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THE CARATHÉODORY METRIC IN PLANE DOMAINS

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Abstract. Let $D \notin O_{AB}$ be a plane domain and let $C_D(z)$ be its analytic capacity at $z \in D$. Let $\mathcal{K}_D(z)$ be the curvature of the Carathéodory metric $C_D(z)|dz|$. We show that $\mathcal{K}_D(z) < -4$ if the Ahlfors function of D with respect to z has a zero other than z. For finite D, $\mathcal{K}_D(z) \leqq -4$ and equality holds if and only if D is simply connected. As a corollary we obtain a result proved first by Suita, namely, that $\mathcal{K}_D(z) \leqq -4$ if $D \notin O_{AB}$. Several other properties related to the Carathéodory metric are proven.

§ 1. Introduction.

Let $D \notin O_{AB}$ be a plane domain and let $C_D(z)$ be its analytic capacity at $z \in D$. Let $\mathcal{K}_D(z)$ be the curvature of the Carathéodory metric $C_D(z)|dz|$. This paper is divided into three main parts. In the first part we study the curvature $\mathcal{K}_D(z)$ and its relation with the curvatures of the Bergman metrics (Theorem 1). As a corollary, we show that the boundary value of $\mathcal{K}_D(z)$ is -4 if D is bounded by C^2 curves.

The second part of this paper is devoted to the discussion of the boundedness of $\mathcal{K}_D(z)$ by -4. Suita [8], using the notion of supporting metrics, has shown that $\mathcal{K}_D(z) \leqq -4$. However, this method of proof seems not to apply for deciding when $\mathcal{K}_D(z) = -4$ occurs. After submitting this paper for publication the author's attention was drawn to the existence of another recent paper of Suita [9] wherein a different method of proof is provided to settle the above question when D is finite. This method is similar to the one we employed in our Theorem 2 and it is based on the "method of minimum integral" with respect to the Szegö kernel function. In fact, with similar methods and without being aware of [9] we were able to extend some of the present results to higher order curvatures [2]. This part is revised accordingly. The author is most indebted to Professor N. Suita for his valuable comments and suggestions, and, in particular, for pointing out the possibility of sharpening our assertions to the form embodied in Theorem 3 of this paper.

Finally, in the third part we study in some detail the curvature $\mathcal{K}_D(z)$ for

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doubly connected domains D. Part of this treatment is patterned after that of Zarankiewicz [10] for the reduced Bergman kernel. Several interesting relationships arise from this study (Theorem 4).

§ 2. Preliminaries.

Let D be a plane domain $\notin O_{AB}$ and let $H(D:\Delta)$ designate the class of all analytic functions from D into the unite disc Δ. Let $\xi\in D$ and set $H_\xi(D:\Delta)=\{f\in H(D:\Delta): f(\xi)=0\}$. The analytic capacity $C(\xi)=C_D(\xi)$ is given by $C(\xi)=\sup\{|f'(\xi)| : f\in H_\xi(D:\Delta)\}$. It follows that the metric $C(z)|dz|$ is conformally invariant. This is exactly the Carathéodory metric for D (cf. Reiffen [6]). There exists a unique function F in $H_\xi(D:\Delta)$, called that Ahlfors function $F(z)=F(z:\xi)$, such that $F'(\xi)=C(\xi)$.

Let $B(z,\bar z)$ and $\tilde B(z,\bar z)$ be the values of the Bergman kernel and reduced Bergman kernel of D respectively. We write $M(z)=M_D(z)=\sqrt{\pi B(z,\bar z)}$ and $\tilde M(z)=\tilde M_D(z)=\sqrt{\pi\tilde B(z,\bar z)}$. It is well known [7] that $\tilde M(z)\leqq C(z)\leqq M(z)$. Consequently, $C(z)|dz|$ is complete if D is bounded by C^2 non-degenerate curves. In fact, by employing a method described in [1, p. 38] and noting that the above three quantities are monotonic with D, one obtains:

PROPOSITION 1. *Let D be a domain bounded by C^2 non-degenerate curves and $t\in\partial D$. Then, within the angular sector $|\arg(z-t)|\leqq\alpha<\pi/2$, $\lim_{z\to t}[2\operatorname{Re}(z-t)C(z)]^2=1$.*

§ 3. The Curvature.

For a positive C^2 function $f(z)$, defined in an open subset of the plane, $K(f)$, given by

$$K(f)=-(2f)^{-1}\Delta\log f, \qquad f=f(z),$$

is the curvature of the metric $f(z)|dz|^2$. Let $g(z)$ be another positive C^2 function defined in the same open set as that of $f(z)$. We note the following identity

$$fg(f+g)[f^2K(f)+g^2K(g)-(f+g)^2K(f+g)]=2\left|f\frac{\partial g}{\partial z}-g\frac{\partial f}{\partial z}\right|^2.$$

Consequently,

$$(f+g)^2K(f+g)\leqq f^2K(f)+g^2K(g). \tag{3.1}$$

By Using a canonical exhaustion process (cf. [8]) it can be shown that $C(z)$ is real analytic and hence we can introduce the curvature of $C(z)|dz|$

$$\mathcal{K}_D(z)=K(C^2)=-C^{-2}\Delta\log C, \qquad C=C(z).$$

Similarly, $\mu_D(z)=K(M^2)$ and $\tilde\mu_D(z)=K(\tilde M^2)$ are the curvatures of $M(z)|dz|$ and

$\tilde{M}(z)|dz|$ respectively.

For future reference we denote by $\mathscr{D}_p$, $1\leqq p<\infty$, the class of all p-connected domains with no degenerate boundary component and we let $\mathscr{D}_p^{(a)}$ designate the subclass of plane domains bounded by p analytic Jordan curves. We now state:

THEOREM 1. *Let $D\notin O_{AD}$. Then*

$$\left(\frac{M}{C}\right)^4 \mu_D(z)\leqq \mathcal{K}_D(z)\leqq \left(\frac{\tilde{M}}{C}\right)^4 \tilde{\mu}_D(z). \tag{3.2}$$

Proof. Assume first that $D\in\mathscr{D}_p^{(a)}$, $1\leqq p<\infty$. (3.2) clearly holds when $p=1$ for, in that case, $\tilde{M}=C=M$ and $\tilde{\mu}_D(z)=\mathcal{K}_D(z)=\mu_D(z)\equiv -4$. Assume now that $1<p<\infty$. Then (cf. Hejhal [5, p. 107])

$$C^2(z)=\tilde{M}^2(z)+k_1(z), \qquad M^2(z)=C^2(z)+k_2(z),$$

where

$$k_i(z)=\sum_{j=1}^{p-1}|\phi_j^{(i)}(z)|^2>0, \qquad z\in D,\ (i=1,2),$$

and $\phi_j^{(i)}(z)$, $1\leqq j\leqq p-1$, $i=1,2$, are analytic in $\bar{D}$. It is easily verified that $K(k_i)\leqq 0$, $i=1,2$. Therefore, employing (3.1), we obtain $M^4K(M^2)\leqq C^4K(C^2)\leqq \tilde{M}^4K(\tilde{M}^2)$ which is exactly (3.2). For the general case $D\notin O_{AB}$, we take a canonical exhaustion $\{D_n\}$ of D, with $D_n\in\mathscr{D}_{p_n}^{(a)}$, $1\leqq p_n<\infty$. The theorem now follows in an obvious way by letting $n\to\infty$.

COROLLARY 1. *Let D be a domain bounded by C^2 non-degenerate, curves and $t\in\partial D$. Then, within the angular sector $|\arg(z-t)|\leqq\alpha<\pi/2$, $\lim_{z\to t}\mathcal{K}_D(z)=-4$.*

Proof. Within the above angular sector we have [1, p. 38]

$$\lim_{z\to t}[2\operatorname{Re}(z-t)M(z)]^2=\lim_{z\to t}[2\operatorname{Re}(z-t)\tilde{M}(z)]^2=1$$

and $\lim_{z\to t}\mu_D(z)=\lim_{z\to t}\tilde{\mu}_D(z)=-4$. The assertion now follows from Proposition 1 and Theorem 1.

§ 4. The Szegö Kernel.

Let $D\in\mathscr{D}_p^{(a)}$. As usual, $H_2=H_2(\partial D)$ stands for the Hardy-Szegö space of D. It is a Hilbert space of analytic functions in D with the scalar product $(f,g)=\int_{\partial D}f(z)\overline{g(z)}|dz|$ and $\|f\|^2=(f,f)$. The integration is carried over the boundary values of the analytic functions f and g (this refers to an arbitrary non-tangential approach). H_2 admits a reproducing kernel $K(z,\bar{\xi})$ which is the classical Szegö kernel for D.

According to a classical result of Garabedian (cf. [1, p. 118]) $C(\xi)=2\pi K(\xi,\bar{\xi})$

and $F(z)=F(z:\xi)=K(z,\bar{\xi})/L(z,\xi)$. Here, $F'(\xi)=C(\xi)$ and $L(z,\xi)$ is the adjoint of $K(z,\bar{\xi})$ satisfying the boundary relation

$$\imath\overline{K(z,\bar{\xi})}\,|dz|=L(z,\xi)dz\,; \qquad z\in\partial D,\quad \xi\in D\,. \tag{4.1}$$

Therefore, $|F(z)|\equiv 1$ for $z\in\partial D$ and $|F(z)|\leqq 1$ throughout D. Moreover the divisor of $L(z,\xi)$ is exactly ξ^{-1} with residue $(2\pi)^{-1}$ and the analytic function $L(z,\xi)-(2\pi)^{-1}(z-\xi)^{-1}$ vanishes at $z=\xi$. The divisor of $F(z)$ is therefore $\xi\overline{b_1(\xi)}\cdots\overline{b_{p-1}(\xi)}$ where $\overline{b_j(\xi)}$, $j=1,\cdots,p-1$, are the $p-1$ (possibly repeated) zeros of $K(z,\bar{\xi})$ (none of which is on ∂D). The functions $b_j(\xi)$ are analytic in ξ.

Quite recently, Hejhal [4] has obtained certain relationships for a large class of kernel functions. One of these relationships reads

$$\frac{\partial^2}{\partial z\partial\bar{\xi}}\log K(z,\bar{\xi})=[2\pi K(z,\bar{\xi})]^2+\sum_{j=1}^{p-1}[2\pi L(z,\overline{b_j(\xi)})]^2\overline{b_j'(\xi)}\,, \tag{4.2}$$

provided ξ is a non-exceptional point of D. Furthermore, Theorem 2 states that $\mathcal{K}_D(\xi)<-4$ for $p>1$. To prove that, one, of course, is tempted to use the above identity. This amounts to in only showing that the second term on the right hand side of (4.2) is strictly positive for $z=\xi$ and $p>1$. Apriori, the truth of such as an assertion is not known and in fact it is only a corollary of Theorem 2.

§ 5. Boundedness of Curvature.

For $\xi\in D$ we write $\delta_D(\xi)=\sup_{z\in\partial D}|z-\xi|$ whence if $\infty\in D$ and $\xi\neq\infty$ $\delta_D(\xi)\leqq\delta\leqq\infty$. Again, assume that $D\in\mathfrak{D}_p^{(a)}$ and $\xi\in D$. Write $A(\xi)=\{f\in H_2: f(\xi)=0, f'(\xi)=1\}$. $A(\xi)$ is a closed convex subset of H_2 and it is not empty for, the function

$$\varphi(z)=\frac{2\pi F(z)K(z,\bar{\xi})}{C^2(\xi)} \tag{5.1}$$

is in $A(\xi)$. Let ψ be the unique solution of the minimal problem $\lambda(\xi)=\min\{\|f\|^2: f\in A(\xi)\}$. Then (cf. Bergman [1, p. 26])

$$\lambda(\xi)=4(K\Delta\log K)^{-1}, \qquad K=K(\xi,\bar{\xi})\,, \tag{5.2}$$

and

$$\psi(z)=K\lambda(\xi)\frac{\partial}{\partial\bar{\xi}}\frac{K(z,\bar{\xi})}{K(\xi,\bar{\xi})}\,. \tag{5.3}$$

Part (i) of the following theorem appears also in [9].

THEOREM 2. *Let* $D\in\mathfrak{D}_p$ *and* ξ *fixed in* D. *Then*

(i) $\mathcal{K}_D(\xi)\leqq -4$ *and equality holds if and only if* $p=1$.

(ii) *If* $D\in\mathfrak{D}_p^{(a)}$, *the identity,* (5.4) $$\int_{\partial D} f(z)\frac{L(z,\xi)^2}{K(z,\bar{\xi})}dz = i\frac{\partial}{\partial\xi}\frac{f(\xi)}{C(\xi)}$$

holds foy all $f \in H_2$ *if and only if* $p=1$.

(iii) *For* $D \in \mathfrak{D}_p^{(a)}$ *and* $p>1$ *we have*

(5.5) $$\mathcal{K}_D(\xi) < -4 - \frac{16\pi^2}{\delta_D^2(\xi)} \frac{|\xi_j-\xi|^2|L(\xi_j,\xi)|^2}{C(\xi)C(\xi_j)},$$

where ξ_j *is any one of the* $(p-1)$ *zeros of* $K(z,\bar{\xi})$, *i. e.*, $\xi_j=\overline{b_j(\xi)}$.

Proof. Since $\mathcal{K}_D(z)$ is conformally invariant we may assume that $D \in \mathfrak{D}_p^{(a)}$. Clearly, $\|\varphi\|^2 \geqq \lambda(\xi)$. But $\|\varphi\|^2 = 2\pi C^{-3}(\xi)$ and thus, using (5.2), $4C^2(\xi) \leqq \Delta \log K = \Delta \log C(\xi)$ which shows that $\mathcal{K}_D(\xi) \leqq -4$. Equality holds if and only if $\varphi=\psi$ which is equivalent to $(f,\varphi)=(f,\psi)$ for each $f \in H_2$. According to (5.3) this is equivalent to

$$(f,\varphi)=K\lambda(\xi)\Big(f, \frac{\partial}{\partial\xi}\frac{K(z,\bar{\xi})}{K(\xi,\bar{\xi})}\Big).$$

Since $\mathcal{K}_D(\xi)=-4$, using the reproducing property, we obtain

$$(f,\varphi)=\frac{1}{C^2(\xi)}\frac{\partial}{\partial\xi}\frac{f(\xi)}{K(\xi,\bar{\xi})}$$

for all $f \in H_2$. According to (4.1) and (5.1)

$$(f,\varphi)=\frac{1}{C^2(\xi)}\frac{2\pi}{i}\int_{\partial D} f(z)\overline{F(z)}L(z,\xi)dz$$

and thus (5.4) holds for all $f \in H_2$. However, for $p>1$, (5.4) does not holds. Indeed, $K(z,\bar{\xi})=(z-\xi_1)\cdots(z-\xi_{p-1})h(z)$ where $h(z)$ and $H(z)=L(z,\xi)(z-\xi)$ do not vanish in D. Let

$$f_0(z)=(z-\xi_2)\cdots(z-\xi_{p-1})(z-\xi)^2.$$

Then $f_0 \in H_2$ and

$$\int_{\partial D} f_0(z)\frac{L(z,\xi)^2}{K(z,\bar{\xi})}dz = \int_{\partial D}\frac{H(z)^2}{h(z)(z-\xi_1)}dz$$

$$= 2\pi i\frac{H(\xi)^2}{h(\xi)} \neq 0.$$

On the other hand, the right hand side of (5.4) is zero for f_0. We therefore proved (i) and (ii). We now show (iii). Let $p>1$ and thus $\xi_j=\overline{b_j(\xi)}$, $1 \leqq j \leqq p-1$. Thus $\xi_j \neq \xi$. Let

$$g_j(z)=(z-\xi)F(z)L(z,\xi_j), \qquad 1 \leqq j \leqq p-1.$$

Since $F(\xi_j)=0$ it is clear that $g_j \in H_2$ and $g_j(\xi)=g_j'(\xi)=0$. Also, by (4.1), $\|g_j\|^2 \leqq \delta_D^2(\xi)K(\xi_j,\bar{\xi}_j)$. The function

$$h_j=\varphi-\frac{(\varphi, g_j)}{\|g_j\|^2}g_j, \qquad 1\leqq j\leqq p-1.$$

belongs to $A(\xi)$ whence

$$\lambda(\xi)\leqq\|h_j\|^2=\|\varphi\|^2-\frac{|(g_j, \varphi)|^2}{\|g_j\|^2}.$$

Here, again, $\|\varphi\|^2=2\pi C^{-3}(\xi)$ and, using (4.1),

$$(g_j, \varphi)=\int_{\partial D}g_j\bar{\varphi}\,|dz|=\frac{2\pi}{C^2(\xi)}\frac{1}{i}\int_{\partial D}(z-\xi)L(z, \xi_j)L(z, \xi)dz$$

$$=\frac{2\pi}{C^2(\xi)}(\xi_j-\xi)L(\xi_j, \xi).$$

Consequently,

$$\lambda(\xi)\leqq\frac{2\pi}{C^3(\xi)}-\frac{4\pi^2}{C^4(\xi)}\frac{|\xi_j-\xi|^2|L(\xi_j, \xi)|^2}{\delta_D^2(\xi)K(\xi_j, \bar{\xi}_j)}.$$

Write

$$A_j=\frac{2\pi}{C(\xi)}\frac{|\xi_j-\xi|^2|L(\xi_j, \xi)|^2}{\delta_D^2(\xi)K(\xi_j, \bar{\xi}_j)}.$$

Then

$$\lambda(\xi)\leqq\frac{2\pi}{C^3(\xi)}(1-A_j)$$

and $0<A_j<1$. Therefore

$$\lambda(\xi)<\frac{2\pi}{C^3(\xi)}(1+A_j)^{-1}.$$

Hence, by (5.2),

$$C^{-2}(\xi)\Delta\log C(\xi)>4(1+A_j)$$

and (5.5) follows. This concludes the proof.

COROLLARY 2. *Let ξ be a non-exceptional point of $D\in\mathfrak{D}_p^{(a)}$, $p>1$. Then* $\sum_{j=1}^{p-1}[L(\xi, \overline{b_j(\xi)})]^2\overline{b_j'(\xi)}>0$.

Proof. This follows from (4.2) and Theorem 2.

COROLLARY 3. *Let $D\notin O_{AB}$. Then $\mathcal{K}_D(z)\leqq-4$.*

Proof. Let $\{D_n\}$ be a canonical exhaustion of D such that ∂D_n consists of a finite number of analytic curves. Then $\mathcal{K}_D(z)=\lim_{n\to\infty}\mathcal{K}_{D_n}(z)$, and since $\mathcal{K}_{D_n}(z)\leqq-4$ for each n the corollary follows.

COROLLARY 4. *Let $D\notin O_{AB}$. Then $\mu_D(z)\leqq-4[C(z)M^{-1}(z)]^4$.*

Proof. By Theorem 1, for $D\notin O_{AB}$, $\mu_D(z)\leqq\mathcal{K}_D(z)[C(z)M^{-1}(z)]^4$ and the corollary follows from Corollary 3.

Corollary 3 was first proved by Suita [8]. He also raised the following:

Conjecture. Let $D\notin O_{AB}$. Then $\mathcal{K}_D(z)\leqq-4$ and equality, at one point, holds if and only if D is conformally equivalent to the unit disc less a (possible) closed set expressed as a countable union of compact N_B sets.

Of course, this conjecture is already verified, in case $D\in\mathfrak{D}_p$, via Theorem 2. The next theorem strengthens the validity of the above coniecture and in fact sharpens the assertions of Theorem 2.

Let $D\notin O_{AB}$ and let $\{D_n\}$ be a canonical exhaustion of D such that ∂D_n consists of a finite number of analytic curves. In every D_n we have the Szegö kernel $K_n(z,\bar{\xi})$, its adjoint $L_n(z,\xi)$ and the Ahlfors function $F_n(z)=F_n(z,\xi)$. Here, as before, $F_n(z)=K_n(z,\bar{\xi})/L_n(z,\xi)$, $F_n'(\xi)=C_n(\xi)=2\pi K_n(\xi,\bar{\xi})$ and $F_n(\xi)=0$. Then, the sequences $\{F_n(z)\}$ and $\{K_n(z,\bar{\xi})\}$ converge uniformly on compacta of D to $F(z)$ and $K(z,\bar{\xi})$ respectively [8]. Of course, $C(\xi)=2\pi K(\xi,\bar{\xi})$. Therefore, $\{L_n(z,\xi)\}$ converges uniformly on compacta of $D-\{\xi\}$ to $L(z,\xi)$.

THEOREM 3. *Let $D\notin O_{AB}$ and $\xi\in D$. Assume the Ahlfors function $F(z)=F(z:\xi)$ has a zero ξ_0 in D other than ξ. Then $\mathcal{K}_D(\xi)<-4$.*

Proof. We may suppose that $\infty\in D$ and $\xi,\xi_0\neq\infty$. Let $\{D_n\}$ be a canonical exhaustion of D as before. Since $F(\xi_0:\xi)=0$, $\xi_0\neq\xi$, it follows from Hurwitz's theorem that, for a sufficiently large n, $F_n(z:\xi)$ has a zero $\xi_n\neq\xi$ near ξ_0. This zero must be a zero of $K_n(z,\bar{\xi})$ and thus $D_n\in\mathfrak{D}_{p_n}^{(a)}$, $p_n>1$, for such a large n. Therefore, by Theorem 2 (iii) and since $\delta_{D_n}(\xi)\leqq\delta_D(\xi)\leqq\delta<\infty$,

$$\mathcal{K}_{D_n}(\xi)<-4-\frac{16\pi^2}{\delta^2}\frac{|\xi_n-\xi|^2|L_n(\xi_n,\xi)|^2}{C_n(\xi)C_n(\xi_n)}$$

for a sufficiently large n. Since $L_n(z,\xi)$ has no zeros in $D_n-\{\xi\}$ it follows by another application of Hurwitz's theorem that $L_n(\xi_n,\xi)\to L(\xi_0,\xi)\neq0$. Letting $n\to\infty$ in the above inequality results in

$$\mathcal{K}_D(\xi)\leqq-4-\frac{16\pi^2}{\delta^2}\frac{|\xi_0-\xi|^2|L(\xi_0,\xi)|^2}{C(\xi)C(\xi_0)}<-4$$

and the assertion follows.

§6. Doubly Connected Domains.

Let $D\in\mathfrak{D}_2$ and assume that its modulus is $1/r$ so that it can be represented as the annulus $R=\{z:r<|z|<1\}$, $0<r<1$. Here we have at our disposal the Weierstrass $\wp$-function with half periods $\omega_1=\pi i$ and $\omega_2=\log r$. The reduced Bergman kernel is given by [1, p. 10]

(6.1) $$\tilde{B}(z, \bar{\xi})=\frac{1}{\pi z\bar{\xi}}\{\wp(\log z\bar{\xi}\,;\,\omega_1, \omega_2)-a\}, \qquad a=-\eta_1/\omega_1,$$

where $2\eta_j$ is the increment of the Weierstrassian ζ-function with respect to the periods $2\omega_j$ $(j=1, 2)$.

Since the sequence $z^n/\sqrt{2\pi(1+r^{2n+1})}$, $n=\cdots, -1, 0, 1, \cdots$, forms an orthonormal basis for $H_2(\partial R)$ the Szegö kernel for R is given by

$$K(z, \bar{\xi})=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(z\bar{\xi})^n}{1+r^{2n+1}}.$$

By a well-known formula of Garabedian [1, p. 119] we have

(6.2) $$4\pi K^2(z, \bar{\xi})=\tilde{B}(z, \bar{\xi})+\alpha w(z)\overline{w(\xi)},$$

where $w(z)=(2z\log r)^{-1}$. Comparing the coefficients of $(z\bar{\xi})^{-1}$ in (6.2) and nothing that the term with $(z\bar{\xi})^{-1}$ does not appear in $\tilde{B}(z, \bar{\xi})$, we obtain

$$\alpha=\frac{8\log^2 r}{\pi}\sum_{n=0}^{\infty}\frac{r^{2n+1}}{(1+r^{2n+1})^2}.$$

As usual, we set $\omega_3=\omega_1+\omega_2$; $e_j=\wp(\omega_j)$, $j=1, 2, 3$, where $\wp(u)=\wp(u\,;\,\omega_1, \omega_2)$. We have the identities (cf. Hancock [3, p. 483])

(6.3) $$-a+e_2=2\sum_{n=0}^{\infty}\frac{r^{2n+1}}{(1-r^{2n+1})^2},$$

(6.4) $$-a+e_3=-2\sum_{n=0}^{\infty}\frac{r^{2n+1}}{(1+r^{2n+1})^2},$$

(6.5) $$-a=2\sum_{n=0}\frac{r^{2n+2}}{(1-r^{2n+2})^2}-\frac{1}{12}.$$

Consequently,

$$\alpha=\frac{4\log^2 r}{\pi}(a-e_3).$$

This, together with (6.1) and (6.2), implies that

$$K^2(z, \bar{\xi})=\frac{1}{4\pi^2 z\bar{\xi}}\{\wp(\log z\bar{\xi}\,;\,\omega_1, \omega_2)-e_3\}.$$

We set $\rho=|z|$, $\rho\in[r, 1]$. The reduced Bergman kernel and the analytic capacity for R, are therefore

$$\tilde{M}^2(\rho)=\rho^{-2}\{\wp(2\log\rho)-a\},$$

$$C^2(\rho)=\rho^{-2}\{\wp(2\log\rho)-e_3\},$$

respectivel·. The curvature of $\tilde{M}(\rho)$, $\tilde{\mu}$, was treated by Zarankiewicz [10].

Here we study the curvature of $C(\rho)$, $\mathcal{K}$, and its relation with $\tilde{\mu}$.

For $f(z)=f(\rho)$, $\rho=|z|$, we have $\Delta f=(1/\rho)(d/d\rho)\rho(d/d\rho)f(\rho)$. Hence, using the above expressions, and after some manipulations, we obtain

$$\tilde{\mu}(z)=-4\left[1+\frac{g_2-12a^2}{4(\mathcal{P}-a)^2}-\frac{g(a)}{2(\mathcal{P}-a)^3}\right] \tag{6.6}$$

and similarly

$$\mathcal{K}(z)=-4\left[1+\frac{g_2-12e_3{}^2}{4(\mathcal{P}-e_3)^2}\right]. \tag{6.7}$$

Here $g(a)=4a^3-g_2a-g_3$.

Now, on the boundary of the rectangle 0, πi, $\log r+\pi i$, $\log r$, $\mathcal{P}(u)$ attains values increasing monotonically from $-\infty$ to $+\infty$ and thus $e_1<e_3<e_2$. Further, by (6.3) and (6.4), we have $e_1<e_3<a<e_2$. Moreover, since

$$e_1+e_2+e_3=0\,,$$

$$g_2=-4(e_1e_2+e_1e_3+e_2e_3)=2(e_1^2+e_2^2+e_3^2)\,,$$

it follows that $e_1<0$, $e_2<0$, $g_2>0$ and

$$g_2-12e_3^2=-4(e_3-e_1)(e_3-e_2)>0\,.$$

Also, for $g(t)=4t^3-g_2t-g_3=4(t-e_1)(t-e_2)(t-e_3)$,

$$g\left(\frac{e_2+e_3}{2}\right)=\frac{3}{2}(e_2-e_3)^2e_1<0$$

and, since $g(t)$ does not have zeros between e_3 and e_2, $g(a)<0$. It can be shown [10] that $g_2-12a^2>0$ and thus $\tilde{\mu}(z)<-4$. This led Suita [7] to conjecture that $\tilde{\mu}_D(z)\leqq-4$ for any $D\notin O_{AD}$.

We are now in a position to state:

THEOREM 4. *There exists an $r_0\in(0,1)$ such that for any $D\in\mathcal{D}_2$ with modulus $1/r\leqq1/r_0$ it holds $\tilde{\mu}_D(z)<\mathcal{K}_D(z)$.*

Proof. Let $h(r)=e_3+a$. Using (6.4) and (6.5) we find that $h(r)$ is decreasing in $r\in(0,1)$ and $h(0)=1/6$, $h(1)=-\infty$. Consequently, there is an $r_0\in(0,1)$ with $h(r_0)=0$ and $h(r)<0$ for $r>r_0$. Hence $e_3\leqq-a$ for $r\geqq r_0$. This, together with $e_3<a$, implies that $g_2-12a^2\geqq g_2-12e_3^2$. The theorem now follows from (6.6) and (6.7), and, by noting that $g(a)<0$ and $0<\mathcal{P}-a<\mathcal{P}-e_3$.

We should remark, however, that the above theorem fails for $r<r_0$. Similarly to Theorem 4 one easily shows:

PROPOSITION 2. *$\mathcal{K}(\rho)$ has only one extremal point in $(r,1)$. This extremal point is at $\sqrt{r}$ and it is a minimal point.*

By direct computation we obtain

$$\mathcal{K}(\sqrt{r})=-4\frac{e_2-e_1}{e_2-e_3}$$

and by [3, pp. 480-481]

$$\mathcal{K}(\sqrt{r})=-\frac{1}{4r}\left(\frac{\sum_{m=-\infty}^{\infty}(-1)^m r^{2m^2}}{\sum_{m=-\infty}^{\infty}(-1)^m r^{2m^2+m}}\right)^8. \tag{6.8}$$

COROLLARY 5. *Let* $D\in\mathcal{D}_2$ *with modulus* $1/r$. *Then* $\mathcal{K}(\sqrt{r})\leqq\mathcal{K}_D(z)<-4$, *where the value of* $\mathcal{K}(\sqrt{r})$ *is given by* (6.8).

Finally, consider the invariant $J(\rho)=(\tilde{M}/C)^2=1+(e_3-a)/(\mathcal{P}-e_3)$. Hence $J(\rho)<1$ (which is a well-known fact for more general domains). Again, $J(\rho)$ has only one extremal point in $(r, 1)$. This extremal point is at $\sqrt{r}$ and it is a minimal point.

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