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KODAI MATH. J.
23 (2000), 105–107

A HYPERSURFACE WHICH DETERMINES LINEARLY NON-DEGENERATE HOLOMORPHIC MAPPINGS

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§1. Introduction

In [S1] and [S2], the author gave hypersurfaces S in $\boldsymbol{P}^n(\boldsymbol{C})$ with the property:

(UA) If two algebraically non-degenerate holomorphic mappings f and g of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ have the same pull-back $f^*S = g^*S$ as a divisor, then $f = g$.

However, the minimal degree of S is of exponential order of n. In this paper, we give another hypersurface S of much lower degree with the stronger property:

(UL) If two linearly non-degenerate holomorphic mappings f and g of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ have the same pull-back $f^*S = g^*S$ as a divisor, then $f = g$.

§2. Fundamental result

We mean by a nonzero entire function an entire function with a point whose value is not zero. For two nonzero entire functions f and g, we say that they are equivalent if the ratio f/g is constant. This introduces an equivalence relation in each set of nonzero entire functions. The following theorem was given by Green [G] and Fujimoto [F]:

THEOREM A. *Let $f_0, \dots, f_n$ be nonzero entire functions such that $f_0^d + \cdots + f_n^d = 0$, where d is a positive integer. If $d \geq n^2$, then*

$$\sum_{f_j \in I} f_j^d = 0$$

for each equivalence class I. Especially each class has at least two elements.

Now, we consider a homogeneous polynomial $P(w_0, w_1)$ of degree d with the following property:

(U1) Let f and g be nonconstant holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$ with representations $\tilde{f} = (f_0, f_1)$ and $\tilde{g} = (g_0, g_1)$, respectively. If $P(f_0, f_1) =$

Received April 8, 1999.

$h^d P(g_0, g_1)$ holds for some meromorphic function h, then $f_j = \omega h g_j$ $(0 \leq j \leq n)$, where $\omega^d = 1$.

The existence of such polynomial is shown in [S2], where the minimal degree is 13.

DEFINITION. A holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ is linearly non-degenerate if its image is not contained in any hyperplane of $\boldsymbol{P}^n(\boldsymbol{C})$. This is equivalent to that $f_0, \dots, f_n$ are linearly independent over $\boldsymbol{C}$, where $(f_0, \dots, f_n)$ is a representation of f in a homogeneous coorditate system of $\boldsymbol{P}^n(\boldsymbol{C})$.

§3. Uniqueness of holomorphic mappings

For a given n, we take an integer q with $q \geq (2n-1)^2$ and define a homogeneous polynomial $P_n(w_0, \dots, w_n)$ of degree dq by

$$P_n(w_0, \dots, w_n) = P(w_0, w_1)^q + P(w_1, w_2)^q + \cdots + P(w_{n-1}, w_n)^q,$$

and so, its minimal degree is $d(2n-1)^2$ which is much smaller than d^n of the minimal degree of the polynomials in [S1] and [S2].

Then, the hypersurface defined by the zero set of P_n has the property (UL):

THEOREM. *Let f and g be linearly non-degenerate holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ with representations $\tilde{f} = (f_0, \dots, f_n)$ and $\tilde{g} = (g_0, \dots, g_n)$, respectively. If*

$$P_n(f_0, \dots, f_n) = \alpha P_n(g_0, \dots, g_n)$$

holds for an entire function α without zeros, then

$$f_j = \gamma g_j \quad (0 \leq j \leq n),$$

where $\gamma^{dq} = \alpha$.

Proof. By linear non-degeneracy, $P(f_j, f_{j+1}) \not\equiv 0$ and $P(g_j, g_{j+1}) \not\equiv 0$ and there are no equivalent pairs both in $\{P(f_j, f_{j+1}) : 0 \leq j \leq n-1\}$ and in $\{P(g_j, g_{j+1}) : 0 \leq j \leq n-1\}$. Hence, by Theorem A, there exist k_0 with $0 \leq k_0 < n$ and ω_0 with $\omega_0^q = 1$ such that

$$P(f_0, f_1) = \omega_0 \beta P(g_{k_0}, g_{k_0+1}), \tag{1}$$

where β is an entire function with $\beta^q = \alpha$. Also by Theorem A, we have

$$P(f_1, f_2) = \omega_1 \beta P(g_{k_1}, g_{k_1+1}) \tag{2}$$

for a k_1 with $0 \leq k_1 < n$, $k_1 \neq k_0$ and an ω_1 with $\omega_1^q = 1$. Fix an entire function γ with $\gamma^d = \beta$. Then, by applying (U1) to (1) and (2), there exist η_0 and η_1 with $\eta_0^d = \eta_1^d = \omega_0$ such that

$$f_0 = \eta_0 \gamma g_{k_0}, \quad f_1 = \eta_0 \gamma g_{k_0+1} \tag{3}$$

and

$$f_1 = \eta_1 \gamma g_{k_1}, \quad f_2 = \eta_1 \gamma g_{k_1+1}. \tag{4}$$

Hence, $\eta_0 g_{k_0+1} = f_1/\gamma = \eta_1 g_{k_1}$. By linear non-degeneracy of g, we get $k_0 < n-1$, $k_1 = k_0 + 1$ and $\eta_0 = \eta_1$. Therefore,

$$P(f_1, f_2) = \omega_0 \beta P(g_{k_0+1}, g_{k_0+2}).$$

is obtained. Successively, we have

$$P(f_j, f_{j+1}) = \omega_0 \beta P(g_{k_0+j}, g_{k_0+j+1}) \quad (j = 0, \ldots, n - k_0 - 1).$$

By applying (U1) to this, as above, there exist η_j with $\eta_j^q = \omega_0$ such that $f_j = \eta_j \gamma g_{k_0+j}$, $f_{j+1} = \eta_j \gamma g_{k_0+j+1}$ $(j = 0, \ldots, n - k_0 - 1)$. If $k_0 \neq 0$, then there exist m with $0 \leq m \leq k_0 - 1$ and ω' with $(\omega')^q = 1$ such that $P(f_{n-k_0}, f_{n-k_0+1}) = \omega' \beta P(g_m, g_{m+1})$, and there exists η' with $(\eta')^d = \omega'$ such that $f_{n-k_0} = \eta' \gamma g_m$, $f_{n-k_0+1} = \eta' \gamma g_{m+1}$. Hence, we get $\eta_{n-k_0-1} \gamma g_n = f_{n-k_0} = \eta' \gamma g_m$, which is a contradiction because of $n \neq m$. Therefore we conclude $k_0 = 0$ and that

$$f_j = \eta_j \gamma g_j, \quad f_{j+1} = \eta_j \gamma g_{j+1} \quad (j = 0, \ldots, n-1).$$

These imply $\eta_0 = \cdots = \eta_{n-1}$. Q.E.D.

Remark. The hypersurface given in [S2] has Kobayashi hyperbolicity. However, our hypersurface is no longer Kobayashi hyperbolic for any q if $n \geq 4$. In fact, a nonconstant holomorphic mapping $f = (\alpha : \zeta\alpha : 0 : \beta : \xi\beta : 0 : \cdots : 0)$ satisfies $f(\boldsymbol{C}) \subset S$, where α and β are entire functions linearly independent over $\boldsymbol{C}$, and ζ and ξ are constants satisfying $P(1,\zeta)^q + P(\zeta,0)^q = 0$ and $P(0,1)^q + P(1,\xi)^q = 0$, respectively. Also, it is not difficult to prove Kobayashi hyperbolicity of S for $q \geq (n-1)^2$ in the case of $n = 2, 3$.

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