ON A CERTAIN QUEUING SYSTEM

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1. Introduction. Recently Professor T. Kawata [1] and Professor T. Homma [2] have discussed on some queuing problem with single server in which a newly arrived customer joined with specified probability p_i when he had found i customers waiting or being served. We will make further discussions on their queuing theory. A queuing system is considered to be specified when we know (1) the input, (2) the queue-discipline and (3) the service mechanism. Throughout this paper, we will assume that (1) the interarrival times of successive customers are distributed by a negative exponential law with a parameter $1/\lambda$, (2) the first join will be first served and (3) the distribution of service times is also a negative exponential law.
To analyse the queuing system, it is very available to use the concept of imbedding Markov chain introduced by D. G. Kendall [3], [4]. Considering discrete time points (epochs) at which services start, the number of customers at successive epochs constitutes a simple Markov chain and the queue becomes a birth process in any interval between two successive epochs.
In this paper, first, the transition probability of the simple Markov chain will be calculated and next some characteristics of the birth process will be investigated and further discussions will be done.

2. The transition probability. To begin with, we will restate precisely three assumptions on the queue.

(1) The interarrival times of successive customers are distributed by a negative exponential law with parameter $1/\lambda$, which is equivalent to that the probability that n customers arrive in any time interval t is

$$\frac{e^{-\lambda t}(\lambda t)^n}{n!}. \tag{2.1}$$

(2) The probability that a newly arrived customer joins in the queue is p_i when he has found i customers waiting or being served already and the customers joining in it do not leave without receiving their services. Here, p_i's satisfy the following condition:

$$p_0 = 1. \tag{2.2}$$

(3) The distribution of service times $B(v)$ follows a negative exponential law with a parameter b, that is

Received August 15, 1956.

$$B(v) = be^{-bv}, \tag{2.3}$$

the mean value being $1/b$.

In these assumptions, condition (2.2) means that any customer receives his service always if no one is waiting when he arrives.

Now, we shall consider the transition probability of our Markov chain $p(i,j) = \Pr(x_{t+1} = j \mid x_t = i)$ where x_t be the number of waiting customers at the epoch t $(t = 1, 2, \cdots)$. To calculate this probability, it is necessary to consider a birth process constituted in an interval between successive epochs. In its interval, let us denote $P_m(t, n)$ as the probability that $n + m$ customers join in the queue at the time t, when n customers joined in it at the time $t = 0$. $P_m(t, n)$ must satisfy the following initial conditions:

$$P_0(0, n) = 1, \tag{2.4}$$

$$P_m(0, n) = 0, \qquad m \neq 0. \tag{2.5}$$

Using the equilibrium condition, it is easily derived [1] that

$$\frac{\partial}{\partial t} P_0(t, n) = -\lambda p_n P_0(t, n), \tag{2.6}$$

$$\frac{\partial}{\partial t} P_m(t, n) = -\lambda p_{n+m} P_m(t, n) + \lambda p_{n+m-1} P_{m-1}(t, n), \qquad m \neq 0. \tag{2.7}$$

The solution of (2.6) is

$$P_0(t, n) = e^{-\lambda p_n t}, \tag{2.8}$$

and the solution of (2.7) will be found in the next section.

Using this probability $P_m(t, n)$, the transition probability $p(i,j)$ can be expressed as follows:

$$p(i,j) = \begin{cases} \int_0^\infty \{P_0(t,1) + P_1(t,1)\} B(t)\, dt & \text{for} \quad i = j = 1, \\ \int_0^\infty P_{j+1-i}(t, i) B(t)\, dt & \\ \qquad \text{for} \quad i - 1 \leqq j \text{ (excluding the case } i = j = 1\text{)}, & \\ 0 \quad \text{for} \quad i - 1 > j. & \end{cases} \tag{2.9}$$

We will find the transition probability $p(i,j)$ with the aid of (2.7) and (2.8). Multiplying both sides of (2.7) by $B(t)$, integrating them from 0 to ∞ and applying the integration by parts on the left side, we have

$$(b + \lambda p_{n+m}) \int_0^\infty P_m(t, n) B(t)\, dt = \lambda p_{n+m-1} \int_0^\infty P_{m-1}(t, n) B(t)\, dt, \qquad m \neq 0. \tag{2.10}$$

Putting $m = n = 1$ with slight modifications, we obtain

$$p(1,1) = \frac{1 + \rho p_1 + \rho p_2}{(1 + \rho p_1)(1 + \rho p_2)}, \tag{2.11}$$

where

$$\rho = \lambda / b, \tag{2.12}$$

which is called the relative traffic density in the case where customers always join in the queue.

If we put $m = 2$ and $n = 1$ in (2.10),

$$p(1,2) = \frac{\rho^2 p_1 p_2}{(1 + \rho p_1)(1 + \rho p_2)(1 + \rho p_3)}, \tag{2.13}$$

and if we put $m > 2$ and $n = 1$, we obtain the following recurrence formula:

$$(1 + \rho p_{m+1}) p(1, m) = \rho p_m p(1, m-1). \tag{2.14}$$

Then we obtain $p(1, j)$ for $j > 1$ as

$$p(1, j) = \frac{\rho^j p_1 p_2 \cdots p_j}{(1 + \rho p_1)(1 + \rho p_2) \cdots (1 + \rho p_{j+1})}. \tag{2.15}$$

Finally, the analogous recurrence formula

$$(1 + \rho p_{j+1}) p(i, j) = \rho p_j p(i, j-1) \tag{2.16}$$

will be obtained for $i > 1$ and with the aid of the relation

$$p(i, i-1) = \int_0^\infty P_0(t, i) B(t)\, dt = \frac{1}{1 + \rho p_i}, \tag{2.17}$$

we have

$$p(i, j) = \frac{\rho^{j+1-i} p_i p_{i+1} \cdots p_j}{(1 + \rho p_i)(1 + \rho p_{i+1}) \cdots (1 + \rho p_{j+1})}. \tag{2.18}$$

Thus we can express the transition probability for all i and j as follows:

$$p(i, j) = \begin{cases} \dfrac{1 + \rho p_1 + \rho p_2}{(1 + \rho p_1)(1 + \rho p_2)}, & \text{for} \quad i = j = 1, \\[2ex] \dfrac{\rho^{j+1-i} p_i p_{i+1} \cdots p_j}{(1 + \rho p_i)(1 + \rho p_{i+1}) \cdots (1 + \rho p_{j+1})}, & \text{for} \quad i - 1 \leqq j, \\[2ex] 0, & \text{for} \quad i - 1 > j. \end{cases} \tag{2.19}$$

Since the transition probability satisfy the relation $p(i, i-1) \neq 0$ and $p(i,$

$i+1)\neq 0$ if $p_i>0$ for every i, our Markov chain is irreducible and aperiodic if $p_i>0$ for every i. By a well known theorem of probability theory, this chain is ergodic or null. In both cases, the limit probability u_j exists for all j without depending i:

$$u_j=\lim_{n\to\infty} p^{(n)}(i,j)\geqq 0, \tag{2.20}$$

where $p^{(n)}(i,j)$ is the n-steps transition probability $\Pr(x_{t+n}=j\mid x_t=i)$. If the chain is null, $u_j=0$ for all j and if it is ergodic, u_j's are determined uniquely by the following equations.

$$\begin{aligned} &u_j>0,\\ &\sum_{i=1}^{\infty} u_i p(i,j)=u_j,\\ &\sum_{j=1}^{\infty} u_j=1. \end{aligned} \tag{2.21}$$

In our process, the second equation of (2.21) can be written as

$$\sum_{i=1}^{j+1} u_i p(i,j)=u_j, \tag{2.22}$$

owing to the relation $p(i,j)=0$ for $i-1>j$. With the aid of (2.16) we have

$$\sum_{i=2}^{\infty} u_i \sum_{j=i}^{\infty}(1+\rho p_{j+1})p(i,j)z^j=\sum_{i=2}^{\infty} u_i\sum_{j=i}^{\infty}\rho p_j p(i,j-1)z^j. \tag{2.23}$$

Changing the order of summations and using (2.22), we obtain

$$\begin{aligned} &\sum_{j=2}^{\infty}(1+\rho p_{j+1})\{u_j-u_{j+1}p(j+1,j)-u_1 p(i,j)\}z^j\\ &\qquad=\sum_{j=2}^{\infty}\rho p_j\{u_{j-1}-u_1 p(1,j-1)\}z^j. \end{aligned} \tag{2.24}$$

Since the coefficients of the power z^j on both sides must be equal, we have

$$\begin{aligned} &(1+\rho p_{j+1})\{u_j-u_{j+1}p(j+1,j)-u_1 p(1,j)\}\\ &\qquad=\rho p_j\{u_{j-1}-u_1 p(1,j-1)\}\qquad \text{for}\quad j\geqq 2. \end{aligned} \tag{2.25}$$

Here, we note the relations

$$\begin{aligned} (1+\rho p_{j+1})p(j+1,j)&=1 &&\text{for}\quad j\geqq 2,\\ \rho p_j p(1,j-1)-(1+\rho p_{j+1})p(1,j)&=0 &&\text{for}\quad i\geqq 3. \end{aligned} \tag{2.26}$$

Then we have the following difference equation:

$$u_{j+1}-(1+\rho p_{j+1})u_j+\rho p_j u_{j-1}=0 \qquad \text{for} \quad j\geqq 3. \tag{2.27}$$

The solution of the equation can be easily found as follows:

$$u_j=\rho p_j u_{j-1} \qquad \text{for} \quad j\geqq 3. \tag{2.28}$$

For u_2, putting $j=1$ in (2.22), we can find

$$u_2=\frac{1-p(1,1)}{p(2,1)}u_1=\frac{\rho^2 p_1 p_2}{1+\rho p_1}u_1. \tag{2.29}$$

Finally, we can express u_j as

$$u_j=\begin{cases} \dfrac{1+\rho p_1}{1+\sum\limits_{j=1}^{\infty}\rho^j p_1 p_2\cdots p_j} & \text{for} \quad j=1, \\[2ex] \dfrac{\rho^j p_1 p_2\cdots p_j}{1+\sum\limits_{j=1}^{\infty}\rho^j p_1 p_2\cdots p_j} & \text{for} \quad j\geqq 2, \end{cases} \tag{2.30}$$

in the ergodic case and we mention that it is known [1] that the ergodic condition is

$$\sum_{j=1}^{\infty}\rho^j p_1 p_2\cdots p_j<\infty. \tag{2.31}$$

3. Solution of $P_m(t, n)$. In the previous section, we introduced the probability $P_m(t, n)$ concerning a birth process. Let us solve it in this section.

Now we will impose a slight strengthen condition on p_i, that is

$$p_m>p_n, \qquad m<n. \tag{3.1}$$

Then the solution of (2.7) is expressed as

$$P_m(t,n)=(-1)^m\left\{\prod_{i=0}^{m-1}p_{n+i}\right\}\sum_{j=0}^{m}\frac{e^{-\lambda p_{n+j}t}}{\prod\limits_{k=0}^{m}{}'(p_{n+j}-p_{n+k})}, \qquad m\neq 0, \tag{3.2}$$

where $\prod\limits_{k=0}^{m}{}'$ denotes multiplication for all $0\leqq k\leqq m$ except $k=j$. To prove this it is sufficient to show that (3.2) satisfies both (2.7) and (2.5), for the uniqueness of the solution is evident in the pure birth process. We shall check this by substitution.

First, substituting (3.2) in both sides of (2.7) respectively, we obtain the followings:

$$\text{left side of } (2.7) = (-1)^{m+1}\lambda\left\{\prod_{i=0}^{m-1} p_{n+i}\right\}\sum_{j=0}^{m}\frac{p_{n+j}e^{-\lambda p_{n+j}t}}{\prod\limits_{k=0}^{m}{}'(p_{n+j}-p_{n+k})},$$

$$\text{right side of } (2.7) = (-1)^{m+1}\lambda\left\{\prod_{i=0}^{m-1} p_{n+i}\right\}\sum_{j=0}^{m}\frac{p_{n+m}e^{-\lambda p_{n+j}t}}{\prod\limits_{k=0}^{m}{}'(p_{n+j}-p_{n+k})}$$

$$+(-1)^{m-1}\lambda\left\{\prod_{i=0}^{m-1} p_{n+i}\right\}\sum_{j=0}^{m}\frac{(p_{n+j}-p_{n+m})e^{-\lambda p_{n+j}t}}{\prod\limits_{k=0}^{m}{}'(p_{n+j}-p_{n+k})}.$$

Comparing coefficients of $e^{-\lambda p_{n+j}t}$ in both sides, we can easily find that (2.7) is satisfied.

Next, we put $t=0$ in (3.2) and we have

$$P_m(0,n) = (-1)^{m}\left\{\prod_{i=0}^{m-1} p_{n+i}\right\}\sum_{j=0}^{m}\frac{1}{\prod\limits_{k=0}^{m}{}'(p_{n+j}-p_{n+k})}.$$

And it is sufficient to show that

$$D = \sum_{j=0}^{m}\frac{1}{\prod\limits_{k=0}^{m}{}'(p_{n+j}-p_{n+k})} = 0. \tag{3.3}$$

To prove this we reduce to common denominator of D, and we have

$$D = \sum_{j=0}^{m}\frac{(-1)^{m-j}\prod\limits_{i>k}^{m}{}'(p_{n+i}-p_{n+k})}{\prod\limits_{i>k}^{m}(p_{n+i}-p_{n+k})}. \tag{3.4}$$

In (3.4), the j-th numerator can be expressed as

$$\prod_{i>k}^{m}{}'(p_{n+i}-p_{n+k}) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ p_n & p_{n+1} & p_{n+2} & \cdots & p_{n+j-1} & p_{n+j+1} & \cdots & p_{n+m} \\ p_n^2 & p_{n+1}^2 & p_{n+2}^2 & \cdots & p_{n+j-1}^2 & p_{n+j+1}^2 & \cdots & p_{n+m}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_n^{m-1} & p_{n+1}^{m-1} & p_{n+2}^{m-1} & \cdots & p_{n+j-1}^{m-1} & p_{n+j+1}^{m-1} & \cdots & p_{n+m}^{m-1} \end{vmatrix} \tag{3.5}$$

with the aid of Vandermonde's determinant, then the sum of numerators is an expansion form of the lowest raw of the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ p_n & p_{n+1} & p_{n+2} & \cdots & p_{n+m} \\ p_n^2 & p_{n+1}^2 & p_{n+1}^2 & \cdots & p_{a+m}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_n^{m-1} & p_{n+1}^{m-1} & p_{n+2}^{m-1} & \cdots & p_{n+m}^{m-1} \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

and this determinant clearly vanishes.

4. Other considerations. In this section, we shall consider the number of customers that do not join in the queue but give up to receive their services when they arrive. Now, we introduce the probability $Q_m(t, n)$. In an interval between any two successive epochs, $Q_m(t, n)$ is defined as the probability that m customers give up their services until t when n customers joined in the queue at the time $t = 0$.

Since the input process is followed by Poisson law, the probability $Q_m(t, n)$ can be expressed as

$$Q_m(t, n) = \sum_{l=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{m+l}}{(m+l)!} P_l(t, n). \tag{4.1}$$

Let us put $Q_m(n)$ as the probability that m customers give up their services between two successive epochs when n customers have joined in the queue at the preceding epoch. Then

$$Q_m(n) = \int_0^{\infty} Q_m(t, n) B(t)\, dt. \tag{4.2}$$

The mean value of leaving customers $Q(n)$ between two successive epochs when n customers joined at the preceding epoch is

$$\tag{4.3} \left\{ \begin{aligned} Q(n) &= \sum_{m=1}^{\infty} m Q_m(n) \\ &= \sum_{l=0}^{\infty} (-1)^l \left\{ \prod_{j=0}^{l-1} p_{n+j} \right\} \\ &\quad \cdot \sum_{j=0}^{l} \frac{1}{\prod_{k=0}^{l}{}' (p_{n+j} - p_{n+k})} \frac{\rho^{l+1}}{(1+\rho+\rho p_{n+j})^l} \frac{1}{(1+\rho+\rho p_{n+j})^2}. \end{aligned} \right.$$

If the chain is ergodic, then the limiting mean value of $Q(n)$ is

$$Q = \sum_{n=1}^{\infty} u_n Q(n). \tag{4.4}$$

We are imdebted to Professor T. Kawata for valuable criticism in preparing this paper.

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