SOME EXPANSION THEOREMS FOR STOCHASTIC PROCESSES, II

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1. Let $F(\lambda)$ be the spectral function of a continuous (weakly) stationary process $\mathcal{E}(t)$ with mean zero, and consider $X(t)=f(t)+\mathcal{E}(t)$, $-\infty<t<\infty$, where $f(t)$ is a numerical valued function. Assume that

(i)
$$\int_{-\infty}^{\infty} |\lambda|^{2r+2\alpha}\, dF(\lambda)<\infty,$$

(ii) $H(u)$ is of bounded variation in $(-\infty, \infty)$,

(iii)
$$\int_{-\infty}^{\infty} |u|^{r+\alpha}\, |dH(u)|<\infty,$$

(iv) $|f(u)|\leqq C(1+|u|^{r+\alpha})$ for all u and a positive constant C, and

(v)
$$f(t+u)=\sum_{k=0}^{r}\frac{f^{(k)}(t)}{k!}u^k+o(|u|^{r+\alpha}),$$

where r is a non-negative integer and α is a constant with $0\leqq\alpha<1$. From the conditions (i)$-$(v) it follows that

$$E\left\{\left|\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right)dH(s)-\sum_{k=0}^{r}\frac{(-1)^k X^{(k)}(t)}{k!\, n^k}\int_{-\infty}^{\infty} s^k\, dH(s)\right|^2\right\}=o\left(\frac{1}{n^{2r+2\alpha}}\right). \tag{1.1}$$

This is an expansion theorem for the integral

$$\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right)dH(s),$$

which has been treated by Kawata [3] for $r=0$ and $\alpha=1/2$ with somewhat different conditions, and extended by the author [1] for $r=0, 1, 2, \cdots$ and $0\leqq\alpha<1$ with the above conditions (i)$-$(v). In this paper, we shall show (1.1) for $X(t)=f(t)+\phi(t)\mathcal{E}(t)$, where $\phi(u)$ is a numerical valued function. If $\phi(s)>0$, $-\infty<s<\infty$, then, for this process $X(t)$, the correlation coefficient of $X(t)$ and $X(s)$ is a function of $t-s$ only. In section 2, Taylor expansion of $\mathcal{E}(t)$ is discussed and, in section 3, the expansion theorem for

$$\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right)dH(s)$$

is given, where $X(t)=f(t)+\phi(t)\mathcal{E}(t)$, $-\infty<t<\infty$.

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2. Let $X(t)$ be a stochastic process with $E\{|X(t)|^2\}<\infty$, $-\infty<t<\infty$. If

$$\underset{\tau\to t}{\text{l.i.m.}}\, X(\tau)=X(t) \quad \text{i.e.} \quad E\{|X(\tau)-X(t)|^2\}\to 0 \quad \text{as} \quad \tau\to t,$$

then $X(t)$ is called to be continuous at t. Let $K(t)$ be of bounded variation in any finite interval and consider a division of an interval (A, B):

$$\Delta: \quad A=t_0<t_1<\cdots<t_n=B.$$

If

$$\underset{\Delta}{\text{l.i.m.}} \sum_{k=1}^{n} X(\tau_k)\,[K(t_k)-K(t_{k-1})]=S \qquad \left(\max_{k=1,2,\cdots,n}(t_k-t_{k-1})\to 0\right),$$

where $t_{k-1}\leqq\tau_k\leqq t_k$, $(k=1, 2, \cdots, n)$, then S is denoted as

$$\int_A^B X(t)\, dK(t).$$

We can easily prove the following

LEMMA 1. *Being continuous in an interval* $[A, B]$, $X(t)$ *is uniformly continuous in* $[A, B]$. *Moreover,*

$$\int_A^B X(t)\, dK(t)$$

exists in the above sense.

LEMMA 2. *If* $X^{(k)}(t)$ $(k=1, 2, \cdots, r)$ *exist in the sense that*

$$X^{(k)}(t)=\underset{h\to 0}{\text{l.i.m.}}\frac{X^{(k-1)}(t+h)-X^{(k-1)}(t)}{h} \qquad (k=1, 2, \cdots, r),$$

where $X^{(0)}(t)\equiv X(t)$, *and* $X^{(k)}(t)$ $(k=1, 2, \cdots, r)$ *are cantinuous in* $[a, b]$, *then*

$$X(b)=X(a)+\frac{X'(a)}{1!}(b-a)+\cdots+\frac{X^{(r-1)}(a)}{(r-1)!}(b-a)^{r-1}$$

(2. 1)

$$+\frac{1}{(r-1)!}\int_a^b (b-t)^{r-1}X^{(r)}(t)\, dt \qquad \text{(a.s.)}.$$

Proof. The existence of the integral in (2. 1) is ensured by Lemma 1. Y being any random variable with $E\{|Y|^2\}<\infty$, the numerical valued function $\varphi(t)\equiv E\{X(t)\cdot\overline{Y}\}$ is differentiable r times and $\varphi^{(r)}(t)=E\{X^{(r)}(t)\cdot\overline{Y}\}$ is continuous in $[a, b]$. So we have

$$\varphi(b)=\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(a)}{k!}(b-a)^k+\frac{1}{(r-1)!}\int_a^b (b-t)^{r-1}\varphi^{(r)}(t)\, dt$$

or

$$E\left\{\left[X(b)-\sum_{k=0}^{r-1}\frac{X^{(k)}(a)}{k!}(b-a)^k-\frac{1}{(r-1)!}\int_a^b (b-t)^{r-1}X^{(r)}(t)\, dt\right]\cdot\overline{Y}\right\}=0.$$

Choosing

$$X(b)-\sum_{k=0}^{r-1}\frac{X^{(k)}(a)}{k\,!}(b-a)^k-\frac{1}{(r-1)\,!}\int_a^b (b-t)^{r-1}X^{(r)}(t)\,dt$$

as Y, we have

$$E\left\{\left|X(b)-\sum_{k=0}^{r-1}\frac{X^{(k)}(a)}{k\,!}(b-a)^k-\frac{1}{(r-1)\,!}\int_a^b (b-t)^{r-1}X^{(r)}(t)\,dt\right|^2\right\}=0,$$

which implies (2. 1).

In the following, let $\mathcal{E}(t)$, $-\infty<t<\infty$, be a continuous (weakly) stationary stochastic process with $E\{\mathcal{E}(t)\}=0$ for all t. We note that $\mathcal{E}(t)$ is continuous in the sense stated at the beginning of this section, if and only if the covariance function $\rho(u)$ of $\mathcal{E}(t)$ is continuous.

THEOREM 1. *Let $F(\lambda)$ be the spectral function of $\mathcal{E}(t)$. If*

$$\int_{-\infty}^{\infty}|\lambda|^{2r+2\alpha}\,dF(\lambda)<\infty, \tag{2. 2}$$

where r is a non-negative integer and α is a constant with $0\leqq\alpha<1$, then

$$E\left\{\left|\mathcal{E}(t+u)-\sum_{k=0}^{r}\frac{\mathcal{E}^{(k)}(t)}{k\,!}u^k\right|^2\right\}=o(|u|^{2r+2\alpha})\ \text{as}\ u\to 0. \tag{2. 3}$$

Proof. It is well known that, under the assumption (2. 2), $\mathcal{E}^{(k)}(t)$ $(k=1, 2, \cdots, r)$ exist and are continuous (weakly) stationary processes whose covariance functions are

$$\rho_k(t)=\int_{-\infty}^{\infty}\lambda^{2k}e^{it\lambda}\,dF(\lambda)$$

respectively. Hence, by Lemma 2, we have with probability 1 that

$$\begin{aligned}R&\equiv\mathcal{E}(t+u)-\sum_{k=0}^{r}\frac{\mathcal{E}^{(k)}(t)}{k\,!}u^k\\ &=\frac{1}{(r-1)\,!}\int_t^{t+u}(t+u-\tau)^{r-1}\mathcal{E}^{(r)}(\tau)\,d\tau-\frac{\mathcal{E}^{(r)}(t)}{r\,!}u^r\\ &=\frac{1}{(r-1)\,!}\int_t^{t+u}(t+u-\tau)^{r-1}[\mathcal{E}^{(r)}(\tau)-\mathcal{E}^{(r)}(t)]\,d\tau\end{aligned} \tag{2. 4}$$

and so

$$\begin{aligned}E\{|R|^2\}=\frac{1}{[(r-1)\,!]^2}\,E\Big\{&\int_t^{t+u}(t+u-\tau)^{r-1}[\mathcal{E}^{(r)}(\tau)-\mathcal{E}^{(r)}(t)]\,d\tau\\ &\cdot\int_t^{t+u}(t+u-\sigma)^{r-1}\overline{[\mathcal{E}^{(r)}(\sigma)-\mathcal{E}^{(r)}(t)]}\,d\sigma\Big\}\\ =\frac{1}{[(r-1)\,!]^2}&\int_t^{t+u}\int_t^{t+u}(t+u-\tau)^{r-1}(t+u-\sigma)^{r-1}\end{aligned}$$

$$\cdot E\{[\mathcal{E}^{(r)}(\tau)-\mathcal{E}^{(r)}(t)]\,\overline{[\mathcal{E}^{(r)}(\sigma)-\mathcal{E}^{(r)}(t)]}\}d\tau d\sigma$$

$$=\frac{1}{[(r-1)!]^2}\int_t^{t+u}\int_t^{t+u}\{\rho_r(\tau-\sigma)-\rho_r(\tau-t)-\rho_r(t-\sigma)+\rho_r(0)\}$$

$$\cdot(t+u-\tau)^{r-1}(t+u-\sigma)^{r-1}\,d\tau d\sigma$$

$$(2.5)\qquad =\frac{1}{[(r-1)!]^2}\int_t^{t+u}\int_t^{t+u}\Big[\int_{-\infty}^{\infty}\{e^{i(\tau-\sigma)\lambda}-e^{i(\tau-t)}-e^{i(t-\sigma)\lambda}+1\}$$

$$\cdot\lambda^{2r}dF(\lambda)\Big](t+u-\tau)^{r-1}(t+u-\sigma)^{r-1}\,d\tau d\sigma$$

$$=\frac{1}{[(r-1)!]^2}\int_t^{t+u}\int_t^{t+u}\Big[\int_{-\infty}^{\infty}(e^{i\tau\lambda}-e^{it\lambda})\overline{(e^{i\sigma\lambda}-e^{it\lambda})}$$

$$\cdot\lambda^{2r}dF(\lambda)\Big](t+u-\tau)^{r-1}(t+u-\sigma)^{r-1}\,d\tau d\sigma$$

$$=\frac{1}{[(r-1)!]^2}\int_{-\infty}^{\infty}\left|\int_t^{t+u}(e^{i\tau\lambda}-e^{it\lambda})(t+u-\tau)^{r-1}\,d\tau\right|^2\lambda^{2r}\,dF(\lambda)$$

$$=\frac{1}{[(r-1)!]^2}\int_{-\infty}^{\infty}\left|\int_0^{u}(e^{i\xi\lambda}-1)(u-\xi)^{r-1}d\xi\right|^2\lambda^{2r}\,dF(\lambda)$$

$$\leqq\frac{1}{[(r-1)!]^2}\int_{-\infty}^{\infty}\left\{\int_0^{|u|}|e^{i\xi\lambda}-1|^{1-\alpha}\xi^{\alpha}(|u|-\xi)^{r-1}\,d\xi\right\}^2|\lambda|^{2r+2\alpha}dF(\lambda).$$

This estimation implies with Minkowski's inequality that

$$(2.6)\qquad \sqrt{E\{|R|^2\}}\leqq\frac{1}{(r-1)!}\int_0^{|u|}\left\{\int_{-\infty}^{\infty}|e^{i\xi\lambda}-1|^{2(1-\alpha)}|\lambda|^{2r+2\alpha}dF(\lambda)\right\}^{\frac{1}{2}}\xi^{\alpha}(|u|-\xi)^{r-1}d\xi.$$

Since $|e^{i\xi\lambda}-1|^{2(1-\alpha)}\to 0$ as $\xi\to 0$ and $|e^{i\xi\lambda}-1|^{2(1-\alpha)}\leqq 2^{2(1-\alpha)}$, by the assumption (2.2) and Lebesgue's theorem it holds that

$$(2.7)\qquad \lim_{\xi\to 0}\int_{-\infty}^{\infty}|e^{i\xi\lambda}-1|^{2(1-\alpha)}|\lambda|^{2r+2\alpha}\,dF(\lambda)=0.$$

Hence, for any positive number ε, there exists a positive number η such that

$$\left\{\int_{-\infty}^{\infty}|e^{i\xi\lambda}-1|^{2(1-\alpha)}|\lambda|^{2r+2\alpha}\,dF(\lambda)\right\}^{\frac{1}{2}}<\varepsilon\quad\text{for}\quad|\xi|<\eta.$$

And so, we have

$$(2.8)\qquad \sqrt{E\{|R|^2\}}<\frac{\varepsilon}{(r-1)!}\int_0^{|u|}\xi^{\alpha}(|u|-\xi)^{r-1}d\xi<\frac{\varepsilon}{(r-1)!}|u|^{r+\alpha}\quad\text{for}\quad 0<|u|<\eta,$$

which proves Theorem 1. (The proof of the case $r=0$ will similarly be done with slight modifications.)

REMARK 1. In the case $\alpha=0$, we can prove (2.3) for $X(t)$ in Lemma 2. Because, choosing a positive number δ such that

$$\sqrt{E\{|X^{(r)}(\tau)-X^{(r)}(t)|^2\}}<\varepsilon \qquad \text{for} \quad |\tau-t|<\delta,$$

we have

$$\sqrt{E\{|R|^2\}} \leqq \frac{1}{(r-1)!}\left|\int_t^{t+u}\sqrt{E\{|X^{(r)}(\tau)-X^{(r)}(t)|^2\}}\cdot|t+u-\tau|^{r-1}d\tau\right|$$

$$\leqq \frac{\varepsilon}{(r-1)!}\left|\int_t^{t+u}|t+u-\tau|^{r-1}d\tau\right| \leqq \frac{\varepsilon}{(r-1)!}|u|^r.$$

In the following, let $f(t)$ and $\phi(t)$ be numerical valued functions.

THEOREM 2. *In addition to the assumptions of Theorem* 1, *we assume for some fixed t that*

$$f(t+u)=\sum_{k=0}^{r}\frac{f^{(k)}(t)}{k!}u^k+o(|u|^{r+\alpha}) \qquad \textit{as} \quad u\to 0 \tag{2.9}$$

and

$$\phi(t+u)=\sum_{k=0}^{r}\frac{\phi^{(k)}(t)}{k!}u^k+o(|u|^{r+\alpha}) \qquad \textit{as} \quad u\to 0. \tag{2.10}$$

Putting $X(s)=f(s)+\phi(s)\mathcal{E}(s)$, $-\infty<s<\infty$, *we have*

$$E\left\{\left|X(t+u)-\sum_{k=0}^{r}\frac{X^{(k)}(t)}{k!}u^k\right|^2\right\}=o(|u|^{2r+2\alpha}) \qquad \textit{as} \quad u\to 0. \tag{2.11}$$

Proof. It can be easily seen that $X^{(k)}$ $(k=1, 2, \cdots, r)$ exist and

$$X^{(k)}(t)=f^{(k)}(t)+\sum_{\nu=0}^{k}\binom{k}{\nu}\phi^{(\nu)}(t)\mathcal{E}^{(k-\nu)}(t) \quad (k=1, 2, \cdots, r).$$

Putting

$$f(t+u)-\sum_{k=0}^{r}\frac{f^{(k)}(t)}{k!}u^k=\Delta_1,\ \phi(t+u)-\sum_{k=0}^{r}\frac{\phi^{(k)}(t)}{k!}u^k=\Delta_2 \text{ and } \mathcal{E}(t+u)-\sum_{k=0}^{r}\frac{\mathcal{E}^{(k)}(t)}{k!}u^k=R$$

by (2.9), (2.10) and Theorem 1 we have

$$|\Delta_1|=o(|u|^{r+\alpha}), \qquad |\Delta_2|=o(|u|^{r+\alpha}) \tag{2.12}$$

and

$$E\{|R|^2\}=o(|u|^{2r+2\alpha}) \qquad \text{as} \quad u\to 0. \tag{2.13}$$

Since

$$X(t+u)=\sum_{k=0}^{r}\frac{f^{(k)}(t)}{k!}u^k+\Delta_1+\left[\sum_{k=0}^{r}\frac{\phi^{(k)}(t)}{k!}u^k+\Delta_2\right]\left[\sum_{k=0}^{r}\frac{\mathcal{E}^{(k)}(t)}{k!}u^k+R\right]$$

$$=\sum_{k=0}^{r}\frac{X^{(k)}(t)}{k!}u^k+\Delta_1+\sum_{\substack{0<k,l\leqq r\\ k+l>r}}\frac{\phi^{(k)}(t)}{k!}\frac{\mathcal{E}^{(l)}(t)}{l!}u^{k+l} \tag{2.14}$$

$$+\Delta_2\sum_{k=0}^{r}\frac{\mathcal{E}^{(k)}(t)}{k!}u^k+R\sum_{k=0}^{r}\frac{\phi^{(k)}(t)}{k!}u^k+\Delta_2 R,$$

by (2.12) and (2.13) we have

$$\sqrt{E\left\{\left|X(t+u)-\sum_{k=0}^{r}\frac{X^{(k)}(t)}{k!}u^k\right|^2\right\}}$$

$$\leqq|\varDelta_1|+\sum_{\substack{0<k,l\leqq r\\ k+l>r}}\frac{|\phi^{(k)}(t)|}{k!}\frac{\sqrt{\rho_k(0)}}{l!}|u|^{k+l}+|\varDelta_2|\sum_{k=0}^{r}\frac{\sqrt{\rho_k(o)}}{k!}\cdot|u|^k \tag{2.15}$$

$$+\sqrt{E\{|R|^2\}}\left\{\sum_{k=0}^{r}\frac{|\phi^{(k)}(t)|}{k!}|u|^k+|\varDelta_2|\right\}$$

$$=o(|u|^{r+\alpha}),$$

which proves Theorem 2.

In the following, we shall give some examples as applications of Lemma 2.

Example 1. For $\mathcal{E}(t)$ in Theorem 1, we assume that

$$\int_{-\infty}^{\infty}\lambda^{2n}\,dF(\lambda)<\infty \qquad (n=1, 2, \cdots). \tag{2.16}$$

Since

$$E\{|R_n|^2\}=\frac{1}{[(n-1)!]^2}\int_{-\infty}^{\infty}\left|\int_0^u(e^{i\xi\lambda}-1)(u-\xi)^{n-1}d\xi\right|^2\lambda^{2n}\,dF(\lambda)$$

$$=\int_{-\infty}^{\infty}\left|e^{iu\lambda}-\sum_{k=0}^{n}\frac{(iu\lambda)^k}{k!}\right|^2dF(\lambda),$$

where

$$R_n=\mathcal{E}(t+u)-\sum_{k=0}^{n}\frac{\mathcal{E}^{(k)}(t)}{k!}u^k,$$

we get that

$$\mathcal{E}(t+u)=\sum_{n=0}^{\infty}\frac{\mathcal{E}^{(n)}(t)}{n!}u^n\equiv\underset{n\to\infty}{\mathrm{l.i.m.}}\sum_{k=0}^{n}\frac{\mathcal{E}^{(k)}(t)}{k!}u^k \tag{2.17}$$

if and only if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\left|e^{iu\lambda}-\sum_{k=0}^{n}\frac{(iu\lambda)^k}{k!}\right|^2dF(\lambda)=0. \tag{2.18}$$

Example 2. For $\mathcal{E}(t)$ in Theorem 1, we assume that there exists a positive constant δ such that

$$\int_{-\infty}^{\infty}\lambda^{2n}\,dF(\lambda)=o\left(\frac{[n!]^2}{\delta^{2n}}\right) \qquad \text{as} \quad n\to\infty. \tag{2.19}$$

Then we have (2.17) for any t and u with $|u|<\delta$, because

$$E\{|R_n|^2\}=\frac{1}{[(n-1)!]^2}\int_{-\infty}^{\infty}\left|\int_0^u(e^{i\xi\lambda}-1)(u-\xi)^{n-1}\,d\xi\right|^2\lambda^{2n}dF(\lambda)$$

$$\leqq \frac{4}{[(n-1)!]^2}\int_{-\infty}^{\infty}\left\{\int_0^{|u|}(|u|-\xi)^{n-1}d\xi\right\}^2\lambda^{2n}\,dF(\lambda)$$

$$(2.20)\qquad =\frac{4|u|^{2n}}{[n!]^2}\int_{-\infty}^{\infty}\lambda^{2n}\,dF(\lambda)$$

$$\leqq \frac{4\delta^{2n}}{[n!]^2}\int_{-\infty}^{\infty}\lambda^{2n}\,dF(\lambda)$$

$$=o(1) \qquad \text{as} \quad n\to\infty \quad \text{for} \quad |u|<\delta.$$

EXAMPLE 3. Let $\{t_n\}$ be a sequence of real numbers with $t_n \neq t_0$ $(n=1, 2, \cdots)$ and $t_n \to t$ as $n\to\infty$. Under the assumption (2.19), the random variable $\mathcal{E}(t)$ for any fixed t is determined with probability 1 by the sequence $\{\mathcal{E}(t_n);\ n=1, 2, \cdots\}$ of the random variables.

EXAMPLE 4. Under the assumption (2.16) for any fixed positive integer n, we have with probability 1 that

$$(2.21)\qquad \underset{h\to 0}{\text{l.i.m.}}\frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathcal{E}(t+kh)=\mathcal{E}^{(n)}(t).$$

Since, by Theorem 1,

$$(2.22)\qquad E\{|R_k|^2\}=o(h^{2n}) \qquad \text{as} \quad h\to 0,$$

where

$$(2.23)\qquad \mathcal{E}(t+kh)=\sum_{\nu=0}^{n}\frac{\mathcal{E}^{(\nu)}(t)}{\nu!}(kh)^{\nu}+R_k,$$

we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathcal{E}(t+kh)$$

$$(2.24)\qquad =\sum_{\nu=0}^{n}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^{\nu}\right]\frac{\mathcal{E}^{(\nu)}(t)}{\nu!}h^{\nu}+\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}R_k$$

$$=\mathcal{E}^{(n)}(t)h^n+\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}R_k$$

and so

$$E\left\{\left|\frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathcal{E}(t+kh)-\mathcal{E}^{(n)}(t)\right|^2\right\}$$

$$(2.25)\qquad \leqq\left\{\frac{1}{|h|^n}\sum_{k=0}^{n}\binom{n}{k}\sqrt{\{|R_k|^2\}}\right\}^2=o(1) \qquad \text{as} \qquad h\to 0,$$

which gives (2.21).

REMARK 2. The result of Example 4 has been proved by Kawata [2] in a different method. From Remark 1, we have (2.21) by assuming the existence and the continuity of $\mathcal{E}^{(n)}(t)$ at the point t without the stationarity of $\mathcal{E}(t)$.

3. In the preceeding section, we have defined the integral

$$\int_A^B X(t)\,dK(t).$$

If it holds that

$$\underset{\substack{A\to-\infty\\ B\to\infty}}{\mathrm{l.i.m.}}\int_A^B X(t)\,dK(t)=I,$$

then I is denoted as

$$\int_{-\infty}^{\infty} X(t)\,dK(t).$$

LEMMA 3. *If $X(t)$ is continuous in $(-\infty,\infty)$ and there exists a non-negative valued function $g(t)$ such that*

$$\sqrt{E\{|X(t)|^2\}}\leqq g(t) \qquad \textit{for all } t \tag{3.1}$$

and

$$\int_{-\infty}^{\infty} g(t)|\,dK(t)\,|<\infty, \tag{3.2}$$

then there exists

$$\int_{-\infty}^{\infty} X(t)\,dK(t).$$

Proof. Since

$$E\left\{\left|\int_A^{A'} X(t)\,dK(t)\right|^2\right\}\leqq\left\{\int_A^{A'} g(t)|\,dK(t)\,|\right\}^2, \tag{3.3}$$

we have by (3.2) that

$$\underset{A,A'\to-\infty}{\mathrm{l.i.m.}}\int_A^{A'} X(t)\,dK(t)=0 \tag{3.4}$$

and

$$\underset{B,B'\to\infty}{\mathrm{l.i.m.}}\int_B^{B'} X(t)\,dK(t)=0, \tag{3.5}$$

which ensure the existence of the integral

$$\int_{-\infty}^{\infty} X(t)\,dK(t).$$

THEOREM 3. *Let $H(u)$ be a numerical valued function of bounded variation in $(-\infty, \infty)$ and we assume that*

$$\int_{-\infty}^{\infty} |u|^{r+\alpha} |dH(u)| < \infty, \tag{3.6}$$

where r is a non-negative integer and α is a constant with $0 \leqq \alpha < 1$. If $X(t)$, $-\infty < t < \infty$, is a continuous stochastic process such that

$$\sqrt{E\{|X(s)|^2\}} \leqq C(1+|s|^{r+\alpha}) \tag{3.7}$$

for all s and some constant C and

$$E\left\{\left|X(t+u) - \sum_{k=0}^{r} \frac{X^{(k)}(t)}{k!} u^k\right|^2\right\} = o(|u|^{2r+2\alpha}) \qquad as \quad u \to 0, \tag{3.8}$$

then it holds that

$$\begin{aligned} &E\left\{\left|\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right) dH(s) - \sum_{k=0}^{r} \frac{(-1)^k X^{(k)}(t)}{k!\, n^k} \int_{-\infty}^{\infty} s^k\, dH(s)\right|^2\right\} \\ &= o\left(\frac{1}{n^{2r+2\alpha}}\right) \qquad as \quad n \to \infty. \end{aligned} \tag{3.9}$$

Proof. (3.6) and (3.7) ensure with Lemma 3 the existence of the integral

$$\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right) dH(s).$$

Put

$$\begin{aligned} I &= \int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right) dH(s) - \sum_{k=0}^{r} \frac{(-1)^k X^{(k)}(t)}{k!\, n^k} \int_{-\infty}^{\infty} s^k\, dH(s) \\ &= \int_{-\infty}^{\infty} \left[X\left(t-\frac{s}{n}\right) - \sum_{k=0}^{r} \frac{X^{(k)}(t)}{k!} \left(-\frac{s}{n}\right)^k \right] dH(s). \end{aligned} \tag{3.10}$$

Then we have that

$$\begin{aligned} n^{r+\alpha}\sqrt{E\{|I|^2\}} &\leqq n^{r+\alpha} \int_{-\infty}^{\infty} \sqrt{E\left\{\left| X\left(t-\frac{s}{n}\right) - \sum_{k=0}^{r} \frac{X^{(k)}(t)}{k!} \left(-\frac{s}{n}\right)^k \right|^2\right\}}\, |dH(s)| \\ &= \int_{-\infty}^{\infty} \psi\left(-\frac{s}{n}\right) \cdot |s|^{r+\alpha} |dH(s)|, \end{aligned} \tag{3.11}$$

where

$$\psi(u) = |u|^{-r-\alpha} \sqrt{E\left\{\left|X(t+u) - \sum_{k=0}^{r} \frac{X^{(k)}(t)}{k!} u^k\right|^2\right\}}$$

for $u \neq 0$ and $\psi(0) = 0$.

Since, by (3.8),

$$\psi(u)=o(1) \qquad \text{as} \quad u\to 0, \tag{3.12}$$

we can choose a positive number δ such that

$$\psi(u)\leqq 1 \qquad \text{for} \quad |u|<\delta. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}
\psi(u) &\leqq |u|^{-r-\alpha}\left[\sqrt{E\{|X(t+u)|^2\}}+\sum_{k=0}^{r}\frac{\sqrt{E\{|X^{(k)}(t)|^2\}}}{k!}|u|^k\right] \\
&\leqq |u|^{-r-\alpha}\left[C(1+|t+u|^{r+\alpha})+\sum_{k=0}^{r}\frac{\sqrt{E\{|X^{(k)}(t)|^2\}}}{k!\,\delta^{r+\alpha-k}}|u|^{r+\alpha}\right] \\
&\leqq C\left[\delta^{-r-\alpha}+\left(1+\frac{|t|}{\delta}\right)^{r+\alpha}\right]+\sum_{k=0}^{r}\frac{\sqrt{E\{|X^{(k)}(t)|^2\}}}{k!\,\delta^{r+\alpha-k}} \\
&\equiv K \qquad \text{for} \quad |u|\geqq\delta,
\end{aligned} \tag{3.14}$$

which implies with (3.13) that

$$\psi\left(-\frac{s}{n}\right)<K+1 \qquad \text{for all } s, \tag{3.15}$$

Next, (3.12) shows that

$$\lim_{n\to\infty}\psi\left(-\frac{s}{n}\right)=0 \qquad \text{for any fixed } s. \tag{3.16}$$

Since K is a constant independent of s and n, it is obtained by (3.6), (3.15), (3.16) and Lebesgue's theorem that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\psi\left(-\frac{s}{n}\right)|s|^{r+\alpha}|dH(s)|=0 \tag{3.17}$$

and so

$$\lim_{n\to\infty} n^{r+\alpha}\sqrt{E\{|I|^2\}}=0, \tag{3.18}$$

which gives (3.9).

From Theorem 2 and Theorem 3, we get the following

THEOREM 4. *Let $f(s)$ and $\phi(s)$ be continuous in* $(-\infty, \infty)$, *$H(s)$ be of bounded variation and $\mathcal{E}(s)$, $-\infty<s<\infty$, be continuous (weakly) stationary stochastic process whose spectral function is $F(\lambda)$. If, in addition to the assumptions* (2.2), (2.9) (2.10) *and* (3.6), *we assume that*

$$|f(s)|\leqq C(1+|s|^{r+\alpha}) \tag{3.19}$$

and

$$|\phi(s)|\leqq C(1+|s|^{r+\alpha}) \tag{3.20}$$

for all s and some positive constant C, then we have

$$E\left\{\left|\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right)dH(s)-\sum_{k=0}^{r}\frac{(-1)^k X^{(k)}(t)}{k!\; n^k}\int_{-\infty}^{\infty} s^k\, dH(s)\right|^2\right\} \tag{3.21}$$
$$=o\left(\frac{1}{n^{2r+2\alpha}}\right) \qquad as \qquad n\to\infty,$$

where $X(s)=f(s)+\phi(s)\mathcal{E}(s)$, $-\infty<s<\infty$.

REMARK 3. The continuity of $f(s)$ and $\phi(s)$ is used to ensure the existence of the integral

$$\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right)dH(s)$$

only.

REMARK 4. Let $\mathcal{E}_\nu(t)$ $(\nu=1, 2, \cdots, N)$ be stationary processes satisfying the conditions similar to the one on $\mathcal{E}(t)$ in Theorem 4 and $\phi_\nu(t)$ $(\nu=1, 2, \cdots, N)$ be numerical valued functions satisfying the conditions similar to the one on $\phi(t)$ in Theorem 4. Then we have (3.21) for

$$X(t)=f(t)+\sum_{\nu=1}^{N}\phi_\nu(t)\mathcal{E}_\nu(t), \quad -\infty<t<\infty.$$

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