FOURIER SERIES X: ROGOSINSKI'S LEMMA

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1. W. W. Rogosinski has proved the following theorem [1]:

THEOREM 1. *If $f(t)$ is continuous at $t = \xi$, then*

$$\frac{1}{2}\{s_n(x_n) + s_n(x_n + \pi/n)\} \to f(\xi), \qquad (n \to \infty) \tag{1}$$

for any sequence (x_n) tending to ξ, where $s_n(t)$ is the n th partial sum of the Fourier series of $f(t)$.

This theorem has many applications.
We shall prove the following

THEOREM 2. *If*

$$\int_0^t (f(x+u) - f(x-u))\,du = o(t), \qquad (t \to 0) \tag{2}$$

uniformly in x in a neighbourhood of a point ξ, then

$$\begin{aligned} &\frac{1}{2}\{s_n(x_n) + s_n(x_n + \pi/n)\} \\ &= \frac{1}{2\pi}\int_{-\pi/n}^{2\pi/n} f(x_n + t)\left(\frac{1}{t} - \frac{1}{t - \pi/n}\right)\sin nt\,dt \\ &\qquad + n\pi\int_0^{\pi/n} (f(x_n + t) + f(x_n - t))c(nt)\sin nt\,dt + o(1) \\ &= \frac{1}{2\pi}\int_{-\pi/n}^{2\pi/n} f(x_n + t)R_n(t)\,dt + o(1), \end{aligned} \tag{3}$$

where[1] $R_n(t) \geqq 0$ *and*

$$c(t) = \sum_{k=1}^{\infty} \frac{1}{(t + (2k-1)\pi)(t + 2k\pi)(t + (2k+1)\pi)}.$$

If $f(t)$ is continuous at $t = \xi$, then, supposing that $f(\xi) = 0$, the right side of (3) tends to zero. Thus (1) holds.

From Theorem 2, we get a sort of converse theorem of Theorem 1; that is,

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Received December 14, 1956.

1) $c(t)$ is continuous and

$$c(0) = \frac{1}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k)(2k+1)} < \frac{1}{2\pi^3}.$$

THEOREM 3. *If $f(t)$ is bounded and* (2) *holds in a neighbourhood of a point ξ, and further if* (1) *holds for any sequence (x_n), tending to ξ, then $f(t)$ is essentially continuous*[2] *at $t=\xi$.*

On the other hand, it is known [2], [3] that if a function $f(x)$, satisfying a certain uniformity condition[3], is continuous at $x=\xi$, then the Fourier series of $f(x)$ converges uniformly at $x=\xi$. Conversely, uniform convergence of the Fourier series of $f(x)$ at $x=\xi$ does not imply the continuity of $f(x)$ at $x=\xi$. For, values of $f(x)$ in a null set do not effect its Fourier series. Then there arises the problem to find conditions for $f(x)$ under which the uniform convergence of its Fourier series at a point implies the essential continuity of $f(x)$ at that point. As an answer to this problem we get the following theorem which is a corollary of Theorem 3.

THEOREM 4. *If $f(x)$ is bounded in a neighbourhood of $x=\xi$ and* (2) *holds uniformly there, and further if the Fourier series of $f(x)$ converges uniformly at $x=\xi$, then $f(x)$ is essentially continuous at $x=\xi$.*

On the other hand, considering the case where $x_n=\pi/n$ in (3), we obtain the following

THEOREM 5. *Suppose that*

$$f(t)=a\psi(t-\xi)+g(t), \tag{4}$$

where $\psi(t)$ is a periodic function with period 2π such that

$$\psi(t)=(\pi-t)/2, \qquad (0<t<2\pi),$$

and where

$$\begin{aligned}&\limsup_{t\downarrow\xi} g(t)=0, &&\liminf_{t\uparrow\xi} g(t)=0,\\ &\liminf_{t\downarrow\xi} g(t)\geqq -a\pi, &&\limsup_{t\uparrow\xi} g(t)\leqq a\pi,\end{aligned} \tag{5}$$

$$\int_0^t |g(\xi+u)|\,du=o(|t|), \tag{6}$$

then the Gibbs phenomenon of the Fourier series of $f(t)$ appears at $t=\xi$. The Gibbs set contains the interval $[a(H+1)\pi/4,\ -a(H+1)\pi/4]$ where

$$H=\frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt=1.17\cdots>1.$$

In this theorem, it is not supposed that the point $t=\xi$ is the simple discontinuity point of $f(t)$. Theorem 5 of this case owes to W. W. Rogosinski

2) $f(t)$ is essentially continuous at a point ξ, if there are no sets E such that for any $\delta>0$, $E\cap(\xi-\delta,\ \xi+\delta)$ is not of zero measure and $f(t)$ does not tend to $f(\xi)$ as t tends to ξ belonging to E.

3) For example, $\int_0^u (f(x+t)-f(x-t))\,dt=o\left(u\Big/\log\frac{1}{u}\right)$ as $u\to 0$, uniformly in x. Cf. [3].

[4].[4]

We can generalize Theorem 5 in the following form.

THEOREM 6. *In Theorem* 5, *if we replace the condition* (6) *by the following condition*:

$$\int_0^t g(\xi + u)du = o(|t|),$$

$$\int_0^t (g(x+u) - g(x-u))du = o(|t|)$$

uniformly for all x in a neighbourhood of ξ, then the Gibbs phenomenon of $f(t)$ appears at $t = \xi$, and the Gibbs set contains the interval $[a(H+1)\pi/4, -a(H+1)\pi/4]$.

Further we prove the following theorem.

THEOREM 7. *Suppose that*

$$f(t) = a\psi(t-\xi) + g(t) + h(t),$$

where $\psi(t)$ is a periodic function with period 2π such that

$$\psi(t) = (\pi - t)/2, \qquad (0 < t < 2\pi),$$

$$\int_0^\pi |g(t+\xi)|t^{-1}dt < \infty \tag{7}$$

and $h(t)$ is of bounded variation and is continuous at ξ, then the Gibbs set of $f(t)$ contains the interval$[a(\pi/2)H, -a(\pi/2)H]$.

More generally, (7) *may be replaced by*

$$\int_0^t g(\xi+u)du = o(t), \qquad \int_{\pi/n}^{\pi} \frac{|g(t) - g(t+\pi/n)|}{t}dt = o(1). \tag{8}$$

THEOREM 8[5]. *Suppose that*

$$f(t) = a\psi(t-\xi) + g(t)$$

where $g(t)$ is odd about $t = \xi$, that is

$$g(\xi - t) = -g(\xi + t)$$

for small t and

$$\int_0^t |g(\xi+u)|du = o(t) \qquad (t > 0), \tag{9}$$

$$\int_0^t |g(\xi+u)|du = o(|t|) \qquad (t < 0), \tag{10}$$

then the Gibbs set of $f(t)$ contains the interval $[a(\pi/2)H, -a(\pi/2)H]$.

We conclude this paper proving the following

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4) This paper has not been available for us, but this result is stated in [5].

5) This is a special case of a theorem of O. Szász [6 Theorem 10].

THEOREM 9. (i) *There is a function which presents Gibbs phenomenon at a point $t = \xi$ and has $t = \xi$ as the second kind discontinuity.* (ii) *There is a function which does not present Gibbs phenomenon at $t = \xi$ and has $t = \xi$ as the second kind discontinuity.*

The first part is almost evident, and in fact follows from Theorems 5 and 6. The second part is proved by constructing an example whose construction is suggested by Theorems 5 − 7.

2. Proof of Theorem 2.[6] We put $\varphi_x(t) = f(x+t) + f(x-t)$ and we can suppose that $\xi = 0$. Then we have

$$\begin{aligned} s_n(x) &= \frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin nt}{t}dt + o(1) \\ &= \frac{1}{\pi}\sum_{k=0}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n} \varphi_x(t)\frac{\sin nt}{t}dt + o(1) \\ &= \frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^k\int_0^{\pi/n}\frac{\varphi_x(t+k\pi/n)}{t+k\pi/n}\sin nt\,dt + o(1). \end{aligned}$$

Accordingly we have

$$\begin{aligned} s_n(x) + s_n(x_n+\pi/n) &= \frac{1}{\pi}\sum_{k=0}^{n-1}\Big\{(-1)^k\int_0^{\pi/n}\frac{\varphi_{x_n}(t+k\pi/n)}{t+k\pi/n}\sin nt\,dt \\ &\qquad + (-1)^k\int_0^{\pi/n}\frac{\varphi_{x_n+\pi/n}(t+k\pi/n)}{t+k\pi/n}\sin nt\,dt\Big\} + o(1) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^k\int_0^{\pi/n}\frac{f(x_n+t+k\pi/n)+f(x_n+t+(k+1)\pi/n)}{t+k\pi/n}\sin nt\,dt \\ &+ \frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^k\int_0^{\pi/n}\frac{f(x_n-t-k\pi/n)+f(x_n-t-(k+1)\pi/n)}{t+k\pi/n}\sin nt\,dt + o(1) \\ &= I + J + o(1). \end{aligned}$$

We shall estimate I. Since J may be quite similarly estimated, we shall omit it. We write

$$\begin{aligned} I &= \frac{1}{\pi}\int_0^{\pi/n}\frac{f(x_n+t)}{t}\sin nt\,dt + \frac{1}{n}\int_0^{\pi/n}\frac{f(x_n+t+\pi/n)}{t(t+\pi/n)}\sin nt\,dt \\ &\quad + \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k\int_0^{\pi/n}\frac{f(x_n+t+(k+1)\pi/n)}{(t+k\pi/n)(t+(k+1)\pi/n)}\sin nt\,dt + o(1) \\ &= I_1 + I_2 + I_3 + o(1). \end{aligned}$$

We can here suppose that n is an odd integer and we put, for the sake of simplicity, $N = (n-1)/2$. Then

$$I_3 = \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k\int_0^{\pi/n}\frac{f(x_n+t+(k+1)\pi/n)}{(t+k\pi/n)(t+(k+1)\pi/n)}\sin nt\,dt$$

6) For the method of proof, see [3].

$$= -\frac{1}{n}\sum_{k=1}^{N}\int_0^{\pi/n}\left\{\frac{f(x_n+t+2k\pi/n)}{(t+(2k-1)\pi/n)(t+2k\pi/n)}\right.$$

$$\left. -\frac{f(x_n+t+(2k+1)\pi/n)}{(t+2k\pi/n)(t+(2k+1)\pi/n)}\right\}\sin nt\,dt$$

$$= -\frac{1}{n}\sum_{k=1}^{N}\int_0^{\pi/n}\frac{f(x_n+t+2k\pi/n)-f(x_n+t+(2k+1)\pi/n)}{(t+(2k-1)\pi/n)(t+2k\pi/n)}\sin nt\,dt$$

$$-\frac{2\pi}{n^2}\sum_{k=1}^{N}\int_0^{\pi/n}\frac{f(x_n+t+(2k+1)\pi/n)}{(t+(2k-1)\pi/n)(t+2k\pi/n)(t+(2k+1)\pi/n)}\sin nt\,dt$$

$$= -I_{31}-I_{32},$$

say. Now, by repeated use of the second mean value theorem,

$$I_{31} = \frac{1}{n}\sum_{k=1}^{N}\frac{n^2}{(2k-1)2k\pi^2}\int_0^{\xi}[f(x_n+t+2k\pi/n)-f(x_n+t+(2k+1)\pi/n]\sin nt\,dt$$

$$= \frac{n}{\pi^2}\sum_{k=1}^{N}\frac{\theta_n}{2k(2k-1)}\int_{\zeta}^{\eta}[f(x_n+t+2k\pi/n)-f(x_n+t+(2k+1)\pi/n)]dt$$

where $0<\zeta<\eta\leqq\xi<\pi/n$ and $0<\theta_n\leqq 1$. Since, by the condition (2),

(11) $$\int_0^t[f(x_n+t+2k\pi/n)-f(x_n+t+(2k+1)\pi/n)]dt = o(1/n)$$

uniformly in k and n, we get $I_{31}=o(1)$.

On the other hand we have, by Abel's lemma,

$$I_{32} = \frac{2\pi}{n^2}\sum_{k=1}^{N}\int_0^{\pi/n}\frac{f(x_n+t+(2k+1)\pi/n)}{(t+(2k-1)\pi/n)(t+2k\pi/n)(t+(2k+1)\pi/n)}\sin nt\,dt$$

$$= \frac{2\pi}{n^2}\int_0^{\pi/n}f(x_n+t+3\pi/n)\sin nt\,dt$$

$$\cdot\left[\sum_{k=1}^{N}\frac{1}{(t+(2k-1)\pi/n)(t+2k\pi/n)(t+(2k+1)\pi/n)}\right]$$

$$-\frac{2\pi}{n^2}\sum_{k=2}^{N}\int_0^{\pi/n}[f(x_n+t+(2k+1)\pi/n)-f(x_n+t+(2k-1)\pi/n)]dt$$

$$\cdot\left[\sum_{j=k}^{N}\frac{\sin nt}{(t+(2j-1)\pi/n)(t+2j\pi/n)(t+(2j+1)\pi/n)}\right]$$

$$= I_{321}-I_{322}.$$

Then we have

$$I_{322} = \frac{2\pi}{n^2}\sum_{k=2}^{N}\sum_{j=k}^{N}\frac{\theta'_n n^3}{(2j-1)2j(2j+1)}$$

$$\cdot\int_{\xi_k}^{\eta_k}[f(x_n+t+(2k+1)\pi/n)-f(x_n+t+(2k-1)\pi/n)]dt,$$

where $0<\xi_k<\eta_k<\pi/n$, $0<\theta'_n\leqq 1$ and hence

$$I_{322} = \frac{1}{n^2}\sum_{k=1}^{N}\sum_{j=k}^{N}\frac{n^3}{j^3}\,o\left(\frac{1}{n}\right) = o\left(\sum_{k=1}^{N}\frac{1}{k^2}\right) = o(1).$$

On the other hand

$$I_{321} = \frac{2\pi}{n^2}\int_0^{\pi/n} f(x_n + t + 3\pi/n) \sin nt\, dt$$

$$\cdot \sum_{k=1}^{N} \frac{n^3}{(nt + (2k-1)\pi)(nt + 2k\pi)(nt + (2k+1)\pi)}$$

$$= 2n\pi\int_0^{\pi/n} f(x_n + t + 3\pi/n)c(nt) \sin nt\, dt$$

$$- 2n\pi\int_0^{\pi/n} f(x_n + t + 3\pi/n) \sin nt\, dt$$

$$\cdot \sum_{k=N+1}^{\infty} \frac{1}{(nt + (2k-1)\pi)(nt + 2k\pi)(nt + (2k+1)\pi)}$$

where the sum in the last term on the right is $O(1/n^2)$ and then the last term is

$$O\Big(\frac{1}{n}\int_0^{\pi/n} |f(x_n + t + 3\pi/n)| \sin nt\, dt\Big)$$

$$= O\Big(\int_0^{\pi/n} t\, |f(x_n + t + 3\pi/n)|\, dt\Big) = o(1).$$

And then

$$I_{321} = 2n\pi\int_0^{\pi/n} f(x_n + t + 3\pi/n)c(nt) \sin nt\, dt + o(1)$$

$$= 2n\pi\int_0^{\pi/n} f(x_n + t)c(nt) \sin nt\, dt + o(1).$$

Summing up the above estimations, we get

$$I = I_1 + I_2 - I_{321} + o(1)$$

$$= \frac{1}{\pi}\int_0^{\pi/n} f(x_n + t)\Big(\frac{\sin nt}{t} - 2n\pi^2 c(nt) \sin nt\Big)dt$$

$$= \frac{1}{\pi}\int_0^{\pi/n} f(x_n + t + \pi/n)\Big(\frac{1}{t} - \frac{1}{t + \pi/n}\Big) \sin nt\, dt + o(1).$$

Similarly we get

$$J = \frac{1}{\pi}\sum_{k=0}^{n-1} (-1)^k \int_0^{\pi/n} \frac{f(x_n - t - k\pi/n) + f(x_n - t - (k-1)\pi/n)}{t + k\pi/n} \sin nt\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi/n} \frac{f(x_n - t + \pi/n)}{t} \sin nt\, dt + \frac{1}{\pi}\int_0^{\pi/n} f(x_n - t)\Big(\frac{1}{t} - \frac{1}{t + \pi/n}\Big) \sin nt\, dt$$

$$+ \frac{1}{n}\sum_{k=1}^{n-1} (-1)^k \int_0^{\pi/n} \frac{f(x_n - t - k\pi/n)}{(t + k\pi/n)(t + (k+1)\pi/n)} \sin nt\, dt + o(1).$$

If we denote the last term by J_3, then

$$J_3 = -\frac{2\pi}{n^2}\sum_{k=1}^{N}\int_0^{\pi/n} \frac{f(x_n - t - 2k\pi/n)}{(t + (2k-1)\pi/n)(t + 2k\pi/n)(t + (2k+1)\pi/n)} \sin nt\, dt + o(1)$$

$$= -\frac{2\pi}{n^2}\int_0^{\pi/n} f(x_n - t - 2\pi/n)\sin nt\,dt$$

$$\cdot \sum_{k=1}^{N} \frac{1}{(t+(2k-1)\pi/n)(t+2k\pi/n)(t+(2k+1)\pi/n)} + o(1)$$

$$= -2n\pi\int_0^{\pi/n} f(x_n - t - 2\pi/n)c(nt)\sin nt\,dt + o(1)$$

$$= -2n\pi\int_0^{\pi/n} f(x_n - t)c(nt)\sin nt\,dt + o(1).$$

Thus we have

$$\frac{1}{2}(s_n(x_n) + s_n(x_n + \pi/n)) = \frac{1}{2}(I + J) + o(1)$$

$$= \frac{1}{2\pi}\int_{\pi/n}^{2\pi/n} f(x_n + t)\left(\frac{1}{t} - \frac{1}{t - \pi/n}\right)\sin nt\,dt$$

$$+ \frac{1}{2\pi}\int_0^{\pi/n} f(x_n + t)\left(\frac{1}{t} - \frac{1}{t - \pi/n} - 2n\pi^2 c(nt)\right)\sin nt\,dt$$

$$+ \frac{1}{2\pi}\int_{-\pi/n}^{0} f(x_n + t)\left(\frac{1}{t} - \frac{1}{t - \pi/n} - 2n\pi^2 c(-nt)\right)\sin nt\,dt$$

$$= \frac{1}{2\pi}\int_{-2\pi/n}^{2\pi/n} f(x_n + t)R_n(t)dt + o(1).$$

This is the required.

3. Proof of Theorem 3. We can suppose that $\xi = 0$ and $f(\xi) = 0$. If the theorem does not hold, then there is a set E of positive outer measure such that for any $\delta > 0$, the set $E \cap (-\delta, \delta)$ is of positive outer measure and $f(t)$ does not tend to $f(0)$ as t tends to zero along E.

We can suppose that E is measurable. For, there is an m, for any n, such that

$$e_n = m^*E_n > 0$$

where

$$E_n = E \cap ((-1/n, -1/m) \cup (1/m, 1/n)) = E \cap I_{m,n}.$$

By Lusin's theorem, $f(t)$ is continuous in $I_{m,n}$ except a measurable set E_n' with measure less than $e_n/2$. Hence

$$m^*(E_n - E_n') > e_n/2.$$

For any x in $E_n - E_n'$ we put

$$E_n(x) = (t; |f(x) - f(t)| < 1/n) \cap I_{m,n},$$

$$F = \bigvee_{x \in E_n - E_n'} (E_n(x) \cap cE_n').$$

Each $E_n(x)$ is open in cE_n' and hence F is also and then is measurable and is of measure $> e_n/2$, since $F \supset E_n - E_n'$. Thus we may suppose that E is measurable.

Further we can suppose that

$$f(x) > \varepsilon > 0 \quad \text{for all } x \text{ in } E.$$

Let x be a density point of E. Then, for any η $(1 > \eta > 0)$ there is a ζ such that

$$\text{meas}\,(E \cap (x - \zeta',\ x + \zeta''))/(\zeta' + \zeta'') > \eta$$

for any $\zeta' < \zeta$, $\zeta'' < \zeta$.

Let $2\pi/n < \zeta$ and $x_n = x$, and let

$$G = E \cap (x_n - 2\pi/n,\ x_n + 2\pi/n).$$

We consider the integral in (3) and write

$$I = \int_{-2\pi/n}^{2\pi/n} f(x_n + t) R_n(t) \sin nt\, dt = \int_G + \int_{cG} = I_1 + I_2$$

where the kernel $R_n(t) \sin nt$ is non-negative. Then we get

$$I_1 \geqq \frac{2}{\pi}\varepsilon \cdot n\,|G| \geqq 8\varepsilon\eta, \qquad |I_2| \leqq M \cdot n\,|E| \leqq 4\pi(1 - \eta)M,$$

M being the bound of $|f(t)|$. If we take $\eta > M/(M + 1/\pi)$, then we have, by (3),

$$\begin{aligned} \frac{1}{2}\{s_n(x_n) + s_n(x_n + \pi/n)\} &> \frac{1}{4\pi} I + o(1) \\ &\geqq \frac{1}{4\pi}(I_1 - |I_2|) + o(1) \geqq \frac{2\varepsilon}{\pi}\Big(\eta - \frac{\pi}{2}M(1 - \eta)\Big) + o(1) \qquad (12) \\ &\geqq \varepsilon\eta/\pi + o(1) > \varepsilon M/(\pi M + 1) + o(1). \end{aligned}$$

Since $x = x_n$ may be taken as near as we please to 0, (12) contradicts (1). Thus the theorem is proved.

4. Proof of Theorem 4. If the Fourier series of $f(x)$ converges uniformly at $x = \xi$, then $s_n(x_n)$ converges to $f(\xi)$ for all (x_n), tending to ξ. Hence (1) holds, and then the assumption of Theorem 2 is satisfied. Thus $f(t)$ is essentially continuous at $t = \xi$.

5. Proof of Theorem 5. We can suppose that $\xi = 0$. Then

$$f(t) = \psi(t) + g(t),$$

and, by the condition (6),

$$G(t) = \int_0^t |g(u)|\, du = o(t).$$

Now

$$\frac{1}{2}(s_n(\pi/n,f)+s_n(2\pi/n,f))$$
$$=\frac{1}{2}(s_n(\pi/n,\psi)+s_n(2\pi/n,\psi))+\frac{1}{2}(s_n(\pi/n,g)+s_n(2\pi/n,g)).$$

As is well known,

$$s_n(\pi/n,\psi)\to\int_0^{\pi}\frac{\sin t}{t}dt=1.851\cdots,$$
$$s_n(2\pi/n,\psi)\to\int_0^{2\pi}\frac{\sin t}{t}dt=1.418\cdots,$$

and hence

$$\frac{1}{2}(s_n(\pi/n,\psi)+s_n(2\pi/n,\psi))\to 1.637\cdots>1.57\cdots=\pi/2.$$

Since there is an x_n $(\pi/n\leqq x_n\leqq 2\pi/n)$ such that

$$\frac{1}{2}(s_n(\pi/n,f)+s_n(2\pi/n,f))=s_n(x_n,f)$$

by the Darboux theorem, if we prove that

$$s_n(\pi/n,g)+s_n(2\pi/n,g)\to 0, \tag{13}$$

then $t=1.637\cdots$ belongs to the Gibbs set, and hence the Gibbs phenomenon appears.

We have

$$s_n(\pi/n,g)+s_n(2\pi/n,g)$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\frac{\sin n(t-\pi/n)}{t-\pi/n}dt+\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\frac{\sin n(t-2\pi/n)}{t-2\pi/n}dt+o(1)$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\Big(\frac{1}{t-2\pi/n}-\frac{1}{t-\pi/n}\Big)\sin nt\,dt+o(1)$$
$$=\frac{1}{n}\int_{-\pi}^{\pi}\frac{g(t)\sin nt}{(t-\pi/n)(t-2\pi/n)}dt+o(1)$$
$$=\frac{1}{n}\Big(\int_{-\pi}^{0}+\int_0^{\pi}\Big)+o(1)=I+J+o(1).$$

We write

$$J=\frac{1}{n}\Big(\int_0^{\pi/n}+\int_{\pi/n}^{3\pi/2n}+\int_{3\pi/2n}^{2\pi/n}+\int_{2\pi/n}^{3\pi/n}+\int_{3\pi/n}^{\pi}\Big)$$
$$=J_1+J_2+J_3+J_4+J_5,$$

where

$$|J_1|\leqq\frac{1}{n}\cdot n\int_0^{\pi/n}|g(t)|\frac{dt}{t-2\pi/n}\leqq\frac{n}{\pi}\int_0^{\pi/n}|g(t)|\,dt=o(1)$$

and similarly $J_2+J_3+J_4=o(1)$. By integration by parts,

$$|J_5| \leq \frac{1}{n}\int_{3\pi/n}^{\pi} \frac{|g(t)|}{t^2} dt = \frac{1}{n}\left(\frac{n}{3\pi}\right)^2 G\left(\frac{3\pi}{n}\right) + \frac{2}{n}\int_{3\pi/n}^{\pi} \frac{G(t)}{t^3} dt = o(1).$$

Since I may be estimated similarly (13) is proved; thus the theorem is proved, except the last sentence.

6. Proof of Theorem 6. From the proof of Theorem 5, we can see that it is sufficient to prove that

$$\frac{1}{n}\int_{-\pi}^{\pi} g(t) \frac{\sin nt}{(t-\pi/n)(t-2\pi/n)} dt = o(1),$$

where

$$\int_0^t g(u)\,du = o(1), \qquad \text{and} \qquad \int_0^t (g(x+u) - g(x-u))\,du = o(t)$$

uniformly in x. We write

$$\frac{1}{n}\int_{-\pi}^{\pi} g(t) \frac{\sin nt}{(t-\pi/n)(t-2\pi/n)} dt = \frac{1}{n}\left(\int_{-\pi}^{0} + \int_0^{\pi}\right) = I + J,$$

$$J = \frac{1}{n}\int_0^{3\pi/2n} + \frac{1}{n}\int_{3\pi/2n}^{3\pi/n} + \frac{1}{n}\int_{3\pi/n}^{\pi} = J_1 + J_2 + J_3.$$

By the second mean value theorem

$$J_1 = \int_{\xi}^{\eta} \frac{g(t)}{t - 2\pi/n} dt = \frac{1}{\eta - 2\pi/n}\int_{\xi'}^{\eta} g(t)\,dt$$

where $0 < \xi < \pi/n < \eta < 3\pi/2n$, $\xi < \xi' < \eta$. Hence $J_1 = o(1)$. Similarly $J_2 = o(1)$,

$$J_3 = \frac{1}{n}\sum_{k=3}^{n-1} (-1)^k \int_0^{\pi/n} \frac{g(t+k\pi/n)\sin nt}{(t+(k-1)\pi/n)(t+(k-2)\pi/n)} dt$$

$$= -\frac{1}{n}\sum_{k=1}^{(n-2)/2}\int_0^{\pi/n}\left[\frac{g(t+(2k+1)\pi/n)}{(t+2k\pi/n)(t+(2k-1)\pi/n)} - \frac{g(t+(2k+2)\pi/n)}{(t+(2k+1)\pi/n)(t+2k\pi/n)}\right] \cdot \sin nt\,dt$$

$$= -\frac{1}{n}\sum_{k=1}^{(n-2)/2}\int_0^{\pi/n} \frac{g(t+(2k+1)\pi/n) - g(t+(2k+2)\pi/n)}{(t+2k\pi/n)(t+(2k-1)\pi/n)} \sin nt\,dt$$

$$-\frac{2\pi}{n^2}\sum_{k=1}^{(n-2)/2}\int_0^{\pi/n} \frac{g(t+(2k+2)\pi/n)}{(t+(2k-1)\pi/n)(t+2k\pi/n)(t+(2k+1)\pi/n)} \sin nt\,dt$$

$$= -J_{31} - J_{32},$$

where

$$J_{31} = \frac{\theta_n}{n}\sum_{k=1}^{(n-2)/2} \frac{1}{(2k\pi/n)((2k-1)\pi/n)} \cdot \int_{\xi_k}^{\eta_k} [g(t+(2k+1)\pi/n) - g(t+(2k+2)\pi/n)]\,dt$$

$$= o\left(\sum_{k=1}^{n} \frac{1}{k^2}\right) = o(1), \qquad (0 < \theta_n < 1,\ 0 < \xi_k < \eta_k < \pi/n),$$

and $J_{32} = o(1)$ by the estimation similar to that of I_{32} in the proof of Theorem 2. Thus $J = o(1)$. Similarly we get $I = o(1)$. Proof of the theorem is now completed.

7. Proof of Theorem 8. Without loss of generality we can suppose that $\xi = 0$. As usual, we denote by $s_n(x, f)$ the n th partial sum of the Fourier series of $f(t)$ at $t = x$. Then it is sufficient to prove that $s_n(\pi/n, g)$ tends to zero as $n \to \infty$, that is

$$I = \int_{-\pi}^{\pi} g(t) \frac{\sin n(t - \pi/n)}{t - \pi/n} dt = o(1), \qquad (n \to \infty). \tag{14}$$

We write

$$I = \int_{-\pi}^{-2\pi/n} + \int_{-2\pi/n}^{2\pi/n} + \int_{2\pi/n}^{\pi} = I_1 + I_2 + I_3.$$

Then, by the condition (9), we have

$$|I_2| \leqq n \int_{-2\pi/n}^{2\pi/n} |g(t)| dt = o(1)$$

and, since $g(t)$ is odd,

$$\begin{aligned}
I_1 + I_3 &= \int_{2\pi/n}^{\pi} g(t) \left[\frac{\sin n(t - \pi/n)}{t - \pi/n} - \frac{\sin n(t + \pi/n)}{t + \pi/n} \right] dt \\
&= -\frac{2\pi}{n} \int_{2\pi/n}^{\pi} g(t) \frac{\sin nt}{(t - \pi/n)(t + \pi/n)} dt \\
&= -2\pi \int_{2\pi/n}^{\pi} \cos nt \, dt \int_t^{\pi} \frac{g(u)}{u^2 - (\pi/n)^2} du \\
&= 2\pi \int_{\pi/n}^{\pi - \pi/n} \cos nt \, dt \int_{t+\pi/n}^{\pi} \frac{g(u)}{u^2 - (\pi/n)^2} du \\
&= \pi \int_{\pi/n}^{2\pi/n} \cos nt \, dt \int_{t+\pi/n}^{\pi} \frac{g(u)}{u^2 - (\pi/n)^2} du \\
&\qquad - \pi \int_{2\pi/n}^{\pi - \pi/n} \cos nt \, dt \int_t^{t+\pi/n} \frac{g(u)}{u^2 - (\pi/n)^2} du \\
&\qquad - \pi \int_{\pi - \pi/n}^{\pi} \cos nt \, dt \int_t^{\pi} \frac{g(u)}{u^2 - (\pi/n)^2} du \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Now

$$\begin{aligned}
|J_2| &\leqq 2\pi \int_{2\pi/n}^{\pi - \pi/n} dt \int_t^{t+\pi/n} \frac{|g(u)|}{u^2} du \\
&\leqq \frac{2\pi^2}{n} \int_{\pi/n}^{\pi} \frac{|g(u)|}{u^2} du \\
&= o(1) + \frac{4\pi^2}{n} \int_{\pi/n}^{\pi} \frac{du}{u^3} \int_0^u |g(t)| dt = o(1)
\end{aligned}$$

and

$$|J_1| + |J_3| \leqq \frac{2\pi}{n} \int_{\pi/n}^{\pi} \frac{|g(u)|}{u^2} du + o(1) = o(1).$$

Thus we get (14), which is the required.

8. **Proof of Theorem 7.** It is sufficient to prove the case that $g(t)$ is even at $t = x$. As in §7, we decompose I into I_1, I_2 and I_3, then $I_2 = o(1)$. But

$$\begin{aligned} I_1 + I_3 &= \int_{2\pi/n}^{\pi} g(t) \left[\frac{\sin n(t - \pi/n)}{t - \pi/n} + \frac{\sin n(t + \pi/n)}{t + \pi/n} \right] dt \\ &= -2 \int_{2\pi/n}^{\pi} \frac{t g(t)}{t^2 - (\pi/n)^2} \sin nt \, dt. \end{aligned}$$

When the condition (7) is satisfied, then the theorem is evident. If the condition (8) is satisfied, we have

$$\begin{aligned} I_1 + I_3 &= 2 \int_{\pi/n}^{\pi - \pi/n} \frac{(t + \pi/n) g(t + \pi/n)}{t(t + 2\pi/n)} \sin nt \, dt \\ &= \int_{\pi/n}^{2\pi/n} \frac{(t + \pi/n) g(t + \pi/n)}{t(t + 2\pi/n)} \sin nt \, dt \\ &\quad - \int_{2\pi/n}^{\pi - \pi/n} \left[\frac{t g(t)}{(t - \pi/n)(t + \pi/n)} - \frac{(t + \pi/n) g(t + \pi/n)}{t(t + 2\pi/n)} \right] \sin nt \, dt \\ &\quad - \int_{\pi - \pi/n}^{\pi} \frac{t g(t)}{t^2 - (\pi/n)^2} \sin nt \, dt \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Now

$$|J_2| \leqq 2 \int_{2\pi/n}^{\pi - \pi/n} \frac{|g(t) - g(t + \pi/n)|}{t} dt + o(1)$$

and $J_1 + J_3 = o(1)$ as in the estimation in §7. Thus the theorem is proved.

9. **Proof of Theorem 9.** Let us define a sequence of integers (n_k) and a sequence of functions $f_k(x)$ by induction. Let $n_1 = 2$, and $f_1(x)$ be a triangular function in $(0, \pi)$ such that

$$f_1(0) = f_1(\pi/2 - \pi/2^2) = f_1(\pi/2 + \pi/2^2) = f_1(\pi) = 0,$$
$$f_1(\pi/2) = 1.$$

If $n_1, \cdots, n_{k-1}$ and $f_1(x), \cdots, f_{k-1}(x)$ are determined, then we define n_k and $f_k(x)$ as follows. Let n_k be an integer $\geqq n_{k-1}^2$ such that

$$|s_n(x, f_{k-1}) - f_{k-1}(x)| \leqq 1/(k-1)^2 \qquad (n \geqq n_k).$$

Further, setting

$$a_k = \pi/n_k - \pi/n_k^2, \qquad b_k = \pi/n_k + \pi/n_k^2,$$
$$\Delta_k = (a_k, b_k),$$

we define $f_k(x)$ such that

$$\begin{aligned} f_k(x) &= \frac{n_k^2}{\pi}(x - a_k) && \text{in} \quad (a_k, \ \pi/n_k), \\ &= \frac{n_k^2}{\pi}(b_k - x) && \text{in} \quad (\pi/n_k, \ b_k), \\ &= 0 && \text{otherwise in} \quad (0, \pi). \end{aligned}$$

We write now

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} f_k(x) && \text{in} \quad (0, \pi), \\ &= -f(-x) && \text{in} \quad (-\pi, 0), \end{aligned}$$

and we shall show that Fourier series of $f(x)$ does not represent the Gibbs phenomenon at $x = 0$, that is,

$$\limsup_{n\to\infty} s_n(x_n) \leqq 1, \qquad \liminf_{n\to\infty} s_n(x_n) \geqq -1$$

for any sequence (x_n), tending to zero. We can here suppose that $x_n > 0$ for all n.

For any n, there is a k such that

$$n_k \leqq n < n_{k+1}.$$

We distinguish two cases.

(i) $0 < x_n \leqq \pi/n$.

$$\begin{aligned} s_n(x_n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin n(t - x_n)}{t - x_n}dt + o(1) \\ &= \frac{1}{\pi}\left(\int_{-\pi/n}^{\pi/n} + \int_{\pi/n}^{\pi} + \int_{-\pi}^{-\pi/n}\right) + o(1) \\ &= I_1 + I_2 + I_3 + o(1). \end{aligned}$$

We have easily

$$|I_1| \leqq \frac{n}{\pi}\int_{-\pi/n}^{\pi/n}|f(t)|\,dt + o(1)$$

by the construction of $f(t)$. Moreover

$$\begin{aligned} I_2 &= \frac{1}{\pi}\sum_{j=1}^{k}\int_{\Delta_j} f_j(t)\frac{\sin n(t - x_n)}{t - x_n}dt \\ &= \frac{1}{\pi}\sum_{j=l}^{k}\int_{\Delta_j} f_j(t)\frac{\sin n(t - x_n)}{t - x_n}dt + o(1) \\ &= \frac{1}{\pi}\sum_{j=l}^{k} I_{2,j} + o(1), \end{aligned}$$

where l may be taken sufficiently large but fixed.

$$\left|\sum_{j=l}^{k-1} I_{2,j}\right| \leqq \sum_{j=l}^{k-1} j^{-2} \leqq l^{-1},$$

and

$$|I_{2,k}| \leqq \frac{n}{\pi} \cdot \frac{\pi}{n_k^2} \leqq 1 \qquad (n \leqq n_k^2)$$

$$\leqq \frac{n_k}{\pi}(1 + o(1))\frac{\pi}{n_k^2} = o(1) \qquad (n > n_k).$$

Similarly $I_3 = o(1)$.

Thus we have

$$|s_n(x_n)| \leqq |I_2| + o(1) \leqq 1 + l^{-1} + o(1),$$

where l is a sufficiently large constant.

(ii) $x_n \geqq \pi/n$.

$$s_n(x_n) = \frac{1}{\pi}\sum_{j=1}^{\infty}\int_{\Delta_j} f(t)\frac{\sin n(t - x_n)}{t - x_n}dt = \sum_{j=1}^{\infty} I_{2,j}$$

$$= \sum_{j=1}^{k-1} I_{2,j} + I_{2,k} + \sum_{j=k+1}^{\infty} I_{2,j} = I' + I_{2,k} + I''.$$

If $x_n \geqq \pi/n_{k-1}$, then $I' \leqq 1 + l^{-1}$, $I'' = o(1)$ and

$$I_{2,k} = \frac{1}{\pi}\int_{\Delta_k} f_k(t)\frac{\sin n(t - x_n)}{t - x_n}dt = \frac{1}{\pi}\,\frac{n_{k-1}}{\pi}\,\frac{\pi}{n_k^2} = o(1).$$

If $\pi/n \leqq x_n \leqq \pi/n_{k-1}$, then we get also

$$\sum_{j=1}^{k-2} I_{2,j} + \sum_{j=k-2}^{\infty} I_{2,j} = o(1).$$

We have

$$I_{2,k} = \frac{n_k^2}{\pi^2}\int_{a_k}^{\pi/n_k}(t - a_k)\frac{\sin n(t - x_n)}{t - x_n}dt + \frac{n_k^2}{\pi^2}\int_{\pi/n_k}^{b_k}(b_k - t)\frac{\sin n(t - x_n)}{t - x_n}dt$$

$$= \frac{n_k^2}{\pi^2}\int_{a_k}^{\pi/n_k}((t - x_n) + (x_n - a_k))\frac{\sin n(t - x_n)}{t - x_n}dt$$

$$+ \frac{n_k^2}{\pi^2}\int_{\pi/n_k}^{b_k}((b_k - x_n) + (t - x_n))\frac{\sin n(t - x_n)}{t - x_n}dt$$

$$= \frac{n_k^2}{\pi^2}\left(\int_{a_k}^{\pi/n_k}\sin n(t - x_n)\,dt - \int_{\pi/n_k}^{b_k}\sin n(t - x_n)\,dt\right)$$

$$+ \frac{n_k^2}{\pi^2}\left((x_n - a_k)\int_{a_k}^{\pi/n_k}\frac{\sin n(t - x_n)}{t - x_n}dt + (b_k - x_n)\int_{\pi/n_k}^{b_k}\frac{\sin n(t - x_n)}{t - x_n}dt\right)$$

$$= I_{2,k,1} + I_{2,k,2}.$$

Now

$$I_{2,k,1} = \frac{n_k^2}{\pi^2 n}\{-(-\cos n(a_k - x_n) + \cos n(\pi/n_k - x_n))$$

$$+ (-\cos n(\pi/n_k - x_n) - \cos n(b_k - x_n))\}$$

$$= \frac{2n_k^2}{\pi^2 n}(\sin n(\pi/n_k - a_k)/2 \cdot \sin n(a_k + \pi/n_k - 2x_n)/2$$

$$- \sin n(b_k - \pi/n_k)/2 \cdot \sin n(b_k + \pi/n_k - 2x_n)/2)$$

$$= \frac{2n_k^2}{\pi^2 n} \sin (n\pi/2n_k^2) \cdot (- \sin n(b_k + \pi/n_k - 2x_n)/2$$

$$- \sin n(a_k + \pi/n_k - 2x_n)/2)$$

$$= - \frac{4n_k^2}{\pi n^2} \sin \frac{n\pi}{2n_k^2} \cdot \sin \frac{n\pi}{4n_k^2} \cdot \cos n\left(\frac{\pi}{n_k} - x_n\right),$$

$$I_{2,k,2} = \frac{n_k^2}{\pi^2}\left\{(x_n - a_k)\int_{n(a_k-x_n)}^{n(\pi/n_k-x_n)} \frac{\sin t}{t}dt + (b_k - x_n)\int_{n(\pi/n_k-x_n)}^{n(b_k-x_n)} \frac{\sin t}{t}dt\right\}.$$

If we take

$$n = n_k^2, \qquad x_n = \pi/n_k,$$

then

$$I_{2,k,1} = -\frac{4}{\pi^2} \sin\frac{\pi}{2} \sin\frac{\pi}{4} = -\frac{2\sqrt{2}}{\pi^2} = -0.285\cdots,$$

$$I_{2,k,2} = \frac{n_k^2}{\pi^2}\left\{\frac{\pi}{n_k^2}\int_{-\pi}^{0} \frac{\sin t}{t}dt + \frac{\pi}{n_k^2}\int_0^{\pi} \frac{\sin t}{t}dt\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}dt = \frac{2}{\pi}\cdot 1.851\cdots = 1.179\cdots,$$

and then

$$I_{2,k} = 1.179\cdots - 0.285\cdots = 0.894\cdots < 1.$$

Let x_n be a point in a neighbourhood of π/n_k. Then

$$\varDelta = \frac{1}{\pi}\int_0^{\pi} f_k(t)\frac{\sin n(t - \pi/n_k)}{t - \pi/n_k}dt - \frac{1}{\pi}\int_0^{\pi} f_k(t)\frac{\sin n(t - x_n)}{t - x_n}dt$$

$$= \frac{1}{\pi}\int_0^{\pi} [f_k(t) - f_k(t - x_n')]\frac{\sin n(t - \pi/n_k)}{t - \pi/n_k}dt$$

where $x_n' = x_n - \pi/n$. If we prove that $\varDelta \geqq 0$, then $I_{2,k} < 1$. We suppose $x_n' > 0$ and put

$$h_k(t) = f_k(t) - f(t - x_n'),$$

then we distinguish three cases.

If $\pi/n_k - \pi/n_k^2 \leqq x_n' \leqq \pi/n_k$, then

$$h_k(t) = \frac{n_k^2}{\pi}(t - a_k) \qquad \text{in} \quad (a_k, a_k + x_n'),$$

$$= \frac{n_k^2}{\pi}x_n' \qquad \text{in} \quad (a_k + x_n', \pi/n_k),$$

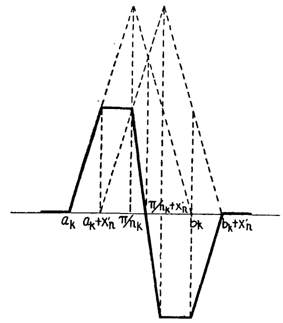
$$= \frac{n_k^2}{\pi}(t - (\pi/n_k + x_n')/2)$$

$$\text{in} \quad (\pi/n_k, \ \pi/n_k + x_n'),$$

$$= -\frac{n_k^2}{\pi} x_n' \qquad \text{in} \quad (\pi/n_k + x_n', b_k),$$

$$= -\frac{n_k^2}{\pi}(b_k - t) \qquad \text{in} \quad (b_k, b_k + x_n'),$$

$$= 0 \qquad \text{otherwise.}$$



Let us prove that

$$\frac{1}{\pi}\int_0^{\pi} h_k(t)\frac{\sin n(t - \pi/n_k)}{t - \pi/n_k}dt \geqq 0. \tag{15}$$

The integral, being considered as a function of θ,

$$\int_{\theta}^{A+\theta} \frac{\sin t}{t} dt$$

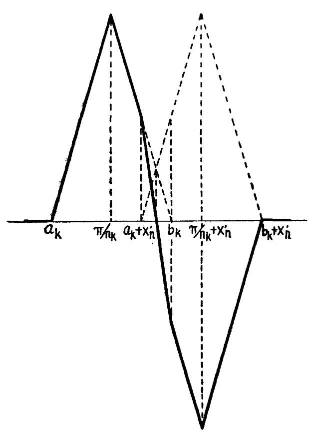
is maximum when $\theta = 0$, for any positive number A. Hence the sum of the integral of (15) on the intervals $(a_k + x_n', \pi/n_k)$ and $(\pi/n_k + x_n', b_k)$ is non-negative. Further, by the second mean value theorem, the sum of the integral of (15) on the intervals $(a_k, a_k + x_n')$ and $(b_k, b_k + x_n')$ is also non-negative. In order to prove that the integral in the remaining interval $(\pi/n_k, \pi/n_k + x_n')$ is non-negative, it is sufficient to show that

$$\int_0^{2na} (a - t)\frac{\sin nt}{t} dt \geqq 0$$

for $a = (x_n' + \pi/n_k)/2$. The left side integral is

$$a\int_0^{2na} \frac{\sin t}{t} dt - \frac{1 - \cos 2an}{n}$$

which is non-negative for larg n and small a. In the case $\pi/n_k \leqq x_n \leqq \pi/n_k + \pi/n_k^2$, we have



$$h_k(t) = \frac{n_k^2}{\pi}(t - a_k) \qquad \text{in} \quad (a_k, \pi/n_k),$$

$$= \frac{n_k^2}{\pi}(b_k - t) \qquad \text{in} \quad (\pi/n_k, \ a_k + x_n'),$$

$$= \frac{n_k^2}{\pi}((a_k + b_k + x_n')/2 - t)$$

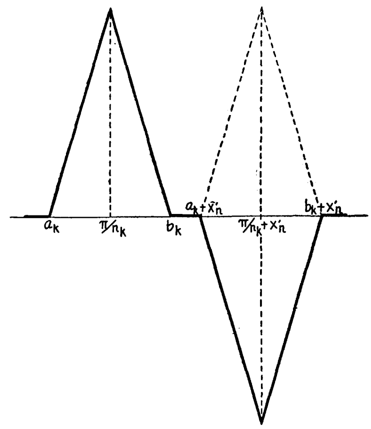
$$\text{in} \quad (a_k + x_n', b_k),$$

$$= -\frac{n_k^2}{\pi}(t - a_k - x_n')$$

$$\text{in} \quad (b_k, \pi/n_k + x_n'),$$

$$= -\frac{n_k^2}{\pi}(t - b_k - x_n')$$

$$\text{in} \quad (\pi/n_k + x_n',\ b_k + x_n').$$



In this case the estimation is similar and we get $I_{2,k} < 1$.

Finally, in the case $x_n' > \pi/n_k + \pi/n_k^2$, $h_k(t)$ becomes the function as in the graph. The estimation of $I_{2,k}$ becomes easier. Thus in the case $x_n' > 0$, we get also the inequality

$$I_{2,k} < 1.$$

If x_n lies in a neighbourhood of π/n_k, then we can easily see that

$$I_{2,k-1} + I_{2,k-1} = o(1).$$

If x_n lies in a neighbourhood of π/n_{k-1} or π/n_{k+1}, then we get

$$I_{2,k-1} < 1, \qquad I_{2,k} + I_{2,k-1} = o(1)$$

or

$$I_{2,k+1} < 1, \qquad I_{2,k-1} + I_{2,k} = o(1),$$

respectively.

We have proved that $I_2 \leqq 1$. Since we can easily see that $I_3 = o(1)$, we have thus proved that

$$\limsup_{n\to\infty} s_n(x_n) \leqq 1.$$

Similarly

$$\liminf_{n\to\infty} s_n(x_n) \geqq -1.$$

This completes the proof of our theorem.

References

[1] G. H. Hardy and W. W. Rogosinski, Fourier series. 1943.

[2] S. Izumi and G. Sunouchi, Notes on Fourier analysis (XLVIII): Uniform convergence of Fourier series. Tôhoku Math. Journ. **3** (1951), 298—305.

[3] M. Satô, Uniform convergence of Fourier series I—VI. Proc. Japan Acad. **30—32** (1954—1956), 528—531, 698—701, 809—813; 261—263, 600—605; 99—104.

[4] W. W. Rogosinski, Schriften der Königsberger gelehrten Gesellschaft **3** (1926).

[5] G. H. Hardy and W. W. Rogosinski, Notes on Fourier series (II): On the Gibbs phenomenon. Journ. London Math. Soc. **18** (1943),

[6] O. Szász, On some trigonometric summability methods and Gibbs' phenomenon. Trans. Amer. Math. Soc. **54** (1943), 483—497.

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