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ON REGULARLY BRANCHED THREE-SHEETED COVERING RIEMANN SURFACES

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§1. Throughout this paper D denotes either the domain $\{z||z|<\infty\}$ or $\{z|0<|z|<\infty\}$, and the notations $T, m, N, N_2, N_1, \bar{N}_1$ and T^*, m^*, etc. on meromorphic function in D are used in the sense of Nevanlinna [6].

Let R be a Riemann surface and $\mathfrak{M}(R)$ the family of non-constant meromorphic functions on R. For $f\in\mathfrak{M}(R)$ we define $P(f)$ to be the number of values which are not taken by f on R and denote $\sup_{f\in\mathfrak{M}(R)} P(f)$ by $P(R)$. We call $P(R)$ Picard's constant of R, following Ozawa. Then for every open surface R we have $P(R)\geqq 2$.

We confine our attention mainly to those open Riemann surfaces R that are regularly branched three-sheeted covering surface defined by $y^3=g(z)$, where $g(z)$ is a single-valued transcendental regular function in D having an infinite number of simple or double zeros. In this case we say that $g(z)$ is admissible for $S_3(D)$ where $S_3(D)$ denotes the class of all such surfaces R, and sometimes we simply say that R is defined by $y^3=g(z)$. Then $\sqrt[3]{g(z)}$ is a three-valued regular algebroid function in D. Hence $P(R)\leqq 6$ from Selberg's theory [11].

In the present paper we shall characterize some of surfaces $R\in S_3(D)$ in an explicit form and study the existence problems of analytic mappings among them. For such work we shall list some notations and lemmas.

LEMMA 1.1. (Borel [1]-Nevanlinna [5]) *Let $a_0(z), a_1(z), \cdots, a_n(z)$ be meromorphic functions and $g_1(z), g_2(z), \cdots, g_n(z)$ regular functions in $r_0\leqq|z|<\infty$. Further suppose that for every j ($j=0, 1, \cdots, n$)*

$$T(r, a_j(z))=o\left(\sum_{\nu=1}^{n} m(r, e^{g_\nu(z)})\right)$$

holds outside a set of finite measure. If the identity

$$\sum_{\nu=1}^{n} a_\nu(z)e^{g_\nu(z)}=a_0(z)$$

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holds, then there are constants $c_1, c_2, \cdots, c_n$, *not all zeros such that*

$$\sum_{\nu=1}^{n} c_\nu a_\nu(z) e^{g_\nu(z)} = 0.$$

From Lemma 1. 1 we have:

LEMMA 1. 2. *Let* $a_0(z), a_1(z), \cdots, a_n(z)$ *be meromorphic in* $r_0 \leqq |z| < \infty$ *and* $e^{g(z)}$ *be transcendental regular function there satisfying* $T(r, a_j(z)) = o(m(r, e^{g(z)}))$, $j=0, 1, \cdots, n$ *as* $r \to \infty$ *outside a set of finite measure. If*

$$\sum_{j=1}^{n} a_j(z) e^{jg(z)} = a_0(z)$$

holds, then $a_j(z) \equiv 0$ *for all* j $(j=0, 1, \cdots, n)$.

Two transcendental regular functions $e^{H(z)}$ and $e^{L(z)}$ in $r_0 \leqq |z| < \infty$ are said to be mutually dependent if $m(r, e^{H(z)-L(z)}) = O(\log r)$ outside a set of finite measure.

Suppose $e^{H(z)}$ and $e^{L(z)}$ are two mutually dependent transcendental regular functions in $r_0 \leqq |z| < \infty$ and let

$$H(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu + a_0 + \sum_{\nu=1}^{\infty} a_{-\nu} z^{-\nu} \qquad \text{and} \qquad L(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu + b_0 + \sum_{\nu=1}^{\infty} b_{-\nu} z^{-\nu}.$$

We write

$$H_p = \sum_{\nu=1}^{\infty} a_\nu z^\nu, \qquad H_0 = a_0 \qquad \text{and} \qquad H_N = \sum_{\nu=1}^{\infty} a_{-\nu} z^{-\nu}.$$

Ozawa proved the following lemma by using Lemma 1. 1.

LEMMA 1. 3. *Let* $a_j(z)$ $(j=0, 1, \cdots, n)$ *be meromorphic and* $a_j(z) \not\equiv 0$ $(j=1, 2, \cdots, n)$ *in* $r_0 \leqq |z| < \infty$ *satisfying* $T(r, a_j(z)) = O(\log r)$ *as* $r \to \infty$. *Let* $e^{g_j(z)}$ $(j=1, 2, \cdots, n)$ *be transcendental regular function there such that*

$$\sum_{j=1}^{n} a_j(z) e^{g_j(z)} = a_0(z).$$

Then the set $\{e^{g_j(z)}\}_{j=1,2,\cdots,n}$ *is divided into a finite number of groups each of which consists of dependent functions, and* $a_0(z) \equiv 0$.

LEMMA 1. 4. (Hiromi and Ozawa [3]) *Let* $L(z)$ *and* $g(z) \not\equiv 0$ *be regular functions in* $1 \leqq |z| < \infty$ *satisfying* $m(r, g) = o(m(r, e^L))$ *as* $r \to \infty$ *outside a set of finite measure. Then* $N_2(r, 0, e^L - g) \sim m(r, e^L)$ *and* $N_1(r, 0, e^L - g) = o(m(r, e^L))$ *as* $r \to \infty$ *outside a set of finite measure.*

LEMMA 1. 5. (Ozawa [8, 9]) *Let* $G(z)$ *be a transcendental regular function of* z *in* $1 \leqq |z| < \infty$ *with infinitely many zeros and* $h(z)$ *regular there, then* $m(r, h) = o(N(r, 0, G \circ h))$ *as* $r \to \infty$ *outside a set of finite measure.*

Let $f_1(z)$ and $f_2(z)$ be two meromorphic functions. Let $N_0(r, 0, f_1, f_2)$ denote the N-function of common zeros of $f_1(z)$ and $f_2(z)$. Niino proved the following:

LEMMA 1.6. (Niino [7]) *Let $H(z)$, $\phi_j(z)$, $(j=1, 2, \cdots, \mu)$ and $\phi_k^*(z)$ $(k=1, 2, \cdots, \nu)$ be regular functions in D satisfying $m(r, \phi_j)=o(m(r, e^H))$ $(j=1, 2, \cdots, \mu)$ and $m(r, \phi_k^*)=o(m(r, e^H))$ $(k=1, 2, \cdots, \nu)$ as $r\to\infty$ outside a set of finite measure. If the polynomials*

$$Q_\mu(h)\equiv h^\mu+\phi_1(z)h^{\mu-1}+\cdots+\phi_\mu(z)$$

and

$$Q_\nu^*(h)\equiv h^\nu+\phi_1^*(z)h^{\nu-1}+\cdots+\phi_\nu^*(z)$$

are irreducible and $Q_\mu(e^H)\not\equiv Q_\nu^*(e^H)$, then

$$N_0(r, 0, Q_\mu(e^H), Q_\nu^*(e^H))=o(m(r, e^H))$$

as $r\to\infty$ outside a set of finite measure.

§2. Let $R\in S_3(D)$ be defined by $y^3=g(z)$. Let f be a three-valued regular algebroid function in D which is single-valued regular on R, and let its defining equation be

$$F(z, f)=f^3-s_1(z)f^2+s_2(z)f-s_3(z)=0, \tag{2.1}$$

where $s_1(z)$, $s_2(z)$ and $s_3(z)$ are single-valued regular functions in D. Let $\omega\neq1$ be a cubic root of 1. We put $p_1=(z, y)$, $p_2=(z, \omega y)$ and $p_3=(z, \omega^2 y)$ and set

$$\begin{cases} f_1(z)=\dfrac{1}{3}\{f(p_1)+f(p_2)+f(p_3)\}, \\ f_2(z)=\dfrac{1}{3y}\{f(p_1)+\omega^2 f(p_2)+\omega f(p_3)\}, \\ f_3(z)=\dfrac{1}{3y^2}\{f(p_1)+\omega f(p_2)+\omega^2 f(p_3)\}. \end{cases} \tag{2.2}$$

Then $f_1(z)$ is a single-valued regular function in D, and $f_2(z)$ and $f_3(z)$ are single-valued regular functions in D except for all the multiple zeros of $g(z)$, at which $f_2(z)$ and $f_3(z)$ have poles at worst in such a way that if z_0 is a zero of $g(z)$ of order $3k+l$ $(0\leqq k, 0\leqq l\leqq2)$, then $f_2(z)$ has at worst a pole at z_0 of order k and $f_3(z)$ has at worst a pole at z_0 of order $2k$ if $l\neq2$ and $2k+1$ if $l=2$.

From (2.2) we have

$$f(p)=f_1(z)+f_2(z)y+f_3(z)y^2, \qquad \text{where} \qquad p=(z, y). \tag{2.3}$$

Conversely, $f(p)$ defined by (2.3) with f_1, f_2 and f_3 having the described properties in the above is regular on R. From (2.3), we see that f satisfies the equation (2.1) with

$$(2.4)\qquad \begin{cases} s_1(z)=3f_1(z),\\ s_2(z)=3f_1^2(z)-3f_2(z)f_3(z)g(z),\\ s_3(z)=f_1^3(z)+f_2^3(z)g(z)+f_3^3(z)g^2(z)-3f_1(z)f_2(z)f_3(z)g(z). \end{cases}$$

Mutō [4] established a necessary and sufficient conditioh for the existence of an analytic map between R_1 and R_2 when $D_1=D_2=\{z||z|<\infty\}$ and $g_i(z)$ $(i=1, 2)$ has no zeros other than an infinite number of simple or double zeros. We extend the result as follows:

LEMMA 2. 1. *A non-trivial analytic map* ϕ *of* R_1 *into* R_2 *exists if and only if there exists a single-valued non-constant regular function* $h(z)$ *in* D_1 *such that either* $\nu^3(z)g_1(z)=g_2\circ h(z)$, *or* $\mu^3(z)g_1^2(z)=g_2\circ h(z)$, *where* $\nu(z)$ *and* $\mu(z)$ *are single-valued meromorphic functions having the properties that their poles are all multiple zeros of* $g_1(z)$ *in such a way that if* a *is a zero of* $g_1(z)$ *of order* $3k+l$ $(0\leqq l\leqq 2)$ *then* a *is at worst a pole of* $\nu(z)$ *of order* k *and* a *is at worst a pole of* $\mu(z)$ *of order* $2k$ *for* $l=0, 1$ *and* $2k+1$ *for* $l=2$.

In particular, if $D_2=\{z|0<|z|<\infty\}$ *then* $h(z)$ *has the form* $z^n e^{K(z)}$ *where* $K(z)$ *is a single-valued regular function in* D_1 *and* n *is an integer* ($n=0$ *when* $D_1=\{z||z|<\infty\}$).

Proof. The proof of this lemma is essentially the same as the proof due to Mutō [4].

If $D_1=D_2$ then R_1 and R_2 are conformally equivalent if and only if $h(z)$ in the lemma is one-to-one and onto. Thus we have the following:

LEMMA 2. 2. *Let* $R\in S_3(D)$ *be defined by* $y^3=g(z)$. *Suppose* $\nu(z)$ *and* $\mu(z)$ *have the same properties as described in the Lemma* 2. 1. *If* $G_1(z)=\nu^3(z)g(z)$ *is admissible for* $S_3(D)$ *and* $R_1\in S_3(D)$ *is defined by* $y^3=G_1(z)$, *then* R *and* R_1 *are conformally equivalent. Similarly, if* $G_2(z)=\mu^3(z)g^2(z)$ *is admissible for* $S_3(D)$ *and* $R_2\in R_3(D)$ *is defined by* $y^3=G_2(z)$, *then* R *and* R_2 *are conformally equivalent.*

Proof. The proof follows directly from the fact that $G_1(z)=G_1\circ z$ and $G_2(z)=G_2\circ z$. (q.e.d.)

§3. In this section we begin with the special type of function $\Omega(z)$ given by

$$\Omega(z)=d_{16}e^{3H+3L}+d_{15}e^{3H+2L}+d_{14}e^{2H+3L}+d_{13}e^{3H+L}+d_{12}e^{2H+2L}+d_{11}e^{H+3L}+d_{10}e^{3H}$$
$$+d_9e^{2H+L}+d_8e^{H+2L}+d_7e^{3L}+d_6e^{2H}+d_5e^{H+L}+d_4e^{2L}+d_3e^{H}+d_2e^{L}+d_1,$$

with the properties:

i) $d_j(z)$ $(j=1, 2, \cdots, 16)$ is meromorphic and $H(z)$, $L(z)$ are non-constant regular functions in D with $H_0=L_0=0$.

ii) If $H_P\neq 0$ then $T(r, d_j)=o(m(r, e^H))$, $(j=1, 2, \cdots, 16)$ and $m(r, e^H)\sim m(r, e^L)$, and if $H_N\neq 0$ then $T^*(r, d_j)=o(m^*(r, e^H))$, $(j=1, 2, \cdots, 16)$ and $m^*(r, e^H)\sim m^*(r, e^L)$ as $r\to\infty$ outside a set of finite measure.

We call d_j ($j=1, 2, \cdots, 16$) a coefficient and denote the term in $\Omega(z)$ with coefficient d_j by $d_j e^{h_j H + l_j L}$ or simply $d_j A_j$. We set

$$A^* = \{(d_{16}), (d_{15}, d_{14}), (d_{13}, d_{12}, d_{11}), (d_{10}, d_9, d_8, d_7), (d_6, d_5, d_4), (d_3, d_2), (d_1)\}$$

and

$$\tilde{A} = \{(d_7), (d_4, d_{11}), (d_2, d_8, d_{14}), (d_1, d_5, d_{12}, d_{16}), (d_3, d_9, d_{15}), (d_6, d_{13}), (d_{10})\}.$$

If $d_k \not\equiv 0$ and all other coefficients in the parenthesis in A^* to which d_k belongs are identically zero, then we write $d_k \equiv d_k^*$, and for the case of $\tilde{A}$ we write $d_k \equiv \tilde{d}_k$.

$$\text{Set} \quad S^* = \{d_k | d_k \equiv d_k^*\} \quad \text{and} \quad \tilde{S} = \{d_k | d_k \equiv \tilde{d}_k\}.$$

Under these notations and conditions on $\Omega(z)$, we have:

LEMMA 3.1. (i) *Suppose* $\Omega(z) \equiv 0$. *Assume* $H_p \not\equiv 0$. *Then* $H_p \equiv -L_p$ *if* $S^* \neq \phi$, *and* $H_p \equiv L_p$ *if* $\tilde{S} \neq \phi$. *If we assume* $H_N \not\equiv 0$, *then* $H_N \equiv -L_N$ *if* $S^* \neq \phi$, *and* $H_N \equiv L_N$ *if* $\tilde{S} \neq \phi$. (ii) $\Omega(z) \not\equiv 0$ *if* $S^* \neq \phi$ *and* $\tilde{S} \neq \phi$.

Proof. (1) Suppose that $d_i A_i + d_j A_j \equiv 0$ with $i \neq j$, $d_i \not\equiv 0$ and $d_j \not\equiv 0$. Assume $H_p \not\equiv 0$. Then $d_i e^{(h_i - h_j)H} \equiv -d_j{}^{(l_j - l_i)L}$. Now

$$T(r, d_i e^{(h_i - h_j)H}) \sim [|h_i - h_j| + o(1)] m(r, e^{H_p})$$

and

$$T(r, -d_j e^{(l_j - l_i)L}) \sim [|l_i - l_j| + o(1)] m(r, e^{L_p}).$$

Hence $m(r, e^{H_p}) \sim m(r, e^{L_p})$ gives $|h_i - h_j| = |l_i - l_j|$.

When $h_i - h_j = l_j - l_i$, we have $H_p \equiv L_p$ in D. When $h_i - h_j = l_i - l_j$ we have $H_p \equiv -L_p$ in D.

(2) Suppose $\Omega(z) \equiv 0$ with some $d_i \not\equiv 0$ for some i. Assume $H_p \not\equiv 0$. Then we claim that $H_p \equiv L_p$ or $H_p \equiv -L_p$ in D. To show this we first notice that $d_i \not\equiv 0$ for at least two i's ($1 \leqq i \leqq 16$). We may assume that $d_i \not\equiv 0$ for at least two i's ($2 \leqq i \leqq 16$). Then we have

$$d_{k_1} A_{k_1} + d_{k_2} A_{k_2} + \cdots + d_{k_r} A_{k_r} + d_1 \equiv 0, \qquad (k_i \neq k_j \quad \text{for} \quad i \neq j \quad \text{and} \quad k_j \geqq 2)$$

which is obtained from $\Omega(z) \equiv 0$ by discarding all the terms $d_i A_i$'s with $d_i \equiv 0$ ($i \geqq 2$). If $d_1 \not\equiv 0$ then by Lemma 1.1 there are some constants $C_i^{(1)}$, at least two of them are not zero such that

$$\sum_{i=1}^{r} C_i^{(1)} d_{k_i} A_{k_i} \equiv 0.$$

If $d_1 \equiv 0$ then $\sum_{i=1}^{r} d_{k_i} A_{k_i} \equiv 0$. In any case we have

$$\sum_{i=1}^{r_1} d_{k_i}^{(1)} A_{k_i} \equiv 0 \qquad (r_1 \geqq 2,\ d_{k_i}^{(1)} \not\equiv 0 \quad \text{for all} \quad k_i\ (1 \leqq i \leqq r_1)). \tag{3.1}$$

If $r_1=2$ then we have the desired results by (1). Suppose $r_1>2$. Write (3. 1) into the form

$$\sum_{i=1}^{r_1-1} d_{k_i}^{(1)}\left(\frac{A_{k_i}}{A_{k_{r_1}}}\right)\equiv -d_{k_{r_1}} \qquad (d_{k_{r_1}}\neq 0)$$

and apply Lemma 1. 1. Then there are some constants $C_i^{(2)}$, at least two of them are not zero, such that

$$\sum_{i=1}^{r_1-1} C_i^{(2)} d_{k_i}^{(1)} A_{k_i}\equiv 0. \tag{3. 2}$$

Let $\sum_{i=1}^{r_2} d_{k_i}^{(2)} A_{k_i}\equiv 0$ be the result obtained from (3. 2) after we discard all the terms with $C_i^{(2)}=0$. Then clearly $2\leqq r_2<r_1$. If $r_2=2$, then we have the desired results. If $r_2>2$ then we repeat the process until we end up with the form: $d_i^{(n)}A_{k_i}+d_j^{(n)}A_{k_j}\equiv 0$ with $i\neq j$, $d_i^{(n)}\neq 0$ and $d_j^{(n)}\neq 0$. Then by (1) we have the results as claimed.

(3) Suppose $\Omega(z)\equiv 0$ and assume $H_p\neq 0$. Let

$$\tag{3. 3} \begin{cases} b_6=d_{16}e^{3H_N+3L_N}, \\ b_5=d_{15}e^{3H_N+2L_N}+d_{14}e^{2H_N+3L_N}, \\ b_4=d_{13}e^{3H_N+L_N}+d_{12}e^{2H_N+2L_N}+d_{11}e^{H_N+3L_N}, \\ b_3=d_{10}e^{3H_N}+d_9e^{2H_N+L_N}+d_8e^{H_N+2L_N}+d_7e^{3L_N}, \\ b_2=d_6e^{2H_N}+d_5e^{H_N+L_N}+d_4e^{2L_N}, \\ b_1=d_3e^{H_N}+d_2e^{L_N}, \\ b_0=d_1, \end{cases}$$

and

$$\tag{3. 4} \begin{cases} c_6=d_7e^{3L_N}, \\ c_5=d_4e^{2L_N}+d_{11}e^{H_N+3L_N}, \\ c_4=d_2e^{L_N}+d_8e^{H_N+2L_N}+d_{14}e^{2H_N+3L_N}, \\ c_3=d_1+d_5e^{H_N+L_N}+d_{12}e^{2H_N+2L_N}+d_{16}e^{3H_N+3L_N}, \\ c_2=d_3e^{H_N}+d_9e^{2H_N+L_N}+d_{15}e^{3H_N+2L_N}, \\ c_1=d_6e^{2H_N}+d_{13}e^{3H_N+L_N} \\ c_0=d_{10}e^{3H_N}. \end{cases}$$

By direct computation and Lemma 1. 2, we see that $b_i\equiv 0$ for all i $(0\leqq i\leqq 6)$ if $H_p\equiv L_p$, and $c_i\equiv 0$ for all i $(0\leqq i\leqq 6)$ if $H_p\equiv -L_p$.

Suppose $S^*\neq\phi$ and assume $H_p\neq 0$. Then $d_i\in S^*$ for some i and obviously $d_i\neq 0$. Hence by (2), $H_p\equiv L_p$ or $H_p\equiv -L_p$. But if $H_p\equiv L_p$, then $b_i\equiv 0$ for all i $(0\leqq i\leqq 6)$.

Hence $d_i \equiv 0$, a contradiction. Hence $H_p \equiv -L_p$. Similarly, if $\tilde{S} \neq \phi$ and $H_p \not\equiv 0$ then $H_p \equiv H_p$. Thus we proved the statement (i) in the lemma in the case $H_p \not\equiv 0$.

(4) If $H_N \not\equiv 0$, then by interchanging p and N, and replacing m by m^*, T by T^* in the above whole argument, we see that $H_N \equiv -L_N$ if $S^* \neq \phi$, and $H_N \equiv L_N$ if $\tilde{S} \neq \phi$.

(5) Suppose $S^* \neq \phi$ and $\tilde{S} \neq \phi$. Assume $\Omega(z) \equiv 0$. If $H_p \not\equiv 0$ then $H_p \equiv -L_p$ and $H_p \equiv L_p$. Hence $H_p \equiv 0$, a contradiction. Hence $H_p \equiv 0$ and $H_N \not\equiv 0$. But then we have again a contradiction. Thus $\Omega(z) \not\equiv 0$. (q.e.d.)

LEMMA 3. 2. *Let* $g_i(z)$ $(i=1, 2)$ *be defined in* D *by*

$$g_i(z) = z^{3n_i} B_{i3}(z) e^{3H_i(z)} + z^{2n_i} B_{i2}(z) e^{2H_i(z)} + z^{n_i} B_{i1}(z) e^{H_i(z)} + B_{i0}(z),$$

with the properties: (i) n_i *is an integer*, $B_{ij}(z)$ $(j=0, 1, 2, 3)$ *meromorphic and* $H_i(z)$ *regular and non-constant in* D *such that* $H_{i0}=0$, $B_{i3} \equiv 1$ *and* $B_{i0} \not\equiv 0$. (ii) *If* $H_{1p} \not\equiv 0$ *then* $m(r, e^{H_1}) \sim m(r, e^{H_2})$ *and* $T(r, B_{ij}) = o(m(r, e^{H_i}))$, $(j=0, 1, 2)$ *and if* $H_{1N} \not\equiv 0$ *then* $m^*(r, e^{H_1}) \sim m^*(r, e^{H_2})$ *and* $T^*(r, B_{ij}) = o(m^*(r, e^{H_i}))$ $(j=0, 1, 2)$ *as* $r \to \infty$ *outside a set of finite measure. Suppose that*

$$g_1(z) = f^3(z) g_2(z), \tag{3. 5}$$

where $f(z)$ *is meromorphic in* D *such that if* $H_{1p} \not\equiv 0$, *then* $T(r, f'/f) = o(m(r, e^{H_1}))$ *and if* $H_{1N} \not\equiv 0$, *then* $T^*(r, f'/f) = o(m^*(r, e^{H_1}))$ *as* $r \to \infty$ *outside a set of finite measure. Then in* D *we have either*

$$H_1(z) \equiv H_2(z) \quad and \quad B_{1j}(z) \equiv B_{2j}(z) z^{(3-j)(n_1-n_2)} \qquad (j=0, 1, 2) \tag{3. 6}$$

or

$$H_1(z) \equiv -H_2(z) \quad and \quad B_{1j}(z) \equiv \frac{B_{2(3-j)}(z)}{B_{20}(z)} z^{(3-j)(n_1+n_2)} \qquad (j=0, 1, 2). \tag{3. 7}$$

Proof. By differentiating $g_1 = f^3 g_2$, we have

$$g_1' g_2 = g_1 \left(3 \frac{f'}{f} g_2 + g_2' \right).$$

Then this can be written as follows:

$$\begin{aligned} \Omega(z) \equiv\ & a_{16} e^{3H_1+3H_2} + a_{15} e^{3H_1+2H_2} + a_{14} e^{2H_1+3H_2} + a_{13} e^{3H_1+H_2} \\ & + a_{12} e^{2H_1+2H_2} + a_{11} e^{H_1+3H_2} + a_{10} e^{3H_1} + a_9 e^{2H_1+H_2} + a_8 e^{H_1+2H_2} \\ & + a_7 e^{3H_2} + a_6 e^{2H_1} + a_5 e^{H_1+H_2} + a_4 e^{2H_2} + a_3 e^{H_1} + a_2 e^{H_2} + a_1 \equiv 0, \end{aligned} \tag{3. 8}$$

where if we set $a_j A_j = a_j e^{h_j H_1 + l_j H_2}$, then $a_j(z)$ is given by

$$a_j = B_{1h_j} B_{2l_j} z^{h_j n_1 + l_j n_2} \left[\frac{B_{1h_j}'}{B_{1h_j}} - \frac{B_{2l_j}'}{B_{2l_j}} + (h_j H_1' - l_j H_2') + \frac{h_j n_1 - l_j n_z}{z} - \frac{3f'}{f} \right] \tag{3. 9}$$

$$(j=1, 2, \cdots, 16).$$

We first notice that from (3. 8) we have

$$\prod_{j=1}^{16} a_j \equiv 0, \tag{3. 10}$$

for otherwise $a_j \not\equiv 0$ for all j, and, in particular, $a_{16} \in S^*$ and $a_{10} \in \tilde{S}$. Hence by Lemma 3. 1, we have $\Omega(z) \not\equiv 0$, a contradiction.

Suppose $a_j(z) \equiv 0$ and $B_{1h_j} B_{2l_j} \not\equiv 0$, then from (3. 9) and (3. 5) we have

$$\Omega_j(z) \equiv \sum_{k=0}^{3} \left[B_{1k} B_{2l_j} z^{kn_1 + l_j n_2} e^{kH_1 + l_j H_2} - C_j B_{1h_j} B_{2k} z^{h_j n_1 + kn_2} e^{h_j H_1 + kH_2} \right] \equiv 0. \tag{3. 11}$$

Suppose $B_{11} \not\equiv 0$. Then $B_{11} B_{2l} \not\equiv 0$ for $l_j = 0, 3$. Assume $a_j \equiv 0$ for $j = 11, 3$. Since $B_{1h_j} B_{2l_j} \not\equiv 0$ for $j = 11, 3$, we have $\Omega_j(z) \equiv 0$ for $j = 11, 3$. But in (3. 11) $d_{16} \in S^*$ and $d_7 \in \tilde{S}$ when $j = 11$, and $d_1 \in S^*$ and $d_{10} \in \tilde{S}$ when $j = 3$. Hence by Lemma 3. 1 $\Omega_j(z) \not\equiv 0$ for $j = 11, 3$. This is a contradiction.

Hence $a_{11} \not\equiv 0$ and $a_3 \not\equiv 0$, if $B_{11} \not\equiv 0$. By a similar reasoning we have the following table:

(3. 12)

If $B_{ij} \not\equiv 0$ then $a_k \not\equiv 0$		If $B_{ij} \equiv 0$ then $a_k \equiv 0$	
B_{11}	a_{11}, a_3	B_{11}	a_{11}, a_8, a_5, a_3
B_{12}	a_{14}, a_6	B_{12}	a_{14}, a_{12}, a_9, a_6
B_{21}	a_{13}, a_2	B_{21}	a_{13}, a_9, a_5, a_2
B_{22}	a_{15}, a_4	B_{22}	a_{15}, a_{12}, a_8, a_4

There are two cases:

Case (I) $B_{11} B_{12} B_{21} B_{22} \not\equiv 0$.

From (3. 12) $a_j \not\equiv 0$ for $j = 15, 14, 13, 11, 6, 4, 3, 2$. Hence (3. 10) reduces to $\Pi_j a_j = 0$ where j runs over {16, 12, 10, 9, 8, 7, 5, 1}. But $\Pi_{0 \leq i, j \leq 3} B_{ij} \not\equiv 0$. Hence $\Pi_j \Omega_j(z) \equiv 0$ where j runs over {16, 12, 10, 9, 8, 7, 5, 1}.

Case (II) $B_{11} B_{12} B_{21} B_{22} \equiv 0$. We divide into two subcases: subcase (A) $B_{11} \equiv 0$ and subcase (B) $B_{11} \not\equiv 0$.

In the subcase (A), $\Omega_j(z) \equiv 0$ with $B_{1h_j} B_{2l_j} \not\equiv 0$ for at least one of j ($j = 16, 12, 10, 9, 7, 1$). In the subcase (B), $\Omega_j(z) \equiv 0$ with $B_{1h_j} B_{2l_j} \not\equiv 0$ for at least one of j ($j = 16, 12, 9, 8, 7, 5, 1$).

By summarizing the results obtained so far, we can conclude that: (3. 8) implies that $\Omega_j(z) \equiv 0$ with $B_{1h_j} B_{2l_j} \not\equiv 0$ for at least one of j ($j = 16, 12, 10, 9, 8, 7, 5, 1$).

We first consider the case when $\Omega_j(z) \equiv 0$ and $B_{1h_j} B_{2l_j} \not\equiv 0$ for $j = 16, 12, 5, 1$. Now $h_j = l_j = 3, 2, 1$ and 0 for $j = 16, 12, 5$ and 1, respectively. Further $\tilde{S} \not\equiv \phi$ in the equation $\Omega_j(z) \equiv 0$ for $j = 16, 12, 5, 1$. Thus for j ($j = 16, 12, 5, 1$), $\Omega_j(z) \equiv 0$ implies that $H_{1p} \equiv H_{2p}$ if $H_{1p} \not\equiv 0$, and $H_{1N} \equiv H_{2N}$, if $H_{1N} \not\equiv 0$, by Lemma 3. 1. Assume $H_{1p} \not\equiv 0$. Then $b_i \equiv 0$ ($i = 0, 1, \cdots, 6$) where b_i's are given by (3. 3). But in (3. 11) we have $k + l_j = h_j + k$ ($k = 0, 1, 2, 3$) since $h_j = l_j$. Hence we have

$$B_{1k} B_{2l_j} z^{kn_1 l_j n_2} e^{kH_{1N} + l_j H_{2N}} \equiv C_j B_{1h_j} B_{2k} z^{h_j n_1 + kn_2} e^{h_j H_{1N} + kH_{2N}} \quad (k = 0, 1, 2, 3). \tag{3. 13}$$

Put $k=0$ and 3 in (3. 13). Since $B_{10}B_{20}B_{1h_j}B_{2l_j}B_{13}B_{23}\not\equiv 0$ and $h_j=l_j$ for $j=16, 12, 5$ and 1, we have

$$B_{1k}\equiv B_{2k}z^{(3-k)(n_1-n_2)} \qquad (k=0, 1, 2).$$

Thus if $H_{1p}\not\equiv 0$ then $\Omega_j(z)\equiv 0$ $(j=16, 12, 5, 1)$ implies that $H_1\equiv H_2$ and $B_{1k}\equiv B_{2k}z^{(3-k)(n_1-n_2)}$ $(k=0, 1, 2)$. If $H_{1p}\equiv 0$, then $H_{1N}\not\equiv 0$. Then by interchanging p and N in the above argument we have the the same results.

It remains to examine the case when $\Omega_j(z)\equiv 0$ and $B_{1h_j}B_{2l_j}\not\equiv 0$ for $j=10, 9, 8, 7$. As before, we note that in (3. 11) $S^*\not\equiv\phi$ for $j=10, 9, 8, 7$. Hence by Lemma 3. 1, we see that $\Omega_j(z)\equiv 0$ $(j=10, 9, 8, 7)$ implies that $H_{1p}\equiv -H_{2p}$ if $H_{1p}\not\equiv 0$, and $H_{1N}\equiv -H_{2N}$ if $H_{1N}\not\equiv 0$.

Assume $H_{1p}\not\equiv 0$. Then $H_{1p}\equiv -H_{2p}$ and hence $c_i\equiv 0$ $(i=0, 1, \cdots, 6)$ where c_i's are given by (3. 4). We rewrite (3. 11) into the following form:

$$\sum_{k=0}^{3}\left(B_{1k}B_{2l_j}z^{kn_1l_jn_2}e^{kH_1+l_jH_2}-C_jB_{1h_j}B_{2(3-k)}z^{h_jn_1+(3-k)n_2}e^{h_jH_1+(3-k)H_2}\right)\equiv 0. \tag{3. 14}$$

By comparing (3. 4) and (3. 14) together with $l_j-k=(3-k)-h_j$ for $k=0, 1, 2, 3$ we have

$$B_{2l_j}B_{1k}z^{kn_1+l_jn_2}e^{kH_{1N}+l_jH_{2N}}\equiv C_jB_{1h_j}B_{2(3-k)}z^{h_jn_1+(3-k)n_2}e^{h_jH_{1N}+(3-k)H_{2N}} \quad (k=0, 1, 2, 3).$$

By essentially the same argument we have

$$B_{1k}\equiv\frac{B_{2(3-k)}}{B_{20}}z^{(3-k)(n_1+n_2)} \qquad (k=0, 1, 2).$$

Now clearly (3. 6) and (3. 7) cannot hold simultaneously, for otherwise $H_1\equiv H_2\equiv 0$, a contradiction. This completes the proof. (q.e.d.)

LEMMA 3. 3. *Suppose that*

$$f^3(z)(e^{H(z)}-B_1(z))(e^{H(z)}-B_2(z))^2=(e^{L(z)}-A_1(z))(e^{L(z)}-A_2(z))^2, \tag{3. 15}$$

with the properties that: (i) $H(z)$ *and* $L(z)$ *are non-constant regular functions in* D *with* $H_0=L_0=0$. (ii) $A_i(z)$ *and* $B_i(z)$ $(i=1, 2)$ *are regular functions in* D *such that* $A_i(z)\not\equiv 0$, $B_i(z)\not\equiv 0$, $A_1(z)\not\equiv A_2(z)$ *and* $B_1(z)\not\equiv B_2(z)$. (iii) $m(r, A_i)=o(m(r, e^L))$ *if* $L_p\not\equiv 0$, *and* $m^*(r, A_i)=o(m^*(r, e^L))$ *if* $L_N\not\equiv 0$, *and* $m(r, B_i)=o(m(r, e^H))$ *if* $H_p\not\equiv 0$, *and* $m^*(r, B_i)=o(m^*(r, e^H))$ *if* $H_N\not\equiv 0$, *as* $r\to\infty$ *outside a set of finite measure.* (iv) $f(z)$ *is a meromorphic function in* D. *Then we have that either* $H(z)\equiv L(z)$ *and* $B_i(z)\equiv A_i(z)$ *for* $i=1, 2$, *or* $H(z)\equiv -L(z)$ *and* $B_i(z)\equiv 1/A_i(z)$ *for* $i=1, 2$.

Proof. Let $f(z)$ be meromorphic and $G(z)$ regular in D such that f^3G is regular. Let $N_3(r, 0, f^3G)$ be the N-function of double zeros of f^3G, counted simply. Then we have

$$N_3(r, 0, f^3G)\leqq N_1(r, 0, G). \tag{3. 16}$$

Set $G(z)=(e^H-B_1)(e^H-B_2)^2$ and $g(z)=(e^L-A_1)(e^L-A_2)^2$. Assume $H_p\not\equiv 0$. Then by Lemma 1.4 and Lemma 1.6 we have $(1-o(1))m(r, e^H)\sim N_2(r, 0, e^H-B_1)-N_0(r, 0, e^H-B_1, e^H-B_2)\leqq N_2(r, 0, G)\leqq N_2(r, 0, e^L-A_1)+N_0(r, 0, e^L-A_1, e^L-A_2)+2N_1(r, 0, e^L-A_1)+2N_1(r, 0, e^L-A_2)\sim(1+o(1))m(r, e^L)$, i.e., $(1-o(1))m(r, e^H)\leqq(1+o(1))m(r, e^L)$ as $r\to\infty$ outside a set of finite measure. On the other hand, $N_3(r, 0, g)\geqq N_2(r, 0, e^L-A_2)-N_0(r, 0, e^L-A_1, e^L-A_2)\sim(1-o(1))m(r, e^L)$ and $N_1(r, 0, G)\leqq N_1(r, 0, e^H-B_1)+N_2(r, 0, e^H-B_2)+N_1(r, 0, e^H-B_2)\sim(1+o(1))m(r, e^H)$. From (3.16), $N_3(r, 0, f^3G)=N_3(r, 0, g)\leqq N_1(r, 0, G)$. Hence $(1-o(1))m(r, e^L)\leqq(1+o(1))m(r, e^H)$ as $r\to\infty$ outside a set of finite measure.

Hence $m(r, e^H)\sim m(r, e^L)$ as $r\to\infty$ outside a set of finite measure.

Let $N_1'(r, 0, G)$ be the N-function of zeros of G of order at least three, counted multiply. Then $N(r, \infty, f')\leqq N_1'(r, 0, G)\leqq 2N_1(r, 0, e^H-B_1)+4N_1(r, 0, e^H-B_2)+3N_0(r, 0, e^H-B_1, e^H-B_2)=o(m(r, e^H))$, and $N(r, 0, f)\leqq N_1'(r, 0, g)\leqq 2N_1(r, 0, e^L-A_1)+4N_1(r, 0, e^L-A_2)+3N_0(r, 0, e^L-A_1, e^L-A_2)=o(m(r, e^L))$. Hence $N(r, \infty, f'/f)=o(m(r, e^H))$ as $r\to\infty$ outside a set of finite measure. But $m(r, f'/f)\leqq O(\log rT(r, f))$ as $r\to\infty$ outside a set of finite measure (Nevanlinna [6]). Clearly, $T(r, f)=O(m(r, e^H)+m(r, e^L))$ as $r\to\infty$. Hence $T(r, f'/f)=m(r, f'/f)+N(r, \infty, f'/f)\leqq O(\log rT(r, f))+o(m(r, e^H))=o(m(r, e^H))$, i.e., $T(r, f'/f)=o(m(r, e^H))$ as $r\to\infty$ outside a set of finite measure. Thus if $H_p\not\equiv 0$, then $m(r, e^H)\sim m(r, e^L)$ and $T(r, f'/f)=o(m(r, e^H))$ as $r\to\infty$ outside a set of finite measure. We already assumed that $m(r, A_i)=o(m(r, e^L))$ and $m(r, B_i)=o(m(r, e^H))$ as $r\to\infty$ outside a set of finite measure. If $H_p\equiv 0$, then $H_N\not\equiv 0$. By a similar argument for N_1^*, N_2^*, m^*, T^*, etc., we have the same results for m^* and T^*. Further $B_1(z)B_2^2(z)\not\equiv 0$ and $A_1(z)A_2^2(z)\not\equiv 0$.

Thus we may apply Lemma 3.2 to (3.15) and we have either

$$\begin{cases} L\equiv H, \\ -(A_1+2A_2)\equiv-(B_1+2B_2), \\ A_2^2+2A_1A_2\equiv B_2^2+2B_1B_2 \\ -A_1A_2^2\equiv -B_1B_2^2, \end{cases}$$

or

$$\begin{cases} L\equiv -H, \\ -(A_1+2A_2)\equiv -\dfrac{B_2^2+2B_1B_2}{B_1B_2^2}, \\ A_2^2+2A_1A_2\equiv \dfrac{B_12B_2}{B_1B_2^2}, \\ -A_1A_2^2\equiv -\dfrac{1}{B_1B_2^2}, \end{cases}$$

i.e., either $H(z)\equiv L(z)$ and $B_i(z)\equiv A_i(z)$ for $i=1, 2$, or $H(z)\equiv -L(z)$ and $B_i(z)\equiv 1/A_i(z)$ for $i=1, 2$. (q.e.d.)

§4 In this section we characterize $R\in S_3(D)$ with $P(R)=6$. The following theorem is an extension of a theorem due to Hiromi and Niino [2]:

THEOREM 4.1. *Let $R\in S_3(D)$. Then $P(R)=6$ if and only if R is conformally equivalent to a surface $S\in S_3(D)$ defined by $y^3=(z^n e^{H(z)}-\gamma)(z^n e^{H(z)}-\delta)^2$, where* (i) *$H(z)$ is a non-constant regular function in D with $H_0=0$.* (ii) *γ and δ are constants such that $\gamma\delta(\gamma-\delta)\neq 0$.* (iii) *$n$ is an integer ($n=0$ if $D=\{z||z|<\infty\}$).*

Proof. Suppose $R\in S_3(D)$ is defined by $y^3=g(z)$ and $P(R)=6$. Then there exists a meromorphic function $f\in\mathfrak{M}(R)$ with $P(f)=6$. We may assume that $0, a_1, a_2, a_3, a_4, \infty$ are the six values which are not taken by f. Then f is a single-valued regular function on R. Hence f satisfies (2.3) and the defining equation of f is given by (2.1) where $s_1(z)$, $s_2(z)$ and $s_3(z)$ satisfy (2.4). By Rémoundos' reasoning [10] we have

$$\begin{pmatrix} F(z,0) \\ F(z,a_1) \\ F(z,a_2) \\ F(z,a_3) \\ F(z,a_4) \end{pmatrix} = \text{(i)} \begin{pmatrix} P_0 \\ P_1 \\ P_2 e^{H_2} \\ P_3 e^{H_3} \\ P_4 e^{H_4} \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} P_2 e^{H_2} \\ P_0 \\ P_1 \\ P_3 e^{H_3} \\ P_4 e^{H_4} \end{pmatrix}$$

where H_j $(j=2,3,4)$ is a non-constant regular function in D with $H_{j0}=0$ and $P_j=b_j z^{n_j}$ $(j=0,1,2,3,4)$ with b_j being a non-zero constant and n_j an integer (all n_j are zero when $D=\{z||z|<\infty\}$).

Case (i). We have

$$(4.1)\quad \begin{cases} -s_3=b_0 z^{n_0}, & (1) \\ a_1^3-a_1^2 s_1+a_1 s_2-s_3=b_1 z^{n_1}, & (2) \\ a_2^3-a_2^2 s_1+a_2 s_2-s_3=b_2 z^{n_2} e^{H_2}, & (3) \\ a_3^3-a_3^2 s_1+a_3 s_2-s_3=b_3 z^{n_3} e^{H_3}, & (4) \\ a_4^3-a_4^2 s_1+a_4 s_2-s_3=b_4 z^{n_4} e^{H_4}. & (5) \end{cases}$$

Eliminating s_1, s_2 and s_3 from (1), (3), (4) and (5), we have

$$a_3 a_4(a_3-a_4) b_2 z^{n_2} e^{H_2}-a_2 a_4(a_2-a_4) b_3 z^{n_3} e^{H_3}+a_2 a_3(a_2-a_3) b_4 z^{n_4} e^{H_4}$$
$$=(b_0 z^{n_0}+a_2 a_3 a_4)(a_2-a_3)(a_2-a_4)(a_3-a_4).$$

We rewrite this equation into the form: $a_2' e^{H_{2p}}+a_3' e^{H_{3p}}+a_4' e^{H_{4p}}=a_1'$, where $T(r, a_j')=O(\log r)$ as $r\to\infty$.

If two of H_{2p}, H_{3p}, and H_{4p} are identically zero then the remaining one is also identically zero. So $H_{2p}\equiv H_{3p}\equiv H_{4p}\equiv 0$ in this case. Suppose $H_{2p}\not\equiv 0$ and $H_{3p}\not\equiv 0$. By Lemma 1.3 we may assume that H_{2p} and H_{3p} are dependent, i.e., $H_{2p}\equiv H_{3p}$ in

$1\leqq|z|<\infty$. Then we have $(a_2'+a_3')e^{H_{2p}}+a_4'e^{H_{4p}}=a_1'$. If $e^{H_{4p}}\equiv 0$ in $1\leqq|z|<\infty$ then it would force $H_{2p}\equiv 0$, a contradiction. Hence $e^{H_{4p}}\not\equiv 0$ in $1\leqq|z|<\infty$. Again by Lemma 1. 3 we have that $a_1'\equiv 0$, $H_{2p}\equiv H_{4p}$ and $a_2'+a_3'+a_4'\equiv 0$ in $1\leqq|z|<\infty$. Thus either $H_{2p}\equiv H_{3p}\equiv H_{4p}\equiv 0$ and $a_2'+a_3'+a_4'=a_1'$ or $H_{2p}\equiv H_{3p}\equiv H_{4p}\not\equiv 0$, $a_1'\equiv 0$ and $a_2'+a_3'+a_4'\equiv 0$ in $1\leqq|z|<\infty$. By a similar argument for the H_{jN} in $a_2'+a_3'+a_4'\equiv a_1'$, we have the following:

$$(4.2)\qquad \begin{cases} H_2\equiv H_3\equiv H_4\equiv H,\ n_0=0,\ n_2=n_3=n_4=n,\ b_0=-a_2a_3a_4 \qquad \text{and} \\ a_3a_4(a_3-a_4)b_2-a_2a_4(a_2-a_4)b_3+a_2a_3(a_2-a_3)b_4=0. \end{cases}$$

Next we eliminate s_1, s_2 and s_3 from (1), (2), (3) and (4) in (4. 1), and then substitute $H_2\equiv H_3\equiv H$, $n_3=n_4=n$ and $n_0=0$. Then we have

$$(4.3)\qquad \begin{cases} -a_1a_3(a_1-a_3)b_2+a_1a_2(a_1-a_2)b_3=0, \\ n_1=0, \\ a_2a_3(a_2-a_3)b_1-(b_0+a_1a_2a_3)(a_1-a_2)(a_1-a_3)(a_2-a_3)=0. \end{cases}$$

From (4. 1), (4. 2) and (4. 3), we have

$$\begin{cases} s_1=\dfrac{1}{a_2(a_1-a_2)}b_2z^ne^H+(a_2+a_3+a_4), \\ s_2=\dfrac{a_1}{a_2(a_1-a_2)}b_2z^ne^H+(a_2a_3+a_3a_4+a_4a_2), \\ s_3=a_2a_3a_4. \end{cases}$$

Case (ii). By a similar argument and computation, we have

$$\begin{cases} s_1=-\dfrac{1}{a_1a_2}b_2z^ne^H+(a_3+a_4), \\ s_2=-\dfrac{a_1+a_2}{a_1a_2}b_2z^ne^H+a_3a_4, \\ s_3=-b_2z^ne^H. \end{cases}$$

By a similar method due to Hiromi and Niino [2] we have

$$(4.4)\qquad \begin{cases} \nu^3g=A(z^ne^H-\gamma)(z^ne^H-\delta)^2, \\ \mu^3g^2=A(z^ne^H-\gamma)(z^ne^H-\delta), \end{cases}$$

where A, γ and δ are non-zero constants with $A\gamma\delta(\gamma-\delta)\not\equiv 0$ and $\nu(z)$ and $\mu(z)$ meromorphic functions in D with the properties as described in Lemma 2. 1.

Let $G_1(z)=(z^ne^H-\gamma)(z^ne^H-\delta)^2$ and $G_2(z)=(z^ne^H-\gamma)^2(z^ne^H-\delta)$. Then $G_1(z)$ and $G_2(z)$ are admissible for $S_3(D)$. Let $R_1\in S_3(D)$ and $R_2\in S_3(D)$ be defined by $y^3=G_1(z)$

and $w^3=G_2(x)$, respectively. Then by Lemma 2.2 we see that R is conformally equivalent to R_1 and R_2.

The proof for the converse is the same as in [2]. (q.e.d.)

§5. For our convenience, we define the following: (i) $(m, H, \alpha, \beta)_D$ is a symbol where $H(z)$ is a non-constant regular function in D with $H_0=0$, m is an integer ($m=0$ when $D=\{z||z|<\infty\}$) and α, β are distinct non-zero constants. (ii) $(m, H, \alpha, \beta)_D \equiv (n, L, \gamma, \delta)_D$ if and only if $m=n$, $H\equiv L$, $\alpha=\gamma$, and $\beta=\delta$. (iii) $f(m, H, \alpha, \beta)_D \equiv (z^m e^H-\alpha)(z^m e^H-\beta)^2$. (iv) $S(m, H, \alpha, \beta)_D$ is a surface in $S_3(D)$ defined by $y^3 = f(m, H, \alpha, \beta)_D$.

Suppose $R\in S_3(D)$ is defined by $y^3=g(z)$ and $P(R)=6$. Then from (4.4) there exists a symbol $(m, H, \alpha, \beta)_D$ such that $g(z)$ satisfies

$$\text{(5.1)} \qquad \begin{cases} \nu_1^3(z)g(z)=f(m, H, \alpha, \beta)_D, \\ \mu_1^3(z)g^2(z)=f(m, H, \beta, \alpha)_D, \end{cases}$$

where $\nu_1(z)$ and $\mu_1(z)$ are meromorphic functions in D with the properties as described in Lemma 2.1.

Suppose there exists another symbol $(n, L, \gamma, \delta)_D$ such that $g(z)$ satisfies

$$\text{(5.2)} \qquad \begin{cases} \nu_2^3(z)g(z)=f(n, L, \gamma, \delta)_D, \\ \mu_2^3(z)g^2(z)=f(n, L, \delta, \gamma)_D, \end{cases}$$

where $\nu_2(z)$ and $\mu_2(z)$ have the same properties as $\nu_1(z)$ and $\mu_1(z)$, respectively. From (5.1) and (5.2) we have

$$\nu^3(z)\left(e^H-\frac{\alpha}{z^m}\right)\left(e^H-\frac{\beta}{z^m}\right)^2=f(n, L, \gamma, \delta)_D, \qquad \text{where} \qquad \nu(z)=\frac{\nu_2(z)}{\nu_1(z)}z^m.$$

By Lemma 3.3, we have either

$$H\equiv L,\ \frac{\alpha}{z^m}\equiv\frac{\gamma}{z^n} \ \text{ and } \ \frac{\beta}{z^m}\equiv\frac{\delta}{z^n}, \qquad \text{or} \qquad H\equiv -L,\ \frac{\alpha}{z^m}\equiv\frac{z^n}{\gamma} \ \text{ and } \ \frac{\beta}{z_m}\equiv\frac{z^n}{\delta}.$$

Thus either $(m, H, \alpha, \beta)_D=(n, L, \gamma, \delta)_D$ or $(m, H, \alpha, \beta)_D=(-m, -L, 1/\gamma, 1/\delta)_D$. By Lemma 2.2 we see that R is conformally equivalent to the following four surfaces in $S_3(D)$:

$$S(n, L, \gamma, \delta)_D, \quad S(n, L, \delta, \gamma)_D, \quad S(-n, -L, \gamma^{-1}, \delta^{-1})_D \quad \text{and} \quad S(-n, -L, \delta^{-1}, \gamma^{-1})_D.$$

Let $(n^*, L^*, \gamma^*, \delta^*)_D$ be the one of the following four symbols: $(n, L, \gamma, \delta)_D$, $(n, L, \delta, \gamma)_D$, $(-n, -L, \gamma^{-1}, \delta^{-1})_D$ and $(-n, -L, \delta^{-1}, \gamma^{-1})_D$ such that $|\gamma^*|\leqq|\delta^*|$ and $0\leqq\arg\gamma^*\leqq\arg\delta^*<2\pi$. We denote $(n^*, L^*, \gamma^*, \delta^*)_D$ simply by $(n, L, \gamma, \delta)_D^*$. Then for given $g(z)$, if $R\in S_3(D)$ is defined by $y^3=g(z)$ and $P(R)=6$, then there corresponds a unique symbol $(n, L, \gamma, \delta)_D^*$ such that g and $f(n, L, \gamma, \delta)_D^*$ have the relation (5.1). To emphasize this fact, we sometimes denote $(n, L, \gamma, \delta)_D^*$ which is determined by

$g(z)$ by $(n, L, \gamma, \delta;\ g(z))_D$.

Now the problem of the existence of analytic maps among surfaces $R \in S_3(D)$ which are defined by $y^3=g(z)$ with $P(R)=6$ may be carried over to the same type of problem among surfaces $S(n, L, \gamma, \delta;\ g(z))_D$.

Let $R_1 \in S_3(D_1)$ and $R_2 \in S_3(D_2)$ be given with $P(R_1)=P(R_2)=6$ where R_i $(i=1, 2)$ is defined by $y^3=g_i(z)$, respectively. Here D_1 is either the domain $\{z \mid |z|<\infty\}$ or $\{z \mid 0<|z|<\infty\}$, and so is D_2. Suppose there is a non-trivial analytic map from $S(m, H, \alpha, \beta;\ g_1(z))_{D_1}$ into $S(n, L, \gamma, \delta;\ g_2(z))_{D_2}$. Then by Lemma 2. 1 there is a single-valued non-constant regular function $h(z)$ in D_1 such that either $\nu^3(z)G_1(z)=G_2\circ h(z)$, or $\mu^3(z)G_1^2(z)=G_2\circ h(z)$, where $\nu(z)$ and $\mu(z)$ have the properties as described in the lemma and $G_1(z)=f(m, H, \alpha, \beta)_{D_1}$ and $G_2(z)=f(n, L, \gamma, \delta)_{D_2}$.

If $\nu^3(z)G_1(z)=G_2\circ h(z)$ holds, then $\nu^3(z)(z^m e^H-\alpha)(z^m e^H-\beta)^2=(h^n e^{L\circ h}-\gamma)(h^n e^{L\circ h}-\delta)^2$, which can be written in the form:

$$f_1^3(z)(e^{Q(z)}-B_1(z))(e^{Q(z)}-B_2(z))^2=(e^{H(z)}-A_1(z))(e^{H(z)}-A_2(z))^2 \tag{5. 3}$$

where

$$f_1(z)=\frac{h(z)e^{(L\circ h)_0}}{z^m\nu(z)},\qquad Q=L\circ h-(L\circ h)_0,\qquad B_1(z)=\frac{\gamma}{h^n(z)e^{(L\circ h)_0}},$$

$$B_2(z)=\frac{\delta}{h^n(z)e^{(L\circ h)_0}},\qquad A_1(z)=\frac{\alpha}{z^m}\quad\text{and}\quad A_2(z)=\frac{\beta}{z^m}.$$

If $\mu^3(z)G_1^2(z)=G_2\circ h(z)$ holds, then

$$f_2^3(z)(e^{Q(z)}-B_1(z))(e^{Q(z)}-B_2(z))^2=(e^{H(z)}-A_2(z))(e^{H(z)}-A_1(z))^2, \tag{5. 4}$$

where

$$f_2(z)=\frac{h(z)e^{(L\circ h)_0}}{z^m(z^m e^H-\beta)\mu(z)},$$

and Q, B_1, B_2, A_1 and A_2 are the same as in (5. 3).

If $D_2=\{z \mid |z|<\infty\}$ then $n=0$ and if $D_2=\{z \mid 0<|z|<\infty\}$, then $h(z)$ omits zero. Hence in both cases, $B_1(z)$ and $B_2(z)$ are regular in D_1. Similarly, $A_1(z)$ and $A_2(z)$ are regular in D_1. Clearly $Q(z)$ is a non-constant regular function in D_1 with $Q_0=0$, and $f_1(z)$ and $f_2(z)$ are both meromorphic functions in D_1.

Suppose $Q_p \not\equiv 0$. Now $e^{Q(z)}-a$ $(a\neq 0)$ is a transcendental regular function in $1\leq|z|<\infty$ with infinitely many zeros. Hence by Lemma 1. 5, $m(r, h)=o(N(r, 0, e^Q-a))$. But $N(r, 0, e^Q-a)\leq N_2(r, 0, e^Q-a)+2N_1(r, 0, e^Q-a)\sim(1+o(1))m(r, e^Q)$. Hence $m(r, h)=o(m(r, e^Q))$. Thus $m(r, B_i)=o(m(r, e^Q))$ for $i=1, 2$, as $r\to\infty$ outside a set of finite measure. Clearly $m(r, A_i)=o(m(r, e^H))$ for $i=1, 2$, if $H_p \not\equiv 0$.

If $Q_p\equiv 0$ then $Q_N \not\equiv 0$. By a similar argument, we have $m^*(r, B_i)=o(m^*(r, e^Q))$, as $r\to\infty$ outside a set of finite measure. If $H_N \not\equiv 0$, then $m^*(r, A_i)=o(m^*(r, e^H))$ for $i=1, 2$.

Hence we may apply Lemma 3. 3 to each of (5. 3) and (5. 4) and have that in the case (5. 3), either $Q\equiv H$, $B_1(z)\equiv A_1(z)$ and $B_2(z)\equiv A_2(z)$, or $Q\equiv -H$, $B_1(z)\equiv 1/A_1(z)$ and $B_2(z)\equiv 1/A_2(z)$, i.e., either $H\equiv L\circ h-(L\circ h)_0$ and $\gamma/\alpha=\delta/\beta\equiv ch^n(z)z^{-m}$, or $H\equiv -L\circ h+(L\circ h)_0$ and $\alpha\gamma=\beta\delta\equiv ch^n(z)z^m$, where $c=e^{(L\circ h)_0}$.

If one of m and n is zero then so is the other. Hence either $m=n=0$ or $mn\neq 0$.

If $m=n=0$, then we have either $H\equiv L\circ h-(L\circ h)_0$ and $\gamma/\alpha=\delta/\beta=c$, or $H\equiv -L\circ h+(L\circ h)_0$ and $\alpha\gamma=\beta\delta=c$. If $mn\neq 0$, then either

$$H\equiv L(az^q),\ m=nq,\ \frac{\gamma}{a^n}=\alpha \quad\text{and}\quad \frac{\delta}{a^n}=\beta,$$

or

$$-H\equiv L(az^q),\ -m=nq,\ \frac{\gamma}{a^n}=\frac{1}{\alpha} \quad\text{and}\quad \frac{\delta}{a^n}=\frac{1}{\beta}.$$

In the case (5. 4) we have the results which are obtained from the results in the case (5. 3), by interchanging α and β.

Thus we proved the necessity part of the following:

THEOREM 5. 1. *Suppose $R_1\in S_3(D_1)$ and $R_2\in S_3(D_2)$ are defined by $y^3=g_1(z)$ and $w^3=g_2(x)$, respectively, and $P(R_1)=P(R_2)=6$. Let $(m, H, \alpha, \beta;\ g_1(z))_{D_1}$* and *$(n, L, \gamma, \delta;\ g_2(z))_{D_2}$ correspond to $g_1(z)$ and $g_2(z)$ respectively.*

Then there exists a non-tivial analytic map from R_1 into R_2 if and only if one of the following two statements is true:

(i) *$m=n=0$ and there exists a single-valued non-constant regular function $h(z)$ in D_1 such that*

$$\left(0,\ L\circ h-(L\circ h)_0,\ \frac{\gamma}{e^{(L\circ h)_0}},\ \frac{\delta}{e^{(L\circ h)_0}}\right)^*_{D_1}=(0, H, \alpha, \beta;\ g_1(z))_{D_1}.$$

(ii) *$mn\neq 0$ and there exist a non-zero integer p and non-zero constant c such that*

$$\left(np,\ L(cz^p),\ \frac{\gamma}{c^n},\ \frac{\delta}{c^n}\right)^*_{D_1}=(m, H, \alpha, \beta;\ g_1(z))_{D_1}.$$

Proof. We need to prove only sufficient part, and it is easy.

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