ON A RENEWAL THEOREM

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1. Introduction. Let $X_i\,(i = 1, 2, \dots)$ be independent random variables having the mean value m, and pnt $S_n = \sum_1^n X_i$. So-called renewal theorem which is of the type as

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} \mathrm{P}(x < S_n \leqq x + h) = \frac{h}{m} \tag{1.1}$$

was proved by Feller [6, 7], Täcklind [12], Doob [5], Blackwell [1, 2], Chung-Pollard [3], Cox [4], Smith [4, 11], Karlin [8], etc., in the case X_i identically distributed under the various conditions.

Recently, Prof. T. Kawata [9] showed (1.1) replacing $\lim_{x\to\infty}$ by $\lim_{\xi\to\infty} \frac{1}{\xi} \cdot \int_{-\infty}^{\xi} \cdots\, dx$ and m by $\lim_{n\to\infty} \sum_1^n \mathrm{E}(X_i)/n$ which is assumed to exist in the case, where X_i are not necessarily identical. In this paper, roughly speaking, we shall discuss the limit of $\sum_{n=1}^{\infty} (n - x/m)\, \mathrm{P}(x < S_n \leqq x + h)$ in the same case as [9] by the method analogous to it.

Now, Prof. Kawata [10] discussed the convergence of

$$\sum_{n=0}^{\infty} n \{\mathrm{P}(x < S_n \leqq x + h) - \mathrm{P}(x < S_{n+1} \leqq x + h)\}. \tag{1.2}$$

Of course $\sum_{n=1}^{\infty} n\, \mathrm{P}(x < \mathrm{S}_n \leqq x + h)$ diverges. Our theorem will show the appearance of its divergence in a sense.

For convenience's sake, we shall devote sections **2** and **3** for preparations.

2. Notations and assumptions. Let $X_i\,(i = 1, 2, \dots)$ be independent random variables having the distributions $F_i(x)$, and let us put

$$S_n = \sum_{i=1}^{n} X_i.$$

Suppose that

$$0 < \mathrm{E}(X_i) = m_i < \infty, \tag{2.1}$$

$$\mathrm{E}(X_i^2) = v_i' < \infty, \tag{2.2}$$

$$M_n = \frac{1}{n} \sum_{i=1}^{n} m_i \to m \qquad (n \to \infty), \tag{2.3}$$

$$V_n' = \frac{1}{n} \sum_{i=1}^{n} v_i' \to v' \qquad (n \to \infty), \tag{2.4}$$

Received October 11, 1956.

and

$$V_n = \frac{1}{n}\sum_{i=1}^{n} v_i \to v \qquad (n \to \infty), \tag{2.5}$$

where

$$\mathrm{E}(X_i - m_i)^2 = v'_i - m_i^2 = v_i. \tag{2.6}$$

Moreover

$$\lim_{A\to\infty}\int_A^\infty x^2 dF_i(x) = 0, \tag{2.7}$$

uniformly with respect to i and there exists an s_0 such that

$$\lim_{A\to\infty}\int_{-\infty}^{-A} e^{-sx} dF_i(x) = 0, \tag{2.8}$$

uniformly with repect to i for $0 < s \leqq s_0$.

The distribution function of S_n will be denoted by $\sigma_n(x)$, i.e.,

$$\sigma_n(x) = F_1(x) * F_2(x) * \cdots * F_n(x). \tag{2.9}$$

Furthermore, we shall put

$$f_i(s) = \int_{-\infty}^{\infty} e^{-sx} dF_i(x), \tag{2.10}$$

$$\varphi_n(s) = \int_{-\infty}^{\infty} e^{-sx} d\sigma_n(x) = \prod_{i=1}^{n} f_i(s), \tag{2.11}$$

and

$$\psi_n(s) = n\varphi_n(s) + \frac{1}{M_n}\varphi'_n(s). \tag{2.12}$$

3. Lemmas. Lemma 1 was given in [9] and we shall omit the proof.

LEMMA 1. *Let* $g(t) \geqq 0$,

$$\int_{-\infty}^{0} e^{-st} g(t)\,dt < \infty \qquad \textit{for} \quad 0 \leqq s \leqq s_0, \tag{3.1}$$

and

$$\int_{-\infty}^{\infty} e^{-st} g(t)\,dt \sim \frac{A}{s^\gamma} \qquad \textit{as} \quad s \to 0, \tag{3.2}$$

for some positive $\gamma > 0$, *then*

$$\int_{-\infty}^{t} g(u)\,du \sim \frac{At^\gamma}{\Gamma(\gamma+1)} \qquad \textit{as} \quad t \to \infty. \tag{3.3}$$

LEMMA 2. *Under the condition* (2.3) $\sim$ (2.8), *there exist the numbers* s_5, N, *for arbitrary small* $\varepsilon > 0$, *such that*

(3.4) $\{(1-\varepsilon)\varphi(s)\}^n \leqq \varphi_n(s) \leqq \{(1+\varepsilon)\varphi(s)\}^n$ *for* $0 \leqq s \leqq s_5,\ n > N,$

where $\varphi(s)$ *is a bilateral Laplace transform* $\int_{-\infty}^{\infty} e^{-sx} d\sigma(x)$ *of a suitable distribution function* $\sigma(x)$.

Proof. From (2.7) and (2.9), there exists a constant C_1 independent of i such that

$$\int_{-\infty}^{\infty} x^2 dF_i(x) < C_1. \tag{3.5}$$

Let ε be any given positive number. Take A so large that

$$\int_A^{\infty} x^2 dF_i(x) < \varepsilon, \quad \int_{-\infty}^{-A} x^2 e^{-s_0 x} dF_i(x) < \varepsilon. \tag{3.6}$$

Now we determine s_1 so that

$$\int_{-\infty}^{-A} x^2 e^{-sx} dF_i(x) < \int_{-\infty}^{-A} e^{-s_0 x} dF_i(x) < \varepsilon \qquad \text{for} \quad 0 \leqq s \leqq s_1 < s_0. \tag{3.7}$$

Further, we take s_2 so that

$$|1 - e^{sA}| < \varepsilon \qquad \text{for} \quad 0 \leqq s \leqq s_2 \leqq s_1. \tag{3.8}$$

Then we have

$$\begin{aligned} f_i(s) &= f_i(0) + s f_i'(0) + \frac{s^2}{2} f_i''(\theta s) \\ &= 1 - s m_i + \frac{s^2}{2} v_i' + \frac{s^2}{2} [f_i''(\theta s) - f_i''(0)], \qquad 0 < \theta < 1, \end{aligned}$$

and

$$\begin{aligned} |f_i''(\theta s) - f_i''(0)| &\leqq \left| \left(\int_{|x|>A} + \int_{|x|\leqq A} \right) (e^{-\theta s x} - 1)\, x^2 dF_i(x) \right| \\ &\leqq \int_{x>A} x^2 dF_i(x) + \int_{x<-A} x^2 e^{-sx} dF_i(x) + \int_{|x|\leqq A} (e^{sA} - 1) x^2 dF_i(x) \\ &< \varepsilon + \varepsilon + (e^{sA} - 1) \int_{-\infty}^{\infty} x^2 dF_i(x) < \varepsilon(2 + C_1). \end{aligned}$$

Hence

$$\begin{aligned} f_i(s) &= 1 - s m_i + \frac{s^2}{2} v_i' + \frac{s^2}{2} \eta_i, \\ |\eta_i| &< \varepsilon(2 + C_1) \qquad\qquad \text{for} \quad 0 \leqq s \leqq s_2 \end{aligned} \tag{3.9}$$

uniformly with respect to i. Write

$$\left\{ \begin{aligned} \log f_i(s) &= \log\left(1 - s m_i + \frac{s^2}{2} v_i' + \frac{s^2}{2} \eta_i\right) \\ &= -s m_i + \frac{s^2}{2} v_i' + \frac{s^2}{2} \eta_i - \frac{1}{2}\left(s m_i - \frac{s^2}{2} v_i' - \frac{s^2}{2} \eta_i\right)^2 - \cdots \end{aligned} \right. \tag{3.10}$$

$$= -sm_i + \frac{s^2}{2}(v_i' - m_i^2) + \frac{s^2}{2}\xi_i.$$

Then there exists an s_3 such that

$$|\xi_i| < \varepsilon \qquad \text{for} \quad 0 \leqq s \leqq s_3 \quad \text{uniformly for } i, \tag{3.11}$$

noticing that m_i, v_i are uniformly bounded.

Now we have

$$\begin{aligned} \log \varphi_n(s) = \sum_{i=1}^{n} \log f_i(s) &= -snM_n + \frac{ns^2}{2} V_n + \frac{s^2}{2}\sum_{i=1}^{n}\xi_i \\ &= n\left(-sM_n + \frac{s^2}{2} V_n + \frac{s^2}{2}\Xi_n\right). \end{aligned} \tag{3.12}$$

Let $\sigma(x)$ be a distribution function with mean m and variance v; m, v being those defined in (2.3) and (2.5) and let its bilateral Laplace transform be $\varphi(x) = \int_{-\infty}^{\infty} e^{-sx} d\sigma(x)$ which is assumed to exist. Then we have

$$\begin{aligned} \log \varphi^n(s) &= n \log \varphi(s) \\ &= n\left[\log(1 + s\varphi'(0)) + \frac{s^2}{2}\varphi''(0) + \frac{s^2}{2}[\varphi''(\theta s) - \varphi''(0)]\right] \\ &= n\left(-sm + \frac{s^2}{2} v + \frac{s^2}{2}\delta\right), \\ &\qquad |\delta| < \varepsilon \qquad \text{for} \quad 0 \leqq s \leqq s_4. \end{aligned} \tag{3.13}$$

Hence, we have

$$\begin{aligned} |\log \varphi_n(s) - \log \varphi^n(s)| &= n\left| -s(M_n - m) + \frac{s^2}{2}(V_n - v) + \frac{s^2}{2}(\Xi_n - \delta)\right| \\ &\leqq n\varepsilon \log(1 + s) \qquad \text{for} \quad n > N, \end{aligned} \tag{3.14}$$

and there exists an s_5 such as

$$(1 + s)^{\varepsilon} < \frac{1}{1 - \varepsilon} \qquad \text{for} \quad 0 \leqq s \leqq s_5.$$

This implies (3.4) directly.

LEMMA 3. *Under the conditions* (2.1), (2.3), (2.8) *and that*

$$\lim_{A \to \infty} \int_{A}^{\infty} x dF_i(x) = 0 \tag{3.15}$$

(*uniformly with respect to i*) *the following relatien holds*:

$$\lim_{s \to 0} s \sum_{n=1}^{\infty} \frac{1}{M_n} \varphi_n(s) = \frac{1}{m^2}. \tag{3.16}$$

Proof. Since $M_n \to m \; (n \to \infty)$, $C_2 > M_n > C_3 > 0$, using the fact that for given $\varepsilon > 0$, there exist an N and an s_6 such that

$$\varphi_n(s) = e^{-su(m+\delta_n+\rho_n)},$$

$$(3.17) \qquad |\delta_n| < \varepsilon \qquad \text{for} \quad n > N,$$

$$|\rho_n| < \varepsilon \qquad 0 \leqq s \leqq s_6,$$

which were given in Lemma 2 of [9], we have

$$(3.18) \qquad \begin{aligned} s\sum_{n=1}^{\infty}\frac{\varphi_n(s)}{M_n} &= s\sum_{n=1}^{N}\frac{\varphi_n(s)}{M_n} + s\sum_{n=N+1}^{\infty}\frac{\varphi_n(s)}{M_n} \\ &\leqq \frac{s}{C_3}N + s\cdot\frac{1}{m-\varepsilon}\cdot\frac{1}{(m-2\varepsilon)s}. \end{aligned}$$

Thus noticing that ε is arbitrary, we get

$$(3.19) \qquad \limsup_{s\to 0}\ s\sum_{n=1}^{\infty}\varphi_n(s) \leqq \frac{1}{m^2}.$$

Similarly since

$$s\sum_{n=1}^{\infty}\frac{\varphi_n(s)}{M_n} \geqq s\sum_{n=N}^{\infty}\frac{\varphi_n(s)}{M_n} \geqq \frac{s}{m+\varepsilon}\left(\frac{1}{(m+2\varepsilon)s} - N\right),$$

we get

$$(3.20) \qquad \liminf_{s\to 0}\ s\sum_{n=1}^{\infty}\frac{\varphi_n(s)}{M_n} \geqq \frac{1}{m^2},$$

which, with (3.19), proves the lemma.

LEMMA 4. *Under the conditions* (2.3) $\sim$ (2.8),

$$(3.21) \qquad \lim_{s\to 0}\ s\sum_{n=1}^{\infty}\psi_n(s) = \frac{v'}{m^3},$$

$\psi_n(s)$ *being the one in* (2.12).

Proof. Since, using (3.10),

$$\begin{aligned} \varphi_n'(s) &= \varphi_n(s)\sum_{i=1}^{n}(\log f_i(s))' \\ &= n\varphi_n(s)[-M_n + sV_n + s\Xi_n], \end{aligned}$$

(3.4) will give the following relation:

$$(3.22) \qquad n(\varphi(s)-\varepsilon)^{n-1}(\varphi'(s)-\varepsilon) \leqq \varphi_n'(s) \leqq n(\varphi(s)+\varepsilon)^{n-1}(\varphi'(s)+\varepsilon),$$

$$\text{for} \quad 0 \leqq s \leqq s_5,\ n > N_1.$$

And by Lemma 2, for $0 \leqq s \leqq s_5$, $n > N_2$, (3.4) holds, then we have

$$(3.23) \qquad \psi_n(s) \leqq \frac{n}{M_n}\{M_n(\varphi(s)+\varepsilon) + \varphi'(s)+\varepsilon\}\{\varphi(s)+\varepsilon\}^{n-1}.$$

On the other hand, there exist N_3 and N_4 such that

$$(3.24)\qquad m-\varepsilon < M_n < m+\varepsilon \qquad \text{for} \quad n > N_3,$$

$$(3.25)\qquad n-\frac{x}{m-\varepsilon} > 0 \qquad \text{for} \quad n > N_4.$$

Putting as

$$(3.26)\qquad N = \max(N_1, N_2, N_3, N_4),$$

for $n > N$, we have

$$(3.27)\qquad \begin{aligned} s\sum_{n=1}^{\infty}\psi_n(s) &= s\sum_{n=1}^{N}\psi_n(s) + s\sum_{n=N+1}^{\infty}\psi_n(s) \\ &\leqq \frac{sN(N+1)}{2}C_4 \\ &\quad + \frac{s^2}{m-\varepsilon}\,\frac{(m+\varepsilon)(\varphi(s)+\varepsilon)+(\varphi'(s)+\varepsilon)}{s}\cdot\frac{1}{(1-\varphi(s)-\varepsilon)^2}, \end{aligned}$$

where $C_4 \geqq |\psi_n(s)|$ $(n \leqq N)$. Now,

$$(3.28)\qquad \lim_{s\to 0}\frac{m\varphi(s)+\varphi'(s)}{s} = \lim_{s\to 0}\frac{\varphi'(s)-\varphi'(0)}{s} = \varphi''(0) = v'.$$

And since ε is arbitrary, we get

$$(3.29)\qquad \limsup_{s\to 0}\, s\sum_{n=1}^{\infty}\psi_n(s) \leqq \frac{v'}{m^3}.$$

Similarly for $n > N$,

$$\begin{aligned} s\sum_{n=1}^{\infty}\psi_n(s) &= s\sum_{n=N+1}^{\infty}\psi_n(s) + s\sum_{n=1}^{N}\psi_n(s) \\ &\geqq s^2\frac{1}{m+\varepsilon}\cdot\frac{(m-\varepsilon)(\varphi(s)-s)+\varphi'(s)-\varepsilon}{s}\cdot\frac{1}{(1-\varphi(s)+\varepsilon)^2} \\ &\quad - \frac{sN(N+1)}{2}C_4 \end{aligned}$$

and therefore

$$(3.30)\qquad \liminf_{s\to 0}\, s\sum_{n=1}^{\infty}\varphi_n(s) \geqq \frac{v'}{m^3}.$$

From (3.29) and (3.30) we have (3.21).

4. Theorem. Using these lemmas we shall prove the following

THEOREM. *If* (2.3) $\sim$ (2.8) *hold, then*

$$(4.1)\qquad \lim_{X\to\infty}\frac{1}{X}\int_{-\infty}^{X}dx\sum_{n=1}^{\infty}\left(n-\frac{x}{M_n}\right)\mathrm{P}(x < S_n \leqq x+h) = \frac{h}{m^2}\left(\frac{v'}{m}-\frac{h}{2}\right).$$

Proof. We shall put

$$G_N = \sum_{n=1}^{N} \left(n - \frac{x}{M_n}\right) \mathrm{P}(x < S_n \leqq x + h) \tag{4.2}$$

and form

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-sx} dG_N(x) &= \sum_{n=1}^{N} \left[\int_{-\infty}^{\infty} e^{-sx} d\{\sigma_n(x+h) - \sigma_n(x)\} \left(n - \frac{x}{M_n}\right)\right] \\
&= \sum_{n=1}^{N} \left[\int_{-\infty}^{\infty} \left(n - \frac{x}{M_n}\right) e^{-sx} d\sigma_n(x+h) - \int_{-\infty}^{\infty} \left(n - \frac{x}{M_n}\right) e^{-sx} d\sigma_n(x)\right. \\
&\quad \left. - \frac{1}{M_n} \int_{-\infty}^{\infty} e^{-sx} \sigma_n(x+h)\, dx + \frac{1}{M_n} \int_{-\infty}^{\infty} e^{-sx} \sigma_n(x)\, dx\right] \\
&= \sum_{n=1}^{N} \left[(e^{sh} - 1) \int_{-\infty}^{\infty} e^{-sx} \left(n - \frac{x}{M_n}\right) d\sigma_n(x) - \frac{(e^{sh} - 1)}{M_n} \int_{-\infty}^{\infty} e^{-sx} \sigma_n(x)\, dx\right. \\
&\quad \left. + \frac{h}{M_n} e^{sh} \int_{-\infty}^{\infty} e^{-sx} d\sigma_n(x)\right].
\end{aligned} \tag{4.3}$$

Now, by integration by parts

$$\int_{-\infty}^{\infty} e^{-sx} \sigma_n(x)\, dx = -\frac{1}{s} \Big[e^{-sx} \sigma_n(x)\Big]_{-\infty}^{\infty} + \frac{1}{s} \int_{-\infty}^{\infty} e^{-sx} d\sigma_n(x), \tag{4.4}$$

and the first term on the right hand of (4.4) is 0 according to (2.8). For,

$$\int_{-\infty}^{-A} e^{-sx} d\sigma_n(x) \to 0 \qquad (A \to \infty),$$

and

$$\int_{-\infty}^{-A} e^{-sx} d\sigma_n(x) = \Big[e^{-sx} \sigma_n(x)\Big]_{-\infty}^{-A} + s \int_{-\infty}^{-A} e^{-sx} \sigma_n(x)\, dx,$$

where the both terms on right hand are non-negative. And hence

$$\lim_{A \to \infty} e^{sA} \sigma_n(-A) = 0. \tag{4.5}$$

Therefore (4.3) implies

$$\int_{-\infty}^{\infty} e^{-sx} dG_N(x) = \sum_{n=1}^{N} \left[(e^{sh} - 1)\left(\psi_n(s) - \frac{1}{M_n s} \varphi_n(s)\right) + \frac{h}{M_n} e^{sh} \varphi_n(s)\right]. \tag{4.6}$$

Since $\sum \psi_n(s)$ and $\sum \varphi_n(s)$ are convergent by Lemma 3 and Lemma 4,

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} e^{-sx} dG_N(x) \tag{4.7}$$

exists, and we have

$$\begin{aligned}
\lim_{N \to \infty} \int_{-\infty}^{\infty} e^{-sx} dG_N(x) &= (e^{sh} - 1) \sum_{n=1}^{\infty} \psi_n(s) + \frac{1 - e^{sh} + he^{sh}}{s} \sum_{n=1}^{\infty} \frac{\varphi_n(s)}{M_n} \\
&\sim h \cdot \frac{v'}{m^3} - \frac{h^2}{2m^2} \qquad (s \to 0).
\end{aligned} \tag{4.8}$$

On the other hand,

$$
(4.9)\quad
\begin{aligned}
\lim_{x\to-\infty} e^{-sx}G_N(x) &= \lim_{x\to-\infty} e^{-sx}\sum_{n=1}^{N}\left(n-\frac{x}{M_n}\right)\mathrm{P}(x<S_n\leqq x+h)\\
&= \lim_{x\to-\infty} e^{-sx}\sum_{n=1}^{N}\left(n-\frac{x}{M_n}\right)\mathrm{P}(S_n\leqq x+h)\\
&\quad -\lim_{x\to-\infty} e^{-sx}\sum_{n=1}^{N}\left(n-\frac{x}{M_n}\right)\mathrm{P}(S_n\leqq x)\\
&= \lim_{x\to-\infty}(e^{sh}-1)e^{-sx}K_N(x)+\frac{h}{M_n}\lim_{x\to-\infty}(e^{-sx}H_N(x))
\end{aligned}
$$

holds, where we denote

$$K_N(x)=\sum_{n=1}^{N}\left(n-\frac{x}{M_n}\right)\mathrm{P}(S_n\leqq x),$$

$$H_N(x)=\sum_{n=1}^{N}\mathrm{P}(S_n\leqq x).$$

In the proof of theorem in [9],

$$(4.10)\qquad \lim_{x\to-\infty} e^{-sx}H_N(x)=0$$

was showed. So we shall show a similar relation concerning $K_N(x)$. From Lemma 4 $\sum\psi_n(s)$ converges, so we can put as

$$\psi_n(s)<C_4,$$

and get

$$\int_{-\infty}^{\infty} e^{-sx}dK_N(x)=\sum_{n=1}^{N}\psi_n(s)<NC_4.$$

By an argument analogus to [9], we have

$$(4.11)\qquad \lim_{x\to-\infty} e^{-sx}K_N(x)=0.$$

Combining (4.7), (4.10) and (4.11) we get

$$(4.12)\qquad \lim_{x\to-\infty} e^{-sx}G_N(x)=0.$$

And then

$$(4.13)\qquad \int_{-\infty}^{\infty} e^{-sx}dG_N(x)=s\int_{-\infty}^{\infty} e^{-sx}G_N(x)\,dx.$$

Since $G_N(x)$ increases as $N\to\infty$ and tends to a non-decreasing function, the existence of the limit (4.7), together with (4.13), shows that

$$(4.14)\qquad \lim_{N\to\infty}\int_{-\infty}^{\infty} e^{-sx}G_N(x)\,dx=\int_{-\infty}^{\infty} e^{-sx}G(x)\,dx$$

exists for $0\leqq s\leqq s_6$, and

$$(4.15)\qquad s\int_{-\infty}^{\infty} e^{-sx}G(x)\,dx=\int_{-\infty}^{\infty} e^{-sx}dG(x)$$

exists. Combining (4.8) and (4.15) we have

$$s\int_{-\infty}^{\infty} e^{-sx} G(x)\,dx \sim \frac{hv'}{m^3} - \frac{h^2}{2m^2}. \tag{4.16}$$

Thus by Lemma 1 we get

$$\int_{-\infty}^{X} G(x)\,dx \sim X\cdot\frac{h}{m^2}\left(\frac{v'}{m} - \frac{h}{2}\right)$$

which proves the theorem.

COROLLARY. *If the conditions in* **2** *are satisfied,*

$$\frac{1}{X}\int_{-\infty}^{X} dx \sum_{n=1}^{\infty} n\mathrm{P}(x < S_n \leqq x+h) \sim \frac{X\cdot h}{2m^2} \qquad (X\to\infty). \tag{4.17}$$

Proof. Since by Theorem in [9],

$$\frac{1}{X}\int_{-\infty}^{X} dx \sum_{n=1}^{\infty} \frac{x}{M_n}\mathrm{P}(x < S_n \leqq x+h) \sim \frac{X\cdot h}{2m^2}, \tag{4.18}$$

(4.17) is immediate from (4.1).

In conclusion, I express my sincerest thanks to Professor T. Kawata who has suggested this investigation and given valuable advices.

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