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# ON THE SOLUTION OF THE FUNCTIONAL EQUATION $f\circ g(z)=F(z)$

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**1.** In this note we shall prove the following three theorems:

THEOREM 1. *The functional equation $f\circ g(z)=F(z)$ has no pair of two transcendental entire solutions $f$ and $g$, if $F(z)$ is an entire function of finite order and it has a finite Picard exceptional value.*

THEOREM 2. *The functional equation $f\circ g(z)=F(z)$ has no pair of two transcendental entire solutions $f$ and $g$, if $F(z)$ is an entire function of finite order and it admits two perfectly branched values. Here a perfectly branched value $w$ of $F$ means that $F(z)-w$ has a finite number of simple zeros and has an infinite number of multiple zeros.*

THEOREM 3. *The functional equation $f\circ g(z)=F(z)$ has no pair of two transcendental entire solutions $f$ and $g$, if $F(z)$ is an entire function of finite order and there are $p$ disjoint continuous curves $\Gamma_j$ which extend to infinity and on which all the zeros of $F$ lie and along which $F$ is bounded.*

**2. Proof of theorem 1.** By Pólya's result [3] and by our assumption on the order of $F(z)$ and by the transcendency of $f(z)$ and $g(z)$, $f(z)$ is of order zero and $g(z)$ is of finite order. Let $A$ be a finite Picard exceptional value of $F(z)$. Since $f(z)$ is of order zero and transcendential, $f(z)=A$ has an infinite number of solutions $\{z_j\}$. Hence there is an infinite number of roots of the equation $g(z)=z_j$ excepting at most one $z_j$. These solutions are the solutions of the equation $F(z)=A$. This is impossible.

**3. Valiron's theorem.** In order to prove theorem 2 we need the following theorem, which had been prove by Valiron [5, p. 76]. For completeness we shall give a detailed proof of it, which is essentially the same as Valiron's.

VALIRON'S THEOREM. *If the order of $f(z)$ is finite and not a positive integral multiple of* 1/2, *there can only be one perfectly branched value.*

*Proof.* Assume that there are two such values, which we may suppose to be 1 and $-1$. Consider

$$\theta(z)=\frac{Q_1(z)Q_2(z)f'(z)^2}{f(z)^2-1},$$

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where $Q_1$ and $Q_2$ are polynomials formed by the simple zeros of $f(z)-1$ and $f(z)+1$, respectively. The $\theta(z)$ does not have any pole in $|z|<\infty$. Now consider the proximity function $m(r, \theta)$ of $\theta$. Then

$$\begin{aligned} m(r, \theta) &\leqq O(\log r)+m\left(r, \frac{f'}{f-1}\right)+m\left(r, \frac{f'}{f+1}\right) \\ &= O(\log r)+O(\log rm(r, f))+O(\log rm(r, f)) \\ &= O(\log r), \end{aligned}$$

since $f(z)$ is of finite order. Hence $\theta(z)$ must be a polynomial. This implies that

$$\begin{aligned} \frac{f'(z)^2}{f(z)^2-1} &= \frac{\theta(z)}{Q_1(z)Q_2(z)} \\ &= A_p z^p+\cdots+A_0+\sum_{j=1}^{n} \frac{B_j}{z-w_j}, \end{aligned}$$

since $Q_1Q_2$ has only simple zeros $\{w_j\}$.

If $A_p\neq 0$, by making the indefinite integral around $z=\infty$ and taking the inverse function we have

$$f(z)=\cos\left(\sqrt{A_p}\, z^{(p+2)/2}\Phi_1(z)\right),$$

where $\Phi_1(z)$ is regular at $z=\infty$.

If $A_p=\cdots=A_0=0$ and $\sum_{j=1}^{n} B_j\neq 0$, then

$$f(z)=\cos\left(\sqrt{\sum_{j=1}^{n} B_j}\, z^{1/2}\Phi_2(z)\right),$$

where $\Phi_2(z)$ is regular at $z=\infty$.

If $A_p=\cdots=A_0=0=\sum_{j=1}^{n} B_j$, then

$$f(z)=\begin{cases} \cos\left(\sqrt{\sum_{j=1}^{n} B_j w_j}\, \log z+\Phi_3(z)\right), & \text{when } \sum_{j=1}^{n} B_j w_j\neq 0, \\ \cos(\Phi_3((z)), & \text{when } \sum_{j=1}^{n} B_j w_j=0, \end{cases}$$

where $\Phi_3(z)\to 0$ as $z\to\infty$. Hence in these cases for a suitable $k$

$$\frac{f(z)}{z^k}\to 0$$

as $z\to\infty$. Therefore $f(z)$ does reduce to a polynomial, which may be omitted in our present case. Thus the order of $f(z)$ must be an integral (positive) multiple of 1/2.

**4. Proof of theorem 2.** Let $w_1$ and $w_2$ be two perfectly branched values of $F$. Consider the equations $f(z)=w_j$, $j=1, 2$. If $f(z)=w_1$ has an infinite number of simple zeros $\{z_{1,n}\}$, then $g(z)=z_{1,n}$ must have only a finite number of simple zeros. Hence $g(z)$ has an infinite number of perfectly branched values $\{z_{1,n}\}$, which contradicts the well-known Nevanlinna's ramification relation [2]. Hence $w_1$ must be a perfectly branched value of $f(z)$. The same holds for $w_2$. Then by Valiron's theorem the order of $f(z)$ must be a positive integral multiple of 1/2, which contradicts that the order of $f(z)$ is equal to zero.

An application. The functional equation $f\circ g(z)=\sin z$ has no transcendental entire solutions $f$ and $g$, since $\sin z$ has two perfectly branched values 1 and $-1$.

**5. Proof of theorem 3.** $f(z)$ has zero order by Pólya's theorem. Hence there is an infinite number of zeros of $f(z)$. Let $\{w_j\}$ be the set of zeros of $f(z)$ and $\{z_{jn}\}$ be the set of the solutions of $g(z)=w_j$. By the assumption the set $\{z_{jn}\}$ lies on $\cup_1^p \Gamma_j$. Hence the set $\{w_j\}$ lies on $\cup_1^p g(\Gamma_j)$. Evidently $w_j$ tends to infinity when $j$ tends to $\infty$. Hence at least one $g(\Gamma_j)$ is not bounded. By the assumption $f(z)$ must be bounded on this $g(\Gamma_j)$, which implies that the order of $f$ is not less than 1/2, since for an entire function of order less than 1/2 there is a sequence of values of $r$ tending to infinity through which

$$\min_{|z|=r} |f(z)| \to \infty.$$

This is a contradiction.

An Application. The functional equations $f\circ g(z)=F(z)$ for $F(z)=(e^z-\gamma)(e^z-\delta)$, $(\sin\sqrt{z})/\sqrt{z}$, $(\sin z)^n$, etc. have no transcendential entire solutions.

**6. A variant.** Now we shall give a variant of theorem 1.

THEOREM 4. *The functional equation $f\circ g(z)=F(z)$ has no pair of two transcendental entire solutions $f$ and $g$, if $F(z)$ is an entire function of finite order and $F'(z)$ admits 0 as a Picard exceptional value..*

*Proof.* Evidently $f(z)$ and hence $f'(z)$ are of order zero. Consider the functional equation

$$f'\circ g'(z)\cdot g'(z)=F'(z).$$

Then $f'(w)=0$ has an infinite number of solutions $\{w_j\}$ and $g(z)=w_j$ has an infinite number of solutions $\{z_{j,k}\}$ for each $j$ with at most one exception. Evidently $F'(z_{j,k})=0$, which is a contradiction.

An application. The functional equation $f\circ g(z)=F(z)$ for

$$F(z)=\int_0^z e^{-t^p}dt, \qquad p\text{: a positive integer},$$

has no transcendental entire solutions.

**7. Remarks.** Our theorems cannot use for $f\circ g(z)=1/\Gamma(z)$.

Baker [1] has given an interesting result for the functional equation $f\circ f(z)=F(z)$. For this equation there are lots of bibliography [1], [4].

It should be remarked that our results do lose their effectivity when $F$ is of order less than 1/2. So far as the present author concerns, there seems to be no result for an entire function $F$ of order less than 1/2, which is not a polynomial, up to the present time.

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