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SURFACES OF CURVATURES $\lambda=\mu=0$ IN E^4

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Introduction. A complete surface of Gaussian curvature $G=0$ in Euclidean space E^3 of dimension 3 is a cylinder. This fact was proved by W. S. Massey. The purpose of this paper is to prove the following theorem:

THEOREM. *A complete surface M^2 in Euclidean space E^4 of dimension 4 with the curvatures $\lambda=\mu=0$ is a cylinder.*

The definition of a cylinder in an Euclidean space is given by the following: Through each point of a curve on it, there passes a straight line which has the constant direction and the curve is not equal to one of these straight lines. The method of the proof is due to the idea of the Frenet-frames given by Professor Ōtsuki. In § 1, we study the local properties of M^2 in E^4 without completeness. A global study of complete surfaces of the curvatures $\lambda=\mu=0$ is given in § 2 with the aid of the universal covering space. The above theorem will be proved in this section. The author expresses his deep gratitude to Professor Ōtsuki who encouraged him and gave him a lot of useful suggestions.

§ 1. In the following we consider 2 dimensional, connected, oriented and class C^4 Riemannian manifold M^2 immersed in E^4 with the principal and secondary curvatures $\lambda=\mu=0$. By the definition of the Frenet-frame (p, e_1, e_2, e_3, e_4) for any surface M^2 in E^4, we have the following:

$$dp=e_1\omega_1+e_2\omega_2, \quad de_A=\Sigma\omega_{Aj}e_j+\omega_{A3}e_3+\omega_{A4}e_4, \quad A=1, 2, 3, 4, \quad j=1, 2, \tag{1.1}$$

$$\omega_{13}\wedge\omega_{24}+\omega_{14}\wedge\omega_{23}=0, \tag{1.2}$$

$$\omega_{13}\wedge\omega_{23}=\lambda\omega_1\wedge\omega_2, \quad \omega_{14}\wedge\omega_{24}=\mu\omega_1\wedge\omega_2, \tag{1.3}$$

$$\lambda+\mu=G, \quad \lambda\geqq\mu, \tag{1.4}$$

where ω_1, ω_2 and $\omega_{12}=-\omega_{21}$ are the basic forms and the connection form of M^2 respectively, λ and μ are the principal and secondary curvatures of the surface respectively, and G is the Gaussian curvature of M^2. In our case we cannot define the uniquely determined Frenet-frame, but we suitably take such a frame (p, e_1, e_2, e_3, e_4) from the first. Putting $\omega_{ir}=\Sigma A_{rij}\omega_j$ where $i, j=1, 2,$ $r=3, 4$ and $A_{rij}=A_{rji}$ we get by the hypothesis

$$\operatorname{rank}(A_{3ij})\leqq 1, \quad \operatorname{rank}(A_{4ij})\leqq 1. \tag{1.5}$$

Then we could define the two sets:

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(1. 6) $$M_0=\{p\in M^2 : \mathrm{rank}\,(A_{3ij}(p))=\mathrm{rank}\,(A_{4ij}(p))=0\},$$

(1. 7) $$M_1=\{p\in M^2 : \mathrm{rank}\,(A_{3ij}(p))=1, \text{ or } \mathrm{rank}\,(A_{4ij}(p))=1\}.$$

It is clear that M_0 is closed and M_1 is open.

Now suppose that $M_1\neq\phi$, and $p\in M_1$ and $\mathrm{rank}\,(A_{4ij}(p))=1$. Any tangent vector v is written as $v=\Sigma v^i e_i$. Then we can define the unique direction at p as follows: $\Sigma A_{4ij}v^i v^i=0$. This direction is called the asymptotic direction with respect to e_4. The asymptotic lines with respect to e_4 are defined by the integral curves of the field of unit vectors with this directions. Let e_1 be one of the unit tangent vectors at p with the asymptotic direction with respect to e_4, and we select the unit tangent vector e_2 at p orthogonal to e_1 so that the orientation of (e_1, e_2) is coherent with the one of M^2. Then we can define a field of such frames in a neighborhood U_p of p. According to the definition of the frame, we have

(1. 8) $$(A_{4ij})=\begin{pmatrix}0 & 0\\ 0 & \alpha\end{pmatrix}, \qquad \text{or} \quad \omega_{14}=0, \qquad \omega_{24}=\alpha\omega_2,$$

where α is a continuous everywhere non-zero function defined in U_p. By (1. 1) and $\lambda=0$, we get

(1. 9) $$\omega_{13}=0, \qquad \omega_{23}=\beta\omega_2,$$

where β is a continuous function defined in U_p. Since the second fundamental form with respect to any unit normal vector $e=e_3\cos\theta+e_4\sin\theta$ at p is given as

(1. 10) $$d^2p\cdot e=(\beta\cos\theta+\alpha\sin\theta)\omega_2\omega_2,$$

the asymptotic direction at p is independent of the choice of the unit normal vector at p.

By using (1. 8), (1. 9) and structure equations we have the following:

(1. 11) $$\omega_{12}=\gamma\omega_2, \qquad \omega_1=du, \qquad \omega_{34}=df,$$

where γ, u, f are continuous functions defined in U_p. Putting $e_3^*=e_3\cos\varphi+e_4\sin\varphi$, $e_4^*=-e_3\sin\varphi+e_4\cos\varphi$, we get $\omega_{34}^*=\langle de_3^*, e_4^*\rangle=d(f+\varphi)$. By selecting e_3, e_4 such that

(1. 12) $$\varphi=-f+\text{const.},$$

we have the torsion form $\omega_{34}=0$. Using only such e_3, e_4 we get the following:

(1. 13) $$\begin{cases} dp=e_1du+e_2\omega_2,\\ de_1=\gamma e_2\omega_2,\\ de_2=(-\gamma e_1+\beta e_3+\alpha e_4)\omega_2,\\ de_3=-\beta e_2\omega_2,\\ de_4=-\alpha e_2\omega_2.\end{cases}$$

The above equation shows that any asymptotic line is a straight line or its segment and all the tangent planes are constant along it.

LEMMA 1. *Suppase that $M_1 \neq \phi$, then any asymptotic line extends to infinitely or to the boundary of M^2 in E^4.*

Proof. By virtue of the structure equations, (1. 8), (1. 9) and (1. 11), we have along an asymptotic line as follows:

$$\alpha(s)=\frac{\alpha(0)}{\gamma(0)s+1}, \tag{1. 14}$$

$$\beta(s)=\frac{\beta(0)}{\gamma(0)s+1}, \tag{1. 15}$$

$$\gamma(s)=\frac{\gamma(0)}{\gamma(0)s+1}, \tag{1. 16}$$

where $\alpha(0), \beta(0), \gamma(0)$, are the values of α, β, γ, at $p \in M_1$ $(u=0)$ respectively. Since we can consider that U_p is a convex neighborhood we can define the Frenet-frames in a neighborhood of an asymptotic line with vanishing torsion by virtue of (1. 12). Now let l be the asymptotic line through p and be written as $x=x(s)$, $s=u+\text{const.}$, and $0 \leqq s < s_0$, $x(0)=p$. Suppose that $\lim_{s\uparrow s_0} x(s) \in M^2$, we get at once $\lim_{s\uparrow s_0} x(s) \in M_1$ by (1. 14). This implies a contradiction.

For any set $A, \mathring{A}$ means the largest open set contained in A.

LEMMA 2. *If $\mathring{M}_0 \neq \phi$, then each connected component of $\mathring{M}_0$ is a piece of plane.*

Because we can select e_3 and e_4 such that $\omega_{34}=0$, e_3 and e_4 are constant vectors. This fact implies the lemma.

§ **2.** In this section M^2 is supposed to be complete. A point of a cylinder is called proper if there does not exist any neighborhood of the point which is contained in the plane. A cylinder is called proper if all the points of it are proper. The completeness and Lemma 1, implies that each asymptotic line is a full straight line. Then (1. 16) hold for $-\infty < s < \infty$, but since γ is continuous we must have $\gamma \equiv 0$, i.e., $de_1=0$. On the other hand we have $de_2=(\beta e_3+\alpha e_4)\ \omega_2 \neq 0$. Then we have the following lemma:

LEMMA 3. *If $M_1 \neq \phi$, then M_1 consists of proper cylinders.*

Let us denote by M_1' the set of all boundary points of M_1 in M^2. Since M^2 is complete and $G=0$, the universal covering space of M^2 is a Euclidean plane E^2, and the covering map is written as π. A connected open set contained in the plane is called stripe if for each point of the set, there exists a straight line which has the constant direction and is contained entirely in the set.

LEMMA 4. *If $\mathring{M}_0 \neq \phi$, then $\mathring{M}_0$ consists of stripes in E^4.*

Proof. If $M_1 \neq \phi$, then M^2 is a plane in E^4. Suppose that $M_1 \neq \phi$. Since $\pi^{-1}(M_1)$ consists of parallel stripes in E^2, $\overset{\circ}{\overbrace{E^2-\pi^{-1}(M_1)}}$ also consists of parallel stripes in E^2. On the other hand we have by the property of covering map,

$$\widehat{E^2-\pi^{-1}(\mathring{M}_1)}=\widehat{\pi^{-1}(\mathring{M}_0)}=\pi^{-1}(\mathring{M}_0). \tag{2 1}$$

Let V be a connected component of M_0. Since $\pi^{-1}(V)$ consists of parallel stripes in E^2, V must be a stripe in E^4 by Lemma 2.

LEMMA 5. *If $M_1' \neq \phi$, then for each point of M_1' there exists a unique straight line contained entirely in M_1'.*

Proof. Because π is local isometry, we get the following;

$$\pi^{-1}(M_1')=(\pi^{-1}(M_1))'. \tag{2. 2}$$

Let q be any fixed point of M_1', then there exists a sequence $\{q_n\}_n$ in M_1 such that $\lim q_n=q$. Let $\tilde{q}$ be a point in E^2 such that $\pi(\tilde{q})=q$, and $\tilde{q}_n$ be a point of E^2 such that $\lim \tilde{q}_n=\tilde{q}$, $\pi(\tilde{q})=q_n$. There exists a unique straight line $\tilde{l}_n$ through each q_n in $\pi^{-1}(M_1)$ with the fixed direction such that $\pi(\tilde{l}_n)$ is the asymptotic line of M^2 through q_n. Now let $\tilde{l}$ be a straight line through $\tilde{q}$ in E^2 which is parallel to $\tilde{l}_n$, then we have the following:

$$\tilde{l}\subset(\pi^{-1}(M_1))'. \tag{2. 3}$$

Because π is local isometry $\pi(\tilde{l})$ is a geodesic in M^2 and the above discussion shows that the arclength of $\pi(\tilde{l})$ between q and $r\epsilon\pi(\tilde{l})$ is equal to the distance between q and r in E^4, i.e., $\pi(\tilde{l})$ is a straight line in E^4.

Let us prove the theorem. We have proved that for each point of M^2 there exists a unique straight line contained in M^2. We can take a coordinate system (x, y) in E^2 such that all the images of y=const. under π are these straight lines in M^2. We can define a unit tangent vector field $\bar{e}_1$ over M^2 such that

$$\bar{e}_1=d\pi\left(\frac{\partial}{\partial x}\right), \tag{2. 4}$$

where $d\pi$ is the differential map of π. Then we get by Lemma 3,

$$d(\bar{e}_{1|M_1})=d(e_1)=0, \tag{2. 5}$$

because $\bar{e}_1=\pm e_1$. By virtue of Lemma 4, we get

$$\alpha(\bar{e}_{1|\mathring{M}_0})=0. \tag{2. 6}$$

Therefore we have $d(\bar{e}_1)=0$ in M^2.

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