z)$ and that $F'(z)=\infty$ has at least one root. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Consider the derived functional equation $f'\circ g(z)\cdot g'(z)=F'(z)$. We similarly have

$$f'(w)=(w-w_1)^{-1}e^{L(w)}$$

with a positive integer n and an entire function $L(w)$. Further we have

$$g(z)=w_1+P(z)e^{M(z)} \qquad \text{and} \qquad g'(z)=Q(z)e^{N(z)}$$

with polynomials P, Q and entire functions M, N. Therefore

$$F'(z)=P(z)^{-n}e^{-nM(z)}e^{L\circ(w_1+P(z)e^{M(z)})}Q(z)e^{N(z)}.$$

By its form we have $\hat{\rho}_{F'}\geqq 1$, which is a contradiction.

THEOREM 7. *Let $F(z)$ be a meromorphic function with $\rho_F<1$. Assume that $F(z)$ is not entire and it has ∞ as its Picard exceptional value. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Consider the equation $f(w)=\infty$. It has just one root w_1. Further $g(z)$ has w_1 as a Picard exceptional value. Therefore we have

$$f(w)=(w-w_1)^{-n}f^*(w), \qquad g(z)=w_1+P(z)e^{L(z)}$$

and

$$F(z)=P(z)^{-n}e^{-nL(z)}f^*\circ(w_1+P(z)e^{L(z)}),$$

where $f^*(w)$, $L(z)$ are entire functions and $P(z)$ is a polynomial. Hence by its form we have $\rho_F\geqq 1$, which is a contradiction.

By a similar consideration we can prove the following theorem.

THEOREM 8. *Let $F(z)$ be a meromorphic function with $\rho_F<1$. Assume that $F'(z)$ is not entire and it admits ∞ as a Picard exceptional value. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Let us use $N_2(r;\,0, F)$ as the N-function of simple zeros of F.

THEOREM 9. *Let $F(z)$ be a meromorphic function which attains all the values in the Riemann sphere and has ∞ as a Picard exceptional value. Assume further that the order of $N(r;\,0, F)$ {or more precisely that of $N_2(r;\,0, F)$ assuming that there is at least one simple zero} is less than 1. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. We shall prove this under the $\{\cdots\}$ part. It should be firstly remarked that the order of $N_2(r;\,0, a+P\exp(M))$ is equal to that of $\exp(M(z))$, where a is a non-zero constant and $P(z)$ is a polynomial. Of course we need the second fundamental theorem in order to prove the above fact. Now $f(w)$ must have the following form

$$(w-w_1)^{-n}f^*(w),$$

where f^* is a transcendental entire function and n is a positive integer and w_1 is only one pole of $f(w)$. Further $g(z)$ must have the form

$$g(z)=w_1+P(z)e^{M(z)}$$

with a non-constant polynomial P and an entire function M. Hence

$$F(z)=f\circ g(z)=P(z)^{-n}e^{-nM(z)}f^*\circ(w_1+P(z)e^{M(z)}).$$

Consider $f^*(w)=0$. Then there is at least one simple root w_2 ($\neq w_1$). Then by the above remark we have

$$\rho_{N_2(r;\,0,\,w_1-w_2+P\exp(M))}=\rho_{e^M},$$

which is not less than 1. On the other hand

$$N_2(r;\ 0,\ F)=N_2(r;\ 0, f^*\circ(w_1+Pe^M)),$$

which is not less than that of $P(z)\exp(M(z))+w_1-w_2$. Therefore we have arrived at a contradiction.

LEMMA. *Let $F(z)$ be a transcendental entire function of the form $P(z)e^{M(z)}$ with a polynomial P and an entire function M. Assume that there is a complex number $a\neq 0$ for which almost all the roots of $F(z)=a$ lie on a straight line. Then $P(z)$ is a constant and $M(z)=\alpha z+\beta$.*

Proof. Edrei proved the following theorem in [1]: Let $f(z)$ be an entire function the zeros of which are real. Furthermore, assume that for some integer $n(\geqq 0)$, the zeros of $f^{(n)}(z)-1$ are all real. Then the order of $f(z)$ is finite and does not exceed one.

By this theorem we have firstly that the order of F does not exceed 1. Hence $M(z)$ must be of the form $\alpha z+\beta$ and the equation reduces to

$$P(z)e^{\alpha z+\beta}=a.$$

Assume $P(z)=A_mz^m+\cdots+A_0$, $A_m\neq 0$. In this case we may assume that $\alpha=1$ and $P(z)=z^m+\cdots+A_0$. Now consider z in a sector S_0: $-\pi/2+\varepsilon\leqq\arg z\leqq\pi/2-\varepsilon$ for an arbitrary positive ε. When z tends to ∞ in S_0,

$$P(z)e^z\to\infty$$

uniformly. Let S_1 be the sector which is symmetric to S_0 with respect to the origin. When z tends to ∞ in S_1,

$$P(z)e^z\to 0$$

uniformly. Hence almost all the roots of the equation

$$(z^m+\cdots+A_0)e^z=A$$

must lie on a vertical line. Assume it is $z=x_0+iy$. Then

$$\{(x_0+iy)^m+\cdots+A_0\}e^{x_0+iy}=A$$

holds for an infinite number of values $\{y_j\}$ of y. However this does not hold, since taking the absolute values of both sides we have the equation

$$|(x_0+iy)^m+\cdots+A_0||e^{x_0+iy}|=|A|$$

and then the left hand side term tends to infinity. This is a contradiction.

This implies that $A_m=\cdots=A_1=0$ and $A_0\neq 0$, that is, $P(z)$ is a constant. This is the desired result.

THEOREM 10. *Let $F(z)$ be a meromorphic function whose image covers the Riemann sphere. Assume that ∞ is a Picard exceptional value of F and almost all the zeros of $F(z)$ lie on a straight line l. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Firstly we have

$$f(w)=(w-w_1)^{-n}f^*(w), \qquad g(z)=w_1+P(z)e^{M(z)}$$

with entire functions f^*, M, a positive integer n and a polynomial $P(z)$. Now consider $f^*(w)=0$. If it has an infinite number of roots, $g(z)$ must be a polynomial as in theorem 1, which is a contradiction. Hence $f^*(w)=0$ has only a finite number of roots, which implies that

$$f^*(w)=Q(w)e^{L(w)}.$$

Here $Q(w)$ is a non-constant polynomial and $L(w)$ is entire. Let w_2 be a zero of $Q(w)$. In this case $w_2\neq w_1$. Then consider the equation

$$w_2=w_1+P(z)e^{M(z)}.$$

almost all the roots of this equation lie on l. Hence by Lemma $P(z)$ must be a constant C and $M(z)=\alpha z+\beta$. Then in this case we have

$$F(z)=C^{-n}e^{-n(\alpha z+\beta)}Q\circ(w_1+Ce^{\alpha z+\beta})e^{L\circ(w_1+Ce^{\alpha z+\beta})}.$$

Hence $F(z)$ is an entire function, which contradicts the assumption.

Similarly we have the following

THEOREM 11. *Let $F(z)$ be a meromorphic function. Assume that ∞ is a Picard exceptional value of F' and almost all the zeros of $F'(z)$ lie on a straight line l and further F' covers the Riemann sphere. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

THEOREM 12. *Let $F(z)$ be a meromorphic function almost all whose A-points lie on a straight line for an A. Assume that*

$$\overline{\lim_{r\to\infty}}\frac{N(r;\,A,F)}{T(r,F)}>0.$$

Then the functional equation $F(z)=f\circ g(z)$ is not transcendentally solvable.

Proof. As in theorem 1 the number of roots of $f(w)=A$ is finite. Let $w_1,\cdots,$

w_n be its roots. In this case n must be positive. If this is not the case, there is no A-points of F, which is evidently a contradiction. Then

$$N(r;\ A, F)=\sum_{1}^{n} N(r;\ w_j, g)\leqq n m(r, g).$$

If we can prove that

$$\varlimsup_{r\to\infty} \frac{m(r, g)}{T(r, F)}=0,$$

we have a contradiction. Now returning to the expression of $f(w)$ we have

$$f(w)-A=\prod_{1}^{n}(w-w_j)\cdot\frac{1}{f^*(w)},$$

where $f^*(w)$ is an entire function. Hence we have

$$F(z)-A=\prod_{1}^{n}(g(z)-w_j)\cdot\frac{1}{f^*\circ g(z)}.$$

By this form and by Pólya's method we have

$$m(r, g)=o(m(r, f^*\circ g))$$

and

$$T(r, F)=m(r, f^*\circ g)(1+\varepsilon), \qquad \lim_{r\to\infty}\varepsilon=0.$$

This implies the desired result:

$$\varlimsup_{r\to\infty} \frac{m(r, g)}{T(r, F)}=0.$$

Similarly we have the following theorem.

THEOREM 13. *Let $F(z)$ be a meromorphic function. Assume that almost all the roots of $F(z)=0$ lie on a straight line and that*

$$\varlimsup_{r\to\infty} \frac{N(r;\ 0, F')}{T(r, F')}>0.$$

Then the functional equation $F(z)=f\circ g(z)$ is not transcendentally solvable.

REFERENCE

[1] EDREI, A., Meromorphic functions with three radially distributed values. Trans. Amer. Math. Soc. **78** (1955), 276–293.

[2] OZAWA, M., On the solution of the functional equation $f\circ g(z)=F(z)$, III. Kōdai Math. Sem. Rep. **20** (1968), 257–263.

[3] PÒLYA, G., On an integral function of an integal function. Journ. London Math. Soc. **1** (1926), 12–15.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.