COMPOSITIONS OF SEMIGROUPS

By Miyuki Yamada

If a semigroup S is homomorphic with a semilattice L, then there exists a decomposition to a semilattice which is isomorphic with L ([1], [2]), i.e., there exists, for each element $\delta \in L$, a subset S_δ of S, such that

(1) $$S = \bigcup_{\delta \in L} S_\delta,$$

(2) $$S_\delta \cap S_\beta = \phi \quad \text{for any} \quad \alpha, \beta \in L,\ \alpha \neq \beta,$$

(3) $$S_\alpha S_\beta \subset S_{\alpha\beta} \quad \text{for any} \quad \alpha, \beta \in L,$$

where $S_\alpha S_\beta$ denotes the set $\{xy \mid x \in S_\alpha,\ y \in S_\beta\}$.

Conversely, let L be a given semilattice and S_δ be, for each $\delta \in L$, a given semigroup. Let S be the direct sum, i.e., the disjoint sum of all S_δ; $S = \cdot\sum_{\delta \in L} S_\delta$. Then let us consider the problem of constructing every possible semigroup $S(\circ)$ which consists of all elements of S and in which a product $\circ$ is defined such that

$$(\mathrm{C}) \quad \begin{cases} (1) \text{ for any } \delta \in L,\ S_\delta \text{ is a subsemigroup of } S(\circ);\ a_\delta \circ b_\delta = a_\delta b_\delta \\ \qquad \text{for any elements } a_\delta, b_\delta \in S_\delta, \\ (2) \text{ for any } \alpha, \beta \in L,\ S_\alpha \circ S_\beta \subset S_{\alpha\beta}. \end{cases}$$

We shall call such a semigroup $S(\circ)$ a *compound semigroup of* $\{S_\delta\}_{\delta \in L}$ (the collection of all S_δ) *by* L.

To save repetition, we shall adhere throughout the paper to the following notation. L will denote any semilattice and α, β, γ, δ etc. elements of L. For each $\delta \in L$, S_δ will denote any semigroup and in § 2 it will be a semigroup having a two-sided unit (= an identity element) but no other idempotent. For each $\delta \in L$, the small letters a_δ, b_δ, c_δ, d_δ etc. having an index δ will denote elements of S_δ. S will denote the direct sum of all S_δ; $S = \cdot\sum_{\delta \in L} S_\delta$. $S(\circ)$ will denote the set S in which a binary operation $\circ$ is defined.

§ 1. Existence theorem.

The referee raises the pertinent question of how to tell, for given L and given $\{S_\delta\}_{\delta \in L}$ ($= \{S_\delta \mid \delta \in L\}$), whether a compound semigroup of $\{S_\delta\}_{\delta \in L}$ by L exists, and gives the following example where no compound semigroup

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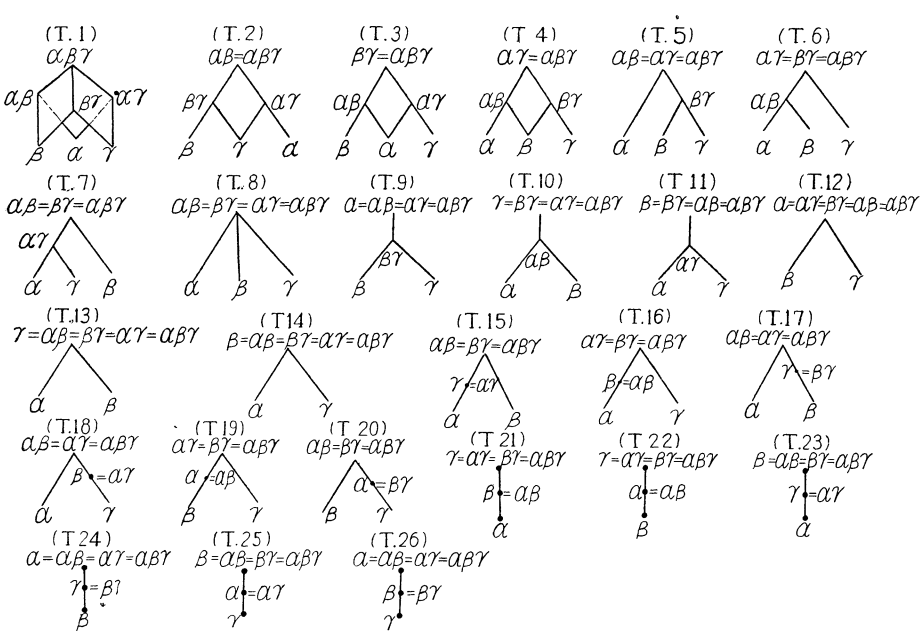
exists:

$$\begin{cases} L=\{\alpha,\ \beta,\ \gamma\};\ \alpha^2=\alpha,\ \beta^2=\beta,\ \gamma^2=\gamma,\ \alpha\beta=\beta\alpha=\gamma,\ \alpha\gamma=\gamma\alpha=\gamma,\ \beta\gamma=\gamma\beta=\gamma. \\ S_\alpha=\{e\};\ e^2=e. \\ S_\beta=\{e'\};\ e'^2=e'. \\ S_\gamma=\{c,\ c^2,\ c^3\ \cdots,\ c^n,\ \cdots\};\ c^i\cdot c^j=c^{i+j}. \end{cases}$$

The author is not able to answer this question. However, at least the following theorem shows that a compound semigroup of $\{S_\delta\}_{\delta\in L}$ by L always exists if each S_δ contains an idempotent.

THEOREM 1. *If each S_δ contains an idempotent, then there exists a compound semigroup of $\{S_\delta\}_{\delta\in L}$ by L.*

To prove Theorem 1, we show next a table of all semilattices each of which is generated by different three elements α, β and γ.



Proof of Theorem 1.

Take up one idempotent e_δ from each S_δ. We define a binary operation $\circ$ in S as follows:

$$a_\alpha \circ b_\beta = \begin{cases} a_\alpha\, b_\beta & \text{if} \quad \alpha=\beta, \\ a_\alpha\, e_\alpha & \text{if} \quad \alpha>\beta, \quad \text{i.e.,} \quad \alpha\beta=\alpha \quad \text{and} \quad \alpha\neq\beta, \\ e_\beta\, b_\beta & \text{if} \quad \alpha<\beta, \quad \text{i.e.,} \quad \alpha\beta=\beta \quad \text{and} \quad \alpha\neq\beta, \\ e_{\alpha\beta} & \text{if} \quad \alpha\nleqq\beta \quad \text{and} \quad \alpha\ngeqq\beta. \end{cases}$$

We may show the resulting system $S(\circ)$ to become a semigroup since the condition (C) is obvious by the definition of the binary operation $\circ$. Therefore we may show only that $S(\circ)$ satisfies the associative law, i.e.,

$$a_\alpha \circ (b_\beta \circ c_\gamma) = (a_\alpha \circ b_\beta) \circ c_\gamma \quad \text{for any} \quad a_\alpha,\ b_\beta,\ c_\gamma \in S(\circ).$$

In case where two or all elements of α, β, γ are just the same element, it is easy to verify the relation in above. Therefore we prove it only in the most complicated case where the subsemilattice generated by α, β, γ is constructing the type (T. 1), because in the other cases (T. 2) $\sim$ (T. 26) we can prove it by the same procedure.

$$(a_\alpha \circ b_\beta) \circ c_\gamma = e_{\alpha\beta} \circ c_\gamma = e_{\alpha\beta\gamma}, \qquad a_\alpha \circ (b_\beta \circ c_\gamma) = a_\alpha \circ e_{\beta\gamma} = e_{\alpha\beta\gamma},$$

whence

$$(a_\alpha \circ b_\beta) \circ c_\gamma = a_\alpha \circ (b_\beta \circ c_\gamma).$$

REMARK. Let L be a semilattice having a minimal element and L' be the totality of all minimal elements of L. Then it is easy to see from the proof of Theorem 1 that, even in case each S_ξ of $\{S_\xi\}_{\xi \in L'}$ has no idempotent, the existence of a compound semigroup of $\{S_\delta\}_{\delta \in L}$ by L is guaranteed if every S_δ of $\{S_\delta\}_{\delta \in L}$, except S_ξ of $\{S_\xi\}_{\xi \in L'}$, contains an idempotent.

§ 2. Determination of all compound semigroups.

If a semigroup has two-sided unit 1 but no other idempotent, we shall call such a semigroup a *hypogroup*. In this section we assume that S_δ is, for each $\delta \in L$, a hypogroup, and completely determine all compound semigroups of $\{S_\delta\}_{\delta \in L}$ by L. Throughout this section e_δ will denote, for each $\delta \in L$, the two-sided unit of S_δ. We shall write $\alpha \leqq \beta$ when $\alpha\beta = \beta$ and $\alpha, \beta \in L$.

Take up a homomorphism $\varphi_{\alpha,\beta}$ of S_α into S_β, for each pair (α, β) of $\alpha, \beta \in L$ such that $\alpha \leqq \beta$. If the totality $\{\varphi_{\alpha,\beta}\}_{\alpha \leqq \beta}$ of all $\varphi_{\alpha,\beta}$ satisfies the following condition (A), then such a system $\{\varphi_{\alpha,\beta}\}_{\alpha \leqq \beta}$ is called a transitive system of homomorphisms induced by $\{S_\delta\}_{\delta \in L}$.

(A) $\quad \varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ for $\alpha \leqq \beta \leqq \gamma$ and $\varphi_{\alpha,\alpha} =$ identity mapping,

where $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}$ denotes a mapping such that

$$\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}(a_\alpha) = \varphi_{\beta,\gamma}(\varphi_{\alpha,\beta}(a_\alpha)) \qquad \text{for any} \quad a_\alpha \in S_\alpha.$$

Let $\{\varphi_{\alpha,\beta}\}_{\alpha \leqq \beta}$ be a transitive system of homomorphisms induced by $\{S_\delta\}_{\delta \in L}$. Define a binary operation $\circ$ by

$$\text{(P)} \qquad a_\alpha \circ b_\beta = \varphi_{\alpha,\alpha\beta}(a_\alpha)\varphi_{\beta,\alpha\beta}(b_\beta)$$

for any $a_\alpha, b_\beta \in S$. Then $S(\circ)$ is a compound semigroup of $\{S_\delta\}_{\delta \in L}$ by L. In fact,

$$(1) \qquad a_\alpha \circ b_\alpha = \varphi_{\alpha,\alpha}(a_\alpha)\varphi_{\alpha,\alpha}(b_\alpha) = a_\alpha b_\alpha,$$

$$(2) \qquad a_\alpha \circ b_\beta = \varphi_{\alpha,\alpha\beta}(a_\alpha)\varphi_{\beta,\alpha\beta}(b_\beta) \in S_{\alpha\beta}.$$

The associativity of $\circ$ is verified as follows:

$$a_\alpha \circ (b_\beta \circ c_\gamma) = a_\alpha \circ (\varphi_{\beta,\beta\gamma}(b_\beta)\varphi_{\gamma,\beta\gamma}(c_\gamma))$$
$$= \varphi_{\alpha,\alpha\beta\gamma}(a_\alpha)\varphi_{\beta\gamma,\alpha\beta\gamma}(\varphi_{\beta,\beta\gamma}(b_\beta)\varphi_{\gamma,\beta\gamma}(c_\gamma))$$
$$= \varphi_{\alpha,\alpha\beta\gamma}(a_\alpha)\varphi_{\beta,\alpha\beta\gamma}(b_\beta)\varphi_{\gamma,\alpha\beta\gamma}(c_\gamma),$$
$$(a_\alpha \circ b_\beta) \circ c_\gamma = (\varphi_{\alpha,\alpha\beta}(a_\alpha)\varphi_{\beta,\alpha\beta}(b_\beta)) \circ c_\gamma = \varphi_{\alpha,\alpha\beta\gamma}(a_\alpha)\varphi_{\beta,\alpha\beta\gamma}(b_\beta)\varphi_{\gamma,\alpha\beta\gamma}(c_\gamma).$$

Conversely, assume that $S(\circ)$ be a compound semigroup of $\{S_\delta\}_{\delta\in L}$ by L. Define mappings $\varphi_{\alpha,\beta}$ as follows:

$$\varphi_{\alpha,\beta}(a_\alpha) = a_\alpha \circ e_\beta \qquad \text{for} \qquad \alpha \leqq \beta.$$

Then $\{\varphi_{\alpha,\beta}\}_{\alpha\leqq\beta}$ becomes a transitive system of homomorphisms induced by $\{S_\delta\}_{\delta\in L}$. In the first place we shall remark that $e_\alpha \circ e_\beta = e_\beta$ if $\alpha \leqq \beta$.
In fact, $\alpha \leqq \beta$ implies $e_\alpha \circ e_\beta \in S_\beta$. Hence,

$$(e_\alpha \circ e_\beta) \circ (e_\alpha \circ e_\beta) = e_\alpha \circ (e_\beta \circ (e_\alpha \circ e_\beta)) = e_\alpha \circ (e_\alpha \circ e_\beta) = e_\alpha \circ e_\beta.$$

Thus $e_\alpha \circ e_\beta$ is an idempotent in S_β and hence $e_\alpha \circ e_\beta = e_\beta$.
Now it is easy to verify that

$$\varphi_{\alpha,\beta}(a_\alpha \circ b_\alpha) = a_\alpha \circ b_\alpha \circ e_\beta = a_\alpha \circ e_\beta \circ b_\alpha \circ e_\beta = \varphi_{\alpha,\beta}(a_\alpha) \circ \varphi_{\alpha,\beta}(b_\alpha),$$
$$\varphi_{\beta,\gamma}(\varphi_{\alpha,\beta}(a_\alpha)) = a_\alpha \circ e_\beta \circ e_\gamma = a_\alpha \circ e_\gamma = \varphi_{\alpha,\gamma}(a_\alpha),$$
$$\varphi_{\alpha,\alpha}(a_\alpha) = a_\alpha \circ e_\alpha = a_\alpha.$$

Furthermore,

$$a_\alpha \circ b_\beta = a_\alpha \circ b_\beta \circ e_{\alpha\beta} = a_\alpha \circ e_{\alpha\beta} \circ b_\beta \circ e_{\alpha\beta} = \varphi_{\alpha,\alpha\beta}(a_\alpha) \circ \varphi_{\beta,\alpha\beta}(b_\beta).$$

Thus we obtain

THEOREM 2. *Let L be a semilattice and S_δ be, for each $\delta \in L$, a hypogroup. Let $\{S_\delta\}_{\delta\in L}$ be the totality of all S_δ, and S be the direct sum of all S_δ. Then every compound semigroup $S(\circ)$ of $\{S_\delta\}_{\delta\in L}$ by L is found as follows. Let $\{\varphi_{\xi,\eta}\}_{\xi\leqq\eta}$ be a transitive system of homomorphisms induced by $\{S_\delta\}_{\delta\in L}$. Then S becomes a compound semigroup of $\{S_\delta\}_{\delta\in L}$ by L if product $\circ$ therein is defined by the following equation* (P):

$$\text{(P)} \qquad a_\alpha \circ b_\beta = \varphi_{\alpha,\alpha\beta}(a_\alpha)\varphi_{\beta,\alpha\beta}(b_\beta)$$

for any $a_\alpha, b_\beta \in S$.

REMARK. If we substitute the words 'commutative semigroup', 'commutative compound semigroup', 'commutative semigroup having only one idempotent' and 'idempotent e_δ' etc. for 'semigroup', 'compound semigroup', 'semigroup having a two-sided unit but no other idempotent (hypogroup)' and 'two-sided unit e_δ' etc. respectively in above discussions, then whole discussion holds in the same manner. Accordingly we have the following theorems.

THEOREM 1′. *Let L be a semilattice and S_δ be, for each $\delta \in L$, a com-*

mutative semigroup. If each S_δ contains an idempotent, then there exists a commutative compound semigroup of $\{S_\delta\}_{\delta\in L}$ by L.

THEOREM 2′. *Let L be a semilattice and S_δ be, for each $\delta \in L$, a commutative semigroup having only one idempotent. Let $\{S_\delta\}_{\delta\in L}$ be the totality of all S_δ, and S be the direct sum of all S_δ. Then every commutative compound semigroup $S(\circ)$ of $\{S_\delta\}_{\delta\in L}$ by L is found as follows. Let $\{\varphi_{\xi,\eta}\}_{\xi\leqq\eta}$ be a transitive system of homomorphisms induced by $\{S_\delta\}_{\delta\in L}$. Then S becomes a commutative compound semigroup of $\{S_\delta\}_{\delta\in L}$ by L if product $\circ$ therein is defined by the following equation* (P):

$$a_\alpha \circ b_\beta = \varphi_{\alpha,\alpha\beta}(a_\alpha)\varphi_{\beta,\alpha\beta}(b_\beta) \tag{P}$$

for any $a_\alpha, b_\beta \in S$.

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DEPARTMENT OF MATHEMATICS,
SHIMANE UNIVERSITY.