

REMARK ON FOURIER COEFFICIENTS OF MODULAR FORMS
OF HALF INTEGRAL WEIGHT BELONGING TO
KOHLEN'S SPACES II

HISASHI KOJIMA

Introduction

In [13], Waldspurger first found that the square of Fourier coefficients $a(n)$ at a square free integer n of modular forms $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$ of half integral weight is essentially proportional to the central value of the zeta function at a certain integer attached to the modular form F if f corresponds to F by the Shimura correspondence Ψ and f is an eigen-function of Hecke operators. Kohnen-Zagier [1], [3] determined explicitly the constant of the proportionality in the case of modular forms belonging to Kohnen's spaces $S_{(2k+1)/2}(N, \chi)$ of weight $(2k+1)/2$ and of square free level N with character χ which is a subspace of $S_{(2k+1)/2}(4N, \chi_1)$, where $S_{(2k+1)/2}(4N, \chi_1)$ means the space of modular cusp forms of half integral weight given in [9]. Kohnen-Zagier [1] (resp. Kohnen [3]) treated the case where $N = 1$ (resp. N is an odd square free integer and χ is the trivial character of level N) (cf. Kojima [4] and [5]).

In [12], Shimura intended to generalize such formulas to the case of Hilbert modular forms f of half integral weight and succeeded in obtaining many general interesting formulas. Among these, some explicit and useful formulas about the proportionality constant were formulated under assumptions that f satisfies the multiplicity one theorem. For modular forms belonging to Kohnen's spaces $S_{(2k+1)/2}(N, \chi)$, Shimura [12] did not give the same explicit formula as that of Kohnen and Zagier [1], [3].

In [6], we represented explicitly the ratio of the square of Fourier coefficients $a(n)$ at a fundamental discriminant n of modular forms $f(z) = \sum_{\varepsilon(-1)^k n \equiv 0, 1(4), n > 0} a(n)e[nz]$ of weight $(2k+1)/2$ belonging to the Kohnen's space $S_{(2k+1)/2}(N, \chi)$ by the central value of the zeta function of the modular form F which is the image of f under the Shimura correspondence Ψ , where χ is any primitive character modulo N and n is equal to 4τ with a square free integer τ satisfying $\tau \equiv 2, 3 \pmod{4}$.

The purpose of this paper is to remove these conditions about χ and n and to

derive a generalization of results in [6] in the case where N is any odd integer, χ is an arbitrary primitive character modulo N and n is an arbitrary positive fundamental discriminant. Though the method of our proof is more complicated, it is the same as that of [6]. To obtain our results we need to modify slightly its method.

Section 0 is a preliminary section. In Section 1, we shall summarize some results about Kohnen's spaces, the Shimura correspondence, theta functions and Hecke operators of Kohnen's spaces. In Section 2, using these and the assumption (1-12), we show that $\langle \Theta(z, w; \zeta_Z), g(2w) \rangle = C_\tau \overline{L_\tau(f)}(z) + D_\tau \overline{L_\tau(f)} | U(4)(z)$ for a modular form $f \in S_{(2k+1)/2}(N, \chi)$ with some constants C_τ and D_τ , where $\Theta(z, w; \zeta_Z)$ is a theta function, $g(w)$ is an element in $S_{2k}(N, \chi^2)$ determined by f under the Shimura correspondence Ψ , $U(4)$ is the operator given by $U(4)(\sum_{n=1}^{\infty} b(n)e[nz]) = \sum_{n=1}^{\infty} b(4n)e[nz]$ and $L_\tau(f)(z) = f(\tau z)\tau^k$. Moreover, executing an explicit and complicate computation of the image $\Psi(f)$ of f under the Shimura correspondence Ψ , we shall verify that $C_\tau = \overline{a(n)}\tilde{c}_1$ and $D_\tau = \overline{a(n)}\tilde{c}_2$ with a fundamental discriminant n , where \tilde{c}_1 and \tilde{c}_2 are explicitly calculable constants. These computations are essential parts for our arguments and these formulas play an important role for our later treatments. In Section 3, using integral formulas of Section 2, the computation of the image of a product of theta series and Eisenstein series by Shimura correspondence and the method of the Rankin's convolution, under some assumptions on f , we shall derive an explicit connection between the square of Fourier coefficients $a(n)$ at an arbitrary positive fundamental discriminant n of a modular form $f(z) = \sum_{\substack{e(-1)^k n \equiv 0, 1(4), n > 0 \\ a(n)e[nz]}}$ belonging to Kohnen's spaces $S_{(2k+1)/2}(N, \chi)$ of half integral weight $(2k+1)/2$ and of an odd level N with arbitrary character χ and the central value of zeta functions associated with the image $\Psi(f)$ of f by the Shimura correspondence Ψ .

We mention that our results give a generalization of some results in Kohnen-Zagier [1] and [3] and Kojima [4], [5] and [6] and our method can be applicable to the case of Maass wave forms of half integral weight (cf. [7]).

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§0. Notation and preliminaries

We denote by \mathbf{Z} , \mathbf{Q} , \mathbf{R} and \mathbf{C} the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For $z \in \mathbf{C}$, we put $e[z] = \exp(2\pi iz)$ and we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg z^{1/2} \leq \pi/2$. Further, we set $z^{k/2} = (\sqrt{z})^k$ for every $k \in \mathbf{Z}$. Let $SL(2, \mathbf{R})$ denote the group of all real matrices of degree 2 with determinant one and \mathfrak{H} the complex upper half plane, i.e.,

$$SL(2, \mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c \text{ and } d \in \mathbf{R} \text{ and } ad - bc = 1 \right\}$$

and

$$\mathfrak{H} = \{z = x + iy \mid x, y \in \mathbf{R} \text{ and } y > 0\}.$$

Define an action of $SL(2, \mathbf{R})$ on \mathfrak{H} by

$$z \rightarrow \gamma(z) = \frac{az + b}{cz + d} \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}) \text{ and for all } z \in \mathfrak{H}.$$

For positive integers M and M' , put

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}) \mid a, b, c \text{ and } d \in \mathbf{Z} \text{ and } c \equiv 0 \pmod{M} \right\}$$

and

$$\Gamma[M, M'] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M') \mid b \equiv 0 \pmod{M} \right\}.$$

We introduce an automorphic factor $j_0(\gamma, z)$ of $\Gamma_0(4)$ determined by

$$j_0(\gamma, z) = \mathfrak{g}_0(\gamma(z))/\mathfrak{g}_0(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \quad \text{and}$$

$$\text{for every } z \in \mathfrak{H} \quad \text{with } \mathfrak{g}_0(z) = \sum_{n=-\infty}^{\infty} e[n^2 z] \text{ and } \mathfrak{g}(z) = \mathfrak{g}_0(z/2).$$

§1. Shimura correspondences of modular forms of half integral weight and theta functions

This section is devoted to summarizing several fundamental facts which we need later. Let k be a positive integer. Let N denote a positive integer and ψ_0 a Dirichlet character modulo $4N$. We denote by $S_{(2k+1)/2}(4N, \psi_0)$ the set of all cusp forms f on \mathfrak{H} such that

$$(1-1) \quad f(\gamma(z)) = \psi_0(d)j_0(\gamma, z)^{2k+1}f(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N).$$

Throughout the rest of the paper we assume that N is an odd integer. Let χ be a Dirichlet character modulo N such that $\chi(-1) = \varepsilon$. Put $\chi_1 = \left(\frac{4\varepsilon}{*}\right)\chi$. We introduce a subspace $S_{(2k+1)/2}(N, \chi)$ of $S_{(2k+1)/2}(4N, \chi_1)$ defined by

(1-2)

$$S_{(2k+1)/2}(N, \chi) = \left\{ f(z) \in S_{(2k+1)/2}(4N, \chi_1) \mid f(z) = \sum_{\varepsilon(-1)^k n \equiv 0, 1(4), n > 0} a(n)e[nz] \right\}.$$

We call $S_{(2k+1)/2}(N, \chi)$ the Kohnen's space of weight $(2k+1)/2$ and of level N with character χ . We denote by $S_{2k}(M, \omega)$ the space of all cusp forms f' of

weight $2k$ and of level M with character ω satisfying

$$f'(\gamma(z)) = \omega(d)(cz + d)^{2k} f'(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M).$$

Here we recall the notation and results in Shimura [12]. Let \mathfrak{b} and \mathfrak{b}' denote integral ideals of \mathcal{Q} and ψ a Hecke character of \mathcal{Q} whose conductor divides $4\mathfrak{b}\mathfrak{b}'$. Let $\mathcal{M}_{(2k+1)/2}(\mathfrak{b}, \mathfrak{b}'; \psi)$ (resp. $\mathcal{S}_{(2k+1)/2}(\mathfrak{b}, \mathfrak{b}'; \psi)$) be the space of modular forms (resp. modular cusp forms) of half integral weight $(2k+1)/2$ given in [12, p. 507]. Let ψ_0 be a Dirichlet character modulo $4N$ such that $\psi_0(-1) = 1$. We choose the Hecke character ψ of \mathcal{Q} such that

$$(1-3) \quad \prod_{p|4N} \psi_p(a) = \left(\frac{-1}{a}\right)^k \psi_0(a)^{-1} \quad \text{for every } a \in (\mathbf{Z}/4N\mathbf{Z})^\times, \psi_p(\mathbf{Z}_p^\times) = 1$$

for every $p \nmid 4N$ and $\psi_a(x) = (\text{sgn}(x))^k (x \in \mathbf{R})$.

For $f' \in \mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi)$, put $L(f')(z) = f'(2z)$. Then the following mapping is bijective (cf. [12, p. 523]).

$$(1-4) \quad L : \mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi) \rightarrow \mathcal{S}_{(2k+1)/2}(4N, \psi_0).$$

Let t denote a positive square free integer. Define a mapping $\Psi_{(2k+1)/2, t}^{4N, \psi_0}$ of $\mathcal{S}_{(2k+1)/2}(4N, \psi_0)$ into $\mathcal{S}_{2k}(2N, \psi_0^2)$ by

$$(1-5) \quad \Psi_{(2k+1)/2, t}^{4N, \psi_0}(f)(z) = \sum_{m=1}^{\infty} \left(\sum_{d|m} \psi_0(d) \left(\frac{-1}{d}\right)^k \left(\frac{t}{d}\right) d^{k-1} a(t(m/d)^2) \right) e[mz]$$

for every $f(z) = \sum_{n=1}^{\infty} a(n)e[nz] \in \mathcal{S}_{(2k+1)/2}(4N, \psi_0)$. The existence of this mapping was first shown by Shimura [9] and it was reformulated by Niwa [8].

Next we recall the definition of theta functions given in [12]. Put

$$V = \left\{ \xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{Q}) \mid \text{tr } \xi = 0 \right\} \quad \text{and}$$

$\mathcal{S}(V) = \{\eta : V \rightarrow \mathbf{C} \mid \eta \text{ is a locally constant function in the sense of [12]}\}$.

Let η be an element of $\mathcal{S}(V)$. Define a theta function $\Theta(z, w; \eta)$ on $\mathfrak{H} \times \mathfrak{H}$ by

$$(1-6) \quad \Theta(z, w; \eta) = y^{1/2} (\Im w)^{-2k} \sum_{\xi \in V} \eta(\xi) [\xi, \bar{w}]^k e[2^{-1} R[\xi, z, w]]$$

for every $(z, w) \in \mathfrak{H} \times \mathfrak{H}$, where

$$[\xi, w] = [\xi, w, w], [\xi, w, w'] = (cww' + dw - aw' - b) \quad \text{and}$$

$$R[\xi, z, w] = (\det \xi)z + \frac{iy}{2} \Im w^{-2} |[\xi, w]|^2 \left(\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V, y = \Im z \right).$$

Throughout the rest of the paper, we take a Dirichlet character ψ_0 with $\psi_0(-1) = 1$ and a positive square free integer τ such that

$$(1-7) \quad (\tau, N) = 1.$$

Put $\varphi = \psi\varepsilon_\tau$ with the Hecke character ε_τ associated with the quadratic field $\mathcal{Q}(\sqrt{\tau})$. For positive rational numbers α and β , put

$$\mathfrak{o}[\alpha, \beta] = \left\{ x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in M_2(\mathcal{Q}) \mid a_x \in \mathbf{Z}, b_x \in \alpha\mathbf{Z}, c_x \in \beta\mathbf{Z} \text{ and } d_x \in \mathbf{Z} \right\}.$$

Moreover, we put $e = 2N$. We consider an element $\eta \in \mathcal{S}(V)$ determined by

$$(1-8) \quad \eta(x) = \begin{cases} 0 & \text{if } x \notin \mathfrak{o}[e^{-1}, e], \\ \sum_{t \in (1/2\tau)\mathbf{Z}/2N\mathbf{Z}} \varphi_a(t) \varphi^*((2t\tau)) e[-b_x t] & \text{otherwise,} \end{cases}$$

where φ^* is the ideal character associated with φ (cf. [12, p. 505]). Introduce a mapping L_τ of $\mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi)$ into $\mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\tau\mathbf{Z}; \psi\varepsilon_\tau)$ defined by

$$(1-9) \quad L_\tau(f)(z) = f(\tau z) \tau^k \quad \text{for every } f(z) \in \mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi).$$

We put $h(z) = L_\tau(f)(z)$. The following lemma is proved by Shimura [12, Propositions 1.4 and 1.5].

LEMMA 1.1. *The notation being as above, the mapping L_τ gives a bijection of $\mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi)$ onto the set*

$$\left\{ h(z) = \sum_{n=1}^{\infty} a'(n) e[nz/2] \in \mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\tau\mathbf{Z}, \psi\varepsilon_\tau) \mid a'(n) = 0 \text{ if } \tau \nmid n \right\}.$$

Moreover,

$$\langle f, f \rangle = \tau^{-(2k+1)/2} \tau \langle h, h \rangle,$$

where for given two cusp forms f, g of weight l with respect to Γ , their inner product $\langle f, g \rangle$ means

$$\langle f, g \rangle = \text{vol}(\Gamma \backslash \mathfrak{H})^{-1} \int_{\Gamma \backslash \mathfrak{H}} \overline{f(z)} g(z) \Im z^l d_{\mathfrak{H}} z \quad \text{with } d_{\mathfrak{H}} z = \frac{dx dy}{y^2} \quad (x = \Re z, y = \Im z).$$

The following proposition is verified by Shimura [10] and [12] (cf. Niwa [8]).

PROPOSITION 1.1. *For $f(z) = \sum_{n=1}^{\infty} a(n) e[nz/2] \in \mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi)$, put $h(z) = L_\tau(f)(z)$. Suppose that ψ_0, τ and ψ satisfy the condition (1-7) and η is the function on V determined by (1-8). Then there exists the element*

$$g_\tau(w) = \sum_{m=1}^{\infty} \left(\sum_{d|m} \psi_0(d) \left(\frac{-1}{d} \right)^k \left(\frac{\tau}{d} \right) d^{k-1} a(\tau(m/d)^2) \right) e[mw]$$

belonging to $S_{2k}(2N, \psi_0^2)$ such that

$$c' g_\tau(w) = \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} h(z) \Theta(z, w; \eta) y^{(2k+1)/2} d_{\mathfrak{H}} z \text{ with } c' = 2^{2+k} i^k 4\tau N.$$

Next we describe basic results of Hecke operators of Kohnen's space. Let χ denote a Dirichlet character modulo N . We denote by $T_{(2k+1)/2, \chi_1}^{4N}(p^2)$ the Hecke operator on $S_{(2k+1)/2}(4N, \chi_1)$ given in [9]. For a positive integer m , we define a function $f|U(m)$ on \mathfrak{H} by

$$f|U(m)(z) = \sum_{n=0}^{\infty} a(mn) e[nz]$$

for every function $f(z) = \sum_{n=0}^{\infty} a(n) e[nz]$ on \mathfrak{H} . We denote by $T_{(2k+1)/2, N, \chi}(p)$ ($p \nmid N$) the Hecke operator on $S_{(2k+1)/2}(N, \chi)$ defined in [2, p. 42]. The following lemma was confirmed by Kohnen [2] (see also Remark in [2, p. 46]).

LEMMA 1.2. *The notation being as above, for $f(z) = \sum_{n \geq 1, \varepsilon(-1)^k n \equiv 0, 1(4)} a(n) e[nz] \in S_{(2k+1)/2}(N, \chi)$, Fourier expansions of $f|T_{(2k+1)/2, N, \chi}(p)(z)$ ($p \nmid N$) and $f|U(p^2)(z)$ ($p|N$) are given as follows:*

$$f|T_{(2k+1)/2, N, \chi}(p)(z) = \sum_{n \geq 1, \varepsilon(-1)^k n \equiv 0, 1(4)} \left(a(p^2 n) + \chi(p) \left(\frac{\varepsilon(-1)^k n}{p} \right) p^{k-1} a(n) + \chi^2(p) p^{2k-1} a(n/p^2) \right) e[nz] \quad (p \nmid N)$$

and

$$f|U(p^2)(z) = \sum_{n \geq 1, \varepsilon(-1)^k n \equiv 0, 1(4)} a(p^2 n) e[nz] \quad (p|N),$$

where $a(n/p^2)$ means 0 if $p^2 \nmid n$.

Observe that $T_{(2k+1)/2, N, \chi}(p)$ coincides with the restriction of $T_{(2k+1)/2, \chi_1}^{4N}(p^2)$ to $S_{(2k+1)/2}(N, \chi)$ for every odd prime p ($p \nmid N$). Now we impose the following assumption.

(1-10) $\varepsilon = \chi(-1)$ satisfies $(-1)^k \varepsilon > 0$, χ is a Dirichlet character modulo N ,

$f \in S_{(2k+1)/2}(N, \chi)$ is an eigenfunction of all Hecke operators

$$T_{(2k+1)/2, N, \chi}(p) \quad (p \nmid N) \quad \text{and} \quad U(p^2) \quad (p|N), \text{ i.e.,}$$

$$f|T_{(2k+1)/2, N, \chi}(p)(z) = \omega(p) f(z) \quad (p \nmid N) \quad \text{and} \quad f|U(p^2)(z) = \omega(p) f(z) \quad (p|N)$$

and $a(4\tau_0) \neq 0$ for some square free positive integer τ_0 such that $\tau_0 \equiv 2, 3(\text{mod } 4)$.

Since $S_{(2k+1)/2}(N, \chi)$ is contained in $S_{(2k+1)/2}(4N, (\frac{4\varepsilon}{*})\chi)$, we have the following diagram.

$$(1-11) \quad S_{(2k+1)/2}(N, \chi) \subset S_{(2k+1)/2}(4N, \psi_0) \xrightarrow{L^{-1}} \mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi)$$

$$\text{with } \psi_0 = \left(\frac{4\varepsilon}{*}\right)\chi.$$

By this relation, we may identify elements $f(z)$ of $S_{(2k+1)/2}(N, \chi)$ with those of $\mathcal{S}_{(2k+1)/2}(\mathbf{Z}, N\mathbf{Z}; \psi)$. In [6], we verified the following lemma.

LEMMA 1.3. *Let $f(z) = \sum_{n \geq 1, \varepsilon(-1)^k n \equiv 0, 1(4)} a(n)e[nz]$ be an element of $S_{(2k+1)/2}(N, \chi)$ satisfying the condition (1-10) and let τ be an integer satisfying the condition that (1-7) and $\tau \equiv 2, 3 \pmod{4}$. Then*

$$\Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\varepsilon}{*})}(f)(w) = \sum_{m=1}^{\infty} \left(\sum_{d|m} \left(\frac{4\tau}{d}\right) \chi(d) d^{k-1} a(\tau(m/d)^2) \right) e[mw]$$

coincides with an element $a(4\tau)g(2w)$ of $S_{2k}(2N, \chi^2)$, where $g(w) = \sum_{n=1}^{\infty} c(n)e[nw]$ belongs to $S_{2k}(N, \chi^2)$ and

$$\sum_{n=1}^{\infty} c(n)n^{-s} = \prod_p (1 - \omega(p)p^{-s} + \chi^2(p)p^{2k-1-2s})^{-1}.$$

It is well known that $f|U(4)(z)$ belongs to $S_{(2k+1)/2}(4N, \chi(\frac{4\varepsilon}{*}))$ for every $f(z) \in S_{(2k+1)/2}(4N, \chi(\frac{4\varepsilon}{*}))$. Here we impose the further condition on f in (1-10).

$$(1-12) \quad \text{If } f' \in S_{(2k+1)/2}\left(4N, \chi\left(\frac{4\varepsilon}{*}\right)\right) \text{ and } f'|T_{(2k+1)/2, \chi(\frac{4\varepsilon}{*})}^{4N}(p^2) = \omega(p)f'$$

for every prime $p(p \nmid 4\tau N)$, then $f'(z)$ equals

$$\tilde{c}f(z) + \tilde{c}'f|U(4)(z)$$

for some constants \tilde{c} and \tilde{c}' and, if $f'' \in S_{(2k+1)/2}\left(4M, \chi\left(\frac{4\varepsilon}{*}\right)\right)$

$$(M|N, M \neq N) \text{ satisfies } f''|T_{(2k+1)/2, \chi(\frac{4\varepsilon}{*})}^{4M}(p^2) = \omega(p)f'' \ (p \nmid 4N\tau),$$

then $f''(z) = 0$ moreover, $4\tau|\mathfrak{h}$, $(\mathfrak{h}^{-1}4N\tau, \tau) = 1$ and $8 \nmid \mathfrak{h}$ if $2 \nmid \tau$, where $\omega(p)$ is the eigenvalue given in Lemma 1.3 and \mathfrak{h} is the conductor of $\varphi = \psi\varepsilon_\tau$ with ψ in (1-11). Moreover, we impose the condition that

$$(1-13) \quad g(w) \text{ is a primitive form of } S_{2k}(N, \chi^2) \text{ and } f|U(4)(z)$$

is not a constant times $f(z)$,

where $f(z)$ and $g(w)$ are the same elements given in Lemma 1.3. Furthermore,

we consider the condition that

$$(1-14) \quad \text{If } 2|\tau, \text{ then } \varphi_2(1+4x) = \varphi_2(1+4x^2) \quad \text{for every } x \in \mathbf{Z}_2,$$

where φ_2 is the restriction of φ to \mathbf{Q}_2^\times and \mathbf{Z}_2 (resp. \mathbf{Q}_2) means the ring of all 2-adic integers (numbers).

§2. Some formulas of theta integrals

This section is devoted to confirming a key proposition concerning theta integrals. For a positive integer a , we define elements $\zeta^{a\mathbf{Z}}$ and $\zeta_{a\mathbf{Z}}$ of $\mathcal{S}(V)$ by

$$(2-1) \quad \zeta^{a\mathbf{Z}}(x) = \begin{cases} \bar{\varphi}_a(b_x)\bar{\varphi}^*((b_x a^{-1}e)) & \text{if } x \in \mathfrak{o}[ae^{-1}, e], \\ 0 & \text{otherwise} \end{cases}$$

and

$$\zeta_{a\mathbf{Z}}(x) = \begin{cases} \bar{\varphi}_a(b_x)\bar{\varphi}^*((b_x a^{-1}e)) & \text{if } x \in \mathfrak{o}[ae^{-1}, e] \text{ and } (b_x a^{-1}e, 4N\tau) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By the same method as that of Shimura [12, Lemma 5.7], we may derive the following lemma (cf. [11]).

LEMMA 2.1. *Suppose that N is odd, τ is a positive square free integer and the conditions (1-7), (1-10), (1-12), (1-13) and (1-14) are satisfied. Let a be a positive integer such that $a\mathfrak{h}|4N\tau$ and $\text{ord}_p(a\mathfrak{h}) = \text{ord}_p(4N\tau)$ for every $p|\tau$. Then there are constants M and M' such that*

$$\langle \Theta(z, w; \zeta^{a\mathbf{Z}}), g(2w) \rangle = Mh(z) + M'h|U(4)(z) \quad \text{with } h(z) = L_\tau(f)(z),$$

where $g(w)$ is the same function given in Lemma 1.3, \mathfrak{h} is the conductor of $\varphi (= \psi\varepsilon_\tau)$ and $\text{ord}_p(d)$ is the p -adic order of an integer d . Moreover, the above integral is equal to 0 if $4\tau|l.c.m.(a\mathfrak{h}, 4)$ and $l.c.m.(a\mathfrak{h}, 4) \neq 4N\tau$.

By virtue of Proposition 1.1, Lemma 1.3 and Lemma 2.1, we may conclude the following proposition.

PROPOSITION 2.1. *Suppose that the assumption in Lemma 2.1 is satisfied. Then we have*

$$A_\tau h_\tau(z) + B_\tau h_\tau|U(4)(z) = \langle \Theta(z, w; \eta), g(2w) \rangle$$

and

$$C_\tau h_\tau(z) + D_\tau h_\tau|U(4)(z) = \langle \Theta(z, w; \zeta_{\mathbf{Z}}), g(2w) \rangle$$

with

$$A_\tau = \frac{\overline{a(4\tau)} \overline{c'}}{\text{vol}(\Gamma[2, 2N\tau] \backslash \mathfrak{H})} \tau^{-(2k+1)/2+1} \\ \times \frac{\langle g(2w), g(2w) \rangle \langle f', f' \rangle - \langle g(w), g(2w) \rangle \langle f, f' \rangle}{\langle f, f \rangle \langle f', f' \rangle - \langle f, f' \rangle \langle f', f \rangle} \text{ if } \tau \equiv 2, 3(4),$$

$$B_\tau = \frac{\overline{a(4\tau)} \overline{c'}}{\text{vol}(\Gamma[2, 2N\tau] \backslash \mathfrak{H})} \tau^{-(2k+1)/2+1} \\ \times \frac{\langle g(w), g(2w) \rangle \langle f, f \rangle - \langle g(2w), g(2w) \rangle \langle f', f \rangle}{\langle f, f \rangle \langle f', f' \rangle - \langle f, f' \rangle \langle f', f \rangle} \text{ if } \tau \equiv 2, 3(4),$$

$$C_\tau = \mu(\mathfrak{h}^{-1}4N\tau)\varphi^*(\mathfrak{h}^{-1}4N\tau)(\varphi_a(-1)\overline{\gamma(\varphi)}\mathfrak{h}^{-1}4N\tau)^{-1}A_\tau \text{ if } \tau \equiv 2, 3(4),$$

$$D_\tau = \mu(\mathfrak{h}^{-1}4N\tau)\varphi^*(\mathfrak{h}^{-1}4N\tau)(\varphi_a(-1)\overline{\gamma(\varphi)}\mathfrak{h}^{-1}4N\tau)^{-1}B_\tau \text{ if } \tau \equiv 2, 3(4),$$

$$A_\tau = \overline{a(\tau)}F_\tau, B_\tau = \overline{a(\tau)}G_\tau \text{ if } \tau \equiv 1(4)$$

$$F_\tau = \overline{c'} \frac{\text{vol}(\Gamma[2, 2\tau N] \backslash \mathfrak{H})^{-1}}{\langle f, f \rangle \langle f', f' \rangle - \langle f, f' \rangle \langle f', f \rangle} \tau^{-(2k+1)/2+1} \\ \times \left(\langle f', f' \rangle \left(\langle g(w), g(2w) \rangle - \left(\frac{\tau}{2} \right) \overline{\chi(2)} 2^{k-1} \langle g(2w), g(2w) \rangle \right) \right. \\ \left. - \langle f, f' \rangle \left(\left(\overline{\omega(2)} - 2^{k-1} \overline{\chi(2)} \left(\frac{\tau}{2} \right) \right) \right. \right. \\ \left. \left. \times \langle g(w), g(2w) \rangle - \overline{\chi(2^2)} 2^{2k-1} \langle g(2w), g(2w) \rangle \right) \right)$$

$$G_\tau = \overline{c'} \frac{\text{vol}(\Gamma[2, 2\tau N] \backslash \mathfrak{H})^{-1}}{\langle f, f \rangle \langle f', f' \rangle - \langle f, f' \rangle \langle f', f \rangle} \tau^{-(2k+1)/2+1} \\ \times \left(- \langle f', f \rangle \left(\langle g(w), g(2w) \rangle - \left(\frac{\tau}{2} \right) \overline{\chi(2)} 2^{k-1} \langle g(2w), g(2w) \rangle \right) \right. \\ \left. + \langle f, f \rangle \left(\left(\overline{\omega(2)} - 2^{k-1} \overline{\chi(2)} \left(\frac{\tau}{2} \right) \right) \right. \right. \\ \left. \left. \times \langle g(w), g(2w) \rangle - \overline{\chi(2^2)} 2^{2k-1} \langle g(2w), g(2w) \rangle \right) \right)$$

$$C_\tau = \mu(\mathfrak{h}^{-1}4N\tau)\varphi^*(\mathfrak{h}^{-1}4N\tau)(\varphi_a(-1)\overline{\gamma(\varphi)}\mathfrak{h}^{-1}4N\tau)^{-1}A_\tau \text{ if } \tau \equiv 1(4),$$

$$D_\tau = \mu(\mathfrak{h}^{-1}4N\tau)\varphi^*(\mathfrak{h}^{-1}4N\tau)(\varphi_a(-1)\overline{\gamma(\varphi)}\mathfrak{h}^{-1}4N\tau)^{-1}B_\tau \text{ if } \tau \equiv 1(4),$$

where η is the same element given in (1-8), $h_\tau(z) = L_\tau(f)(z)$, $h'_\tau(z) = h_\tau|U(4)(z)$, $f' = f|U(4)$ and $\gamma(\varphi)$ means the Gauss sum of φ .

Proof. If $\tau \equiv 2, 3 \pmod{4}$, we can verify in a manner similar to that of [6]. So we may assume that $\tau \equiv 1 \pmod{4}$. We see that

$$(2-2) \quad \Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\tau}{*})}(f)(w) \\ = \sum_{m=1}^{\infty} \left(\sum_{d|m} \chi(d) \left(\frac{4}{d}\right) \left(\frac{\tau}{d}\right) d^{k-1} a(\tau(m/d)^2) \right) e[mw] \in S_{2k}(2N, \chi^2) \\ T_{2k, \chi^2}^{2N}(p) (\Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\tau}{*})}(f)) = \omega(p) \Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\tau}{*})}(f) \quad \text{and} \\ T_{2k, \chi^2}^{2N}(p)g = \omega(p)g \quad \text{for every prime } p \ ((2N, p) = 1),$$

where $T_{2k, \chi^2}^{2N}(p)$ is Hecke operators on $S_{2k}(2N, \chi^2)$. Since g is a primitive form of $S_{2k}(N, \chi^2)$, there are constants c_1 and c_2 such that

$$(2-3) \quad \Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\tau}{*})}(f)(w) = c_1 g(w) + c_2 g(2w).$$

Therefore we obtain

$$(2-4) \quad L\left(s - k + 1, \chi\left(\frac{4}{*}\right)\left(\frac{\tau}{*}\right)\right) \sum_{n=1}^{\infty} a(\tau n^2) n^{-s} = (c_1 + c_2 2^{-s}) L(s, g)$$

with $L(s, g) = \sum_{n=1}^{\infty} c(n) n^{-s}$, where $L(s, \chi(\frac{4}{*})(\frac{\tau}{*}))$ is the Dirichlet L -function associated with $\chi(\frac{4}{*})(\frac{\tau}{*})$. By Lemma 1.2 and the condition (1-10), we have

$$(2-5) \quad L\left(s - k + 1, \chi\left(\frac{\tau}{*}\right)\right) \sum_{n=1}^{\infty} a(\tau n^2) n^{-s} \\ = a(\tau) \prod_p (1 - \omega(p) p^{-s} + \chi(p^2) p^{2k-1-2s})^{-1}$$

(cf. [7, p. 452 and p. 453]). Comparing 2-factors of Dirichlet series (2-4) and (2-5), we can check

$$(2-6) \quad (c_1 + c_2 2^{-s})(1 - \omega(2) 2^{-s} + \chi(2^2) 2^{2k-1-2s})^{-1} \\ = a(\tau) \left(1 - \chi(2) \left(\frac{\tau}{2}\right) 2^{-s+k-1}\right) (1 - \omega(2) 2^{-s} + \chi(2^2) 2^{2k-1-2s})^{-1},$$

which yields that

$$c_1 = a(\tau) \quad \text{and} \quad c_2 = -\left(\frac{\tau}{2}\right) \chi(2) 2^{k-1} a(\tau).$$

Therefore we may justify

$$(2-7) \quad \Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\epsilon}{\tau})}(f)(w) = a(\tau)g(w) - \left(\frac{\tau}{2}\right)\chi(2)2^{k-1}a(\tau)g(2w).$$

Put $\tilde{f}(z) = f|U(4)(z)$. Then we see

$$(2-8) \quad \Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\epsilon}{\tau})}(\tilde{f})(w) = \sum_{m=1}^{\infty} \left(\sum_{d|m} \chi(d) \left(\frac{4}{d}\right) \left(\frac{\tau}{d}\right) d^{k-1} a(4\tau(m/d)^2) \right) e[mw].$$

So we can put

$$(2-9) \quad \Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\epsilon}{\tau})}(\tilde{f})(w) = c'_1 g(w) + c'_2 g(2w) \quad \text{for some constants } c'_1 \text{ and } c'_2.$$

Next we shall determine explicitly c'_1 and c'_2 . For this purpose we need to compute the Dirichlet series

$$(2-10) \quad L\left(s - k + 1, \chi\left(\frac{4}{*}\right)\left(\frac{\tau}{*}\right)\right) \sum_{n=1}^{\infty} a(4\tau n^2) n^{-s}.$$

Consider a power series $H_n(x)$ of x defined by

$$(2-11) \quad H_n(x) = \sum_{m=0}^{\infty} a(\tau 2^{2(m+1)} n^2) x^m.$$

By the definition of $H_n(x)$, we obtain

$$(2-12) \quad \begin{aligned} xH_n(x) &= \sum_{m=0}^{\infty} a(\tau 2^{2m} n^2) x^m - a(\tau n^2) \\ &= a(\tau n^2) \left(1 - 2^{k-1} \chi(2) \left(\frac{\tau}{2}\right) x \right) \\ &\quad \times (1 - \omega(2)x + \chi(2^2) 2^{2k-1} x^2)^{-1} - a(\tau n^2) \end{aligned}$$

for every n ($2 \nmid n$), which shows that

$$(2-13) \quad H_n(x) = a(\tau n^2) \frac{\omega(2) - 2^{k-1} \chi(2) \left(\frac{\tau}{2}\right) - \chi(2^2) 2^{2k-1} x}{1 - \omega(2)x + \chi(2^2) 2^{2k-1} x^2} \quad \text{for every } n \text{ ($2 \nmid n$).$$

Since

$$\sum_{n=1}^{\infty} a(4\tau n^2) n^{-s} = \sum_{(n,2)=1} H_n(2^{-s}) n^{-s},$$

we find

$$c'_1 = a(\tau) \left(\omega(2) - 2^{k-1} \chi(2) \left(\frac{\tau}{2}\right) \right) \quad \text{and} \quad c'_2 = a(\tau) (-\chi(2^2) 2^{2k-1}).$$

Therefore we conclude that

(2-14)

$$\Psi_{(2k+1)/2, \tau}^{4N, \chi(\frac{4\mathfrak{z}}{\tau})}(\tilde{f})(w) = a(\tau) \left(\left(\omega(2) - 2^{k-1} \chi(2) \left(\frac{\tau}{2} \right) \right) g(w) - \chi(2^2) 2^{2k-1} g(2w) \right).$$

For a positive integer a , we put

$$I(a) = \langle \Theta(z, w; \zeta^{aZ}), g(2w) \rangle.$$

By [12, p. 539], (1-12) and Lemma 2.1, we have

$$(2-15) \quad \begin{aligned} \langle \Theta(z, w; \zeta_Z), g(2w) \rangle &= \mu(\mathfrak{h}^{-1}4N\tau) \varphi^*(\mathfrak{h}^{-1}4N\tau) I(\mathfrak{h}^{-1}4N\tau) \\ &= Ch_\tau(z) + Dh'_\tau(z) \end{aligned}$$

and

$$\begin{aligned} \langle \Theta(z, w; \eta), g(2w) \rangle &= \varphi_a(-1) \overline{\gamma(\varphi)}(\mathfrak{h}^{-1}4N\tau) I(\mathfrak{h}^{-1}4N\tau) \\ &= Ah_\tau(z) + Bh'_\tau(z) \end{aligned}$$

for some constants A, B, C and D . We shall determine explicitly these values. By (2-7), we may check

$$(2-16) \quad \begin{aligned} \langle h_\tau(z), \langle \Theta(z, w; \eta), g(2w) \rangle \rangle &= (A \langle h_\tau, h_\tau \rangle + B \langle h_\tau, h'_\tau \rangle) \\ &= \text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})^{-1} \text{vol}(\Gamma[2, 2\tau N] \backslash \mathfrak{H})^{-1} \\ &\quad \times \int_{\Gamma[2\tau, 4N] \backslash \mathfrak{H}} \left\{ \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} \overline{h_\tau(z) \Theta(z, w; \eta) \mathfrak{I}_z^{(2k+1)/2} d_{\mathfrak{H}} z} \right\} g(2w) \mathfrak{I}_w^{2k} d_{\mathfrak{H}} w \\ &= \text{vol}(\Gamma[2, 2\tau N] \backslash \mathfrak{H})^{-1} \overline{c'a(\tau)} (\langle g(w), g(2w) \rangle \\ &\quad - \overline{\left(\frac{\tau}{2} \right) \chi(2) 2^{k-1}} \langle g(2w), g(2w) \rangle). \end{aligned}$$

By (2-14), we may also justify that

$$(2-17) \quad \begin{aligned} A \langle h'_\tau, h_\tau \rangle + B \langle h'_\tau, h'_\tau \rangle &= \text{vol}(\Gamma[2, 2\tau N] \backslash \mathfrak{H})^{-1} \overline{c'a(\tau)} \left(\left(\omega(2) - 2^{k-1} \chi(2) \left(\frac{\tau}{2} \right) \right) \right. \\ &\quad \left. \langle g(w), g(2w) \rangle - \overline{\chi(2^2) 2^{2k-1}} \langle g(2w), g(2w) \rangle \right). \end{aligned}$$

Therefore, by (2-15), (2-16) and (2-17), we conclude our assertions.

§3. Rankin's convolution of theta series, Eisenstein series and final calculation

Put

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e[n^2 z/2] \quad \text{and} \quad L_M(s, \omega) = \sum_{n=1}^{\infty} \omega^*(n\mathbf{Z})n^{-s}$$

for each Hecke character ω of \mathcal{Q} and for each positive integer M , where n runs over all positive integers such that $(n, M) = 1$ and ω^* is the ideal character associated with ω (cf. [12, p. 505]). Let $g(w), f(z), h_\tau$ and $h'_\tau(z)$ be the same functions in Lemma 2.1 and Proposition 2.1.

We consider integrals

$$(3-1) \quad I_1 = \int_{\Gamma \backslash \mathfrak{H}} h_\tau(z) \overline{\vartheta(z) C(z, \bar{s} + 1/2 : k, \bar{\varphi}, \Gamma)} y^{(2k+1)/2} d_{\mathfrak{H}} z \quad (z = x + iy)$$

and

$$I_2 = \int_{\Gamma \backslash \mathfrak{H}} h'_\tau(z) \overline{\vartheta(z) C(z, \bar{s} + 1/2 : k, \bar{\varphi}, \Gamma)} y^{(2k+1)/2} d_{\mathfrak{H}} z \quad (z = x + iy),$$

where $\Gamma = \Gamma[2, 2\tau N]$, $C(z, s : k, \bar{\varphi}, \Gamma) = L_{4N\tau}(2s, \bar{\varphi})E(z, s : k, \bar{\varphi}, \Gamma)$, $C(z, s : k, \bar{\varphi}, \Gamma)$ and $E(z, s : k, \bar{\varphi}, \Gamma)$ means functions given in [12, (4-6) and (4-11)]. By the same method as that of Shimura [12, p. 542], we may check that

$$(3-2) \quad I_1 = \begin{cases} 4a(4\tau)\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}} 2^{-(2s+k)} L(2s+k, g) & \text{if } \tau \equiv 2, 3(4), \\ 4a(\tau)\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}} \left(1 - \left(\frac{\tau}{2}\right)\chi(2)2^{k-1}2^{-(2s+k)}\right) L(2s+k, g) & \text{if } \tau \equiv 1(4) \end{cases}$$

and

$$I_2 = 4a(4\tau)\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}} L(2s+k, g) \quad \text{if } \tau \equiv 2, 3(4),$$

$$I_2 = 4a(\tau)\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}} \times \left(\left(c(2) - 2^{k-1}\chi(2)\left(\frac{\tau}{2}\right) \right) - \chi(2^2)2^{2k-1}2^{-(2s+k)} \right) L(2s+k, g) \quad \text{if } \tau \equiv 1(4),$$

where $L(s, g) = \sum_{n=1}^{\infty} c(n)n^{-s}$ and $g(w) = \sum_{n=1}^{\infty} c(n)e[nw]$.

Next we calculate an integral

$$(3-3) \quad \int_{\Gamma[2\tau, 4N] \backslash \mathfrak{H}} g(2w) \overline{C(w, \bar{s} + 1/2 : \bar{\varphi})E(w, \bar{i} + 1/2 : \bar{\varphi})} \Im w^{2k} d_{\mathfrak{H}} w,$$

where $C(w, s : \bar{\varphi}) = C(w, s : k, \bar{\varphi}, \Gamma[2\tau, 4N])$ and $E(w, t : \bar{\varphi}) = E(w, t : k, \bar{\varphi}, \Gamma[2\tau, 4N])$ are given in [12, (4-6) and (4-11)]. By a method similar to that of [12, p. 550], we find

$$(3-4) \quad \int_{\Gamma[2\tau, 4N] \backslash \mathfrak{H}} g(2w) \overline{C(w, \bar{s} + 1/2 : \bar{\varphi}) E(w, \bar{s} + 1/2, \bar{\varphi})} \mathfrak{I} w^{2k} d_{\mathfrak{H}} w \\ = A(s) L_{4N}(2s + 1, \varphi)^{-1} L(2s + k, g) L(k, g, \bar{\varphi}),$$

where

$$A(s) = 2\tau \overline{\gamma(\bar{\varphi})} \mathfrak{h}^{-1} (4N)^{-2s} i^{-k} 2^{-(2s+k)} (2\pi)^{-k+1} (N/N')^{-k} \sum_{t|N/N'} \\ \mu(t) \varphi^*((t)) t^{k-1} c(N/tN') (2\pi)^{-1/2} 2^{2s+k-(1/2)} (2\tau)^{-2s-2k} \frac{\Gamma(s + (k/2)) \Gamma(k)}{\Gamma(s + (k/2) + (1/2))},$$

where $\mathfrak{h} = 4\tau N'(N'|N)$.

Exchanging the order of integration, we have

$$(3-5) \quad \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} \langle \Theta(z, w; \zeta_Z), g(2w) \rangle \overline{\mathfrak{I}(z) C(z, \bar{s} + 1/2 : k, \bar{\varphi}, \Gamma[2, 2\tau N])} y^{(2k+1)/2} d_{\mathfrak{H}} z \\ = \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} \text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})^{-1} \left\{ \int_{\Gamma[2\tau, 4N] \backslash \mathfrak{H}} \overline{\Theta(z, w; \zeta_Z) g(2w) \mathfrak{I} w^{2k}} d_{\mathfrak{H}} w \right\} \\ \times \overline{\mathfrak{I}(z) C(z, \bar{s} + 1/2 : k, \bar{\varphi}, \Gamma[2, 2\tau N])} y^{(2k+1)/2} d_{\mathfrak{H}} z \\ = \text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})^{-1} \int_{\Gamma[2\tau, 4N] \backslash \mathfrak{H}} \left\{ \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} \mathfrak{I}(z) \Theta(z, w; \zeta_Z) \right. \\ \left. \times \overline{C(z, \bar{s} + 1/2 : k, \bar{\varphi})} y^{(2k+1)/2} d_{\mathfrak{H}} z \right\} g(2w) \mathfrak{I} w^{2k} d_{\mathfrak{H}} w \\ = \langle M'(w, \bar{s}), g(2w) \rangle$$

with

$$M'(w, s) = \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} \mathfrak{I}(z) \Theta(z, w; \zeta_Z) C(z, s + 1/2 : k, \bar{\varphi}, \Gamma[2, 2\tau N]) y^{(2k+1)/2} d_{\mathfrak{H}} z.$$

For further computation, we need the following formula in [12, (7.9a) and (7.13)]

$$(3-6) \quad M'(w, s) = B(s) L_{4N\tau}(2s + 1, \bar{\varphi}) (-1)^k \sum_{q \in 2N\mathbb{Z}/4N\mathbb{Z}} \\ (2N)^{2s+1} C(w, s + (1/2); \bar{\varphi}) E(w, s + (1/2); \bar{\varphi}) \|_{2k\tau_q}$$

with

$$B(s) = 2 \frac{2^{s+(k/2)+(1/2)}}{\pi^{s+(k/2)+(1/2)}} \Gamma\left(s + \frac{k}{2} + \frac{1}{2}\right) \quad \text{and} \quad \tau_q = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix},$$

where $p||_l \alpha(z)$ means $p(\alpha(z))(cz+d)^{-l}$ for a function $p(z)$ on \mathfrak{H} and a $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$. Employing (3-13) and (3-18), we have

$$(3-7) \quad \langle M'(w, \bar{s}), g(2w) \rangle = 2(-1)^k \overline{B(\bar{s})} (2N)^{2s+1} A(s) \text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})^{-1} \\ \times L(2s+k, g) L(k, g, \bar{\varphi}).$$

Combining Proposition 2.1 with (3-5), (3-7), we may derive that

$$(3-8) \quad \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} C_\tau h_\tau(z) \overline{\mathfrak{g}(z) C(z, \bar{s} + 1/2 : k, \bar{\varphi}, \Gamma[2, 2\tau N])} y^{(2k+1)/2} d_{\mathfrak{H}} z \\ + \int_{\Gamma[2, 2\tau N] \backslash \mathfrak{H}} D_\tau h'_\tau(z) \overline{\mathfrak{g}(z) C(z, \bar{s} + 1/2 : k, \bar{\varphi}, \Gamma[2, 2\tau N])} y^{(2k+1)/2} d_{\mathfrak{H}} z \\ = 2(-1)^k \overline{B(\bar{s})} (2N)^{2s+1} A(s) \text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})^{-1} L(2s+k, g) L(k, g, \bar{\varphi}).$$

Applying the formulas (3-2) and (3-8), we multiply both sides of (3-8) by $L(2s+k, g)^{-1}$. Moreover, by Proposition 2.1, (3.2) and (3.8), putting $2s+k=0$, we may deduce that

$$(3-9) \quad |a(4\tau)|^2 \frac{\overline{c'} \mu(\mathfrak{h}^{-1} 4N\tau) \varphi^*(\mathfrak{h}^{-1} 4N\tau) E}{\varphi_a(-1) \gamma(\bar{\varphi}) \mathfrak{h}^{-1} 4N\tau} \\ = \tau^{(2k+1)/2-1} (4N\tau) \mathfrak{h}^{-1} i^k \overline{\gamma(\bar{\varphi})} 2^{1/2} \pi^{-k} (k-1)! \frac{\text{vol}(\Gamma[2, 2N\tau] \backslash \mathfrak{H})}{\text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})} \\ \times (N/N')^{-k} \sum_{t|N/N'} \mu(t) \varphi^*((t)) t^{k-1} c(N/tN') L(k, g, \bar{\varphi}) \quad \text{if } \tau \equiv 2, 3(4)$$

and

$$4|a(\tau)|^2 \tau^k \mu(\mathfrak{h}^{-1} 4N\tau) \varphi^*((\mathfrak{h}^{-1} 4N\tau)) (\varphi_a(-1) \overline{\gamma(\bar{\varphi})} \mathfrak{h}^{-1} 4N\tau)^{-1} \\ \times \left(F_\tau \left(1 - \left(\frac{\tau}{2} \right) \chi(2) 2^{k-1} \right) + G_\tau \left(\left(c(2) - 2^{k-1} \chi(2) \left(\frac{\tau}{2} \right) \right) - \chi(2^2) 2^{2k-1} \right) \right) \\ = (-1)^k 2 \frac{2^{1/2}}{\pi^{1/2}} 2(2N)^{-k+1} 2\tau \overline{\gamma(\bar{\varphi})} \mathfrak{h}^{-1} (4N)^k i^{-k} \\ \times (2\pi)^{-k+1} (N/N')^{-k} \sum_{t|N/N'} \mu(t) \varphi^*((t)) t^{k-1} c(N/tN') (2\pi)^{-1/2} \\ \times 2^{-1/2} (2\tau)^k 2^{-k} (k-1)! \text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})^{-1} L(k, g, \bar{\varphi}) \quad \text{if } \tau \equiv 1(4)$$

with $E = \frac{\langle g(2w), g(2w) \rangle (\langle f', f' \rangle - \langle f', f \rangle) + \langle g(w), g(2w) \rangle (\langle f, f \rangle - \langle f, f' \rangle)}{\langle f, f \rangle \langle f', f' \rangle - \langle f, f' \rangle \langle f', f \rangle}$,

where F_τ (resp. G_τ) is the quantity given in Proposition 2.1 and $f'(z) = f|U(4)(z)$. Consequently, by (3-9), we conclude the following theorem.

THEOREM. *Let the notation be as above. Suppose that N is odd, τ is a positive square free integer and the conditions (1-7), (1-10), (1-12), (1-13) and (1-14) are satisfied. Then we have*

$$\begin{aligned} |a(4\tau)|^2 E &= 2^{-(1/2)-k} N \\ &\times \tau^{(2k+1)/2} \pi^{-k} \mathfrak{h}^{-2} (k-1)! \overline{\gamma(\varphi)\gamma(\bar{\varphi})} (\mu(\mathfrak{h}^{-1}4N\tau)\varphi^*(\mathfrak{h}^{-1}4N\tau))^{-1} (-1)^k \\ &\times \varphi_a(-1)(N/N')^{-k} \sum_{t|N/N'} \mu(t)\varphi^*((t))t^{k-1}c(N/tN')L(k, g, \bar{\varphi}) \text{ if } \tau \equiv 2, 3(4) \end{aligned}$$

and

$$\begin{aligned} |a(\tau)|^2 &\left(F_\tau \left(1 - \left(\frac{\tau}{2} \right) \chi(2) 2^{k-1} \right) \right. \\ &\quad \left. + G_\tau \left(\left(c(2) - 2^{k-1} \chi(2) \left(\frac{\tau}{2} \right) \right) - \chi(2^2) 2^{2k-1} \right) \right) \\ &= i^k 2^{5/2} \pi^{-k} N \tau \mathfrak{h}^{-1} \overline{\gamma(\bar{\varphi})} (N/N')^{-k} (\mu(\mathfrak{h}^{-1}4N\tau)\varphi^*((\mathfrak{h}^{-1}4N\tau)))^{-1} \\ &\quad \times \varphi_a(-1) \overline{\gamma(\varphi)} \mathfrak{h}^{-1} 4N\tau \sum_{t|N/N'} \mu(t)\varphi^*((t))t^{k-1}c(N/tN') \\ &\quad \times (k-1)! \text{vol}(\Gamma[2\tau, 4N] \backslash \mathfrak{H})^{-1} L(k, g, \bar{\varphi}) \text{ if } \tau \equiv 1(4) \end{aligned}$$

where \mathfrak{h} is the conductor of $\varphi(= \psi\epsilon_\tau)$, $\mathfrak{h} = 4\tau N'$ and $\mu(n)$ is the Möbius function of n .

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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
IWATE UNIVERSITY
MORIOKA 020
JAPAN
E-mail: kojima@iwate-u.ac.jp