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ON THE ŁOJASIEWICZ EXPONENT AT INFINITY FOR POLYNOMIAL FUNCTIONS

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1. Introudction

1.1. For $n, q \in \boldsymbol{N} \setminus \{0\}$ we consider the polynomial functions

$$f = f_{n,q} : \boldsymbol{C}^3 \longrightarrow \boldsymbol{C}, \quad f(x, y, z) = f_{n,q}(x, y, z) := x - 3x^{2n+1}y^{2q} + 2x^{3n+1}y^{3q} + yz.$$

We will study some properties of these polynomials, related to their behaviour at infinity, and we will prove that some results, obtained in [6] and [3], [4], [7] for the case of polynomials in two variables, are not true in the case of polynomials in $m \geqq 3$ variables. Also, our polynomials $f_{n,q}$ show that several classes of polynomials, with "good" behaviour at infinity, considered in [8], [9], [14], [10], are distinct.

The first remark on our polynomials is:

1.2. *Remark.* After a suitable polynomial change of coordinates in $\boldsymbol{C}^3$, one can write $f(X, y, Z) = X$. Namely, taking $Z := z - 3x^{2n+1}y^{2q-1} + 2x^{3n+1}y^{3q-1}$, we get: $f(x, y, Z) = x + yZ$. Next, we put $X := x + yZ$ and we obtain $f(X, y, Z) = X$. Thus, there exists a polynomial automorphism $P = (P_1, P_2, P_3) : \boldsymbol{C}^3 \to \boldsymbol{C}^3$ such that $f = P_1$.

1.3. For a polynomial $g : \boldsymbol{C}^m \to \boldsymbol{C}$, we consider grad $g(x) := (\overline{\partial f / \partial x_1(x)}, \dots, \overline{\partial f / \partial x_m(x)})$. If g has non-isolated singularities, the *Łojasiewicz number at infinity*, $L_\infty(g)$, is defined by $L_\infty(g) := -\infty$. When g has only isolated singularities, the Łojasiewicz number at infinity is the supremum of the set

$$\{\nu \in \boldsymbol{R} \mid \exists A > 0, \ \exists B > 0, \ \forall x \in \boldsymbol{C}^m, \text{ if } \|x\| \geqq B, \text{ then } A\|x\|^\nu \leqq \|\operatorname{grad} g(x)\|\}.$$

Equivalent definition is (see for instance [6] or [5], proof of Proposition 1):

$$L_\infty(g) := \lim_{r \to \infty} \frac{\log \varphi(r)}{\log r}, \quad \text{where } \varphi(r) := \inf_{\|x\| = r} \|\operatorname{grad} g(x)\|.$$

The following result is a reformulation of Theorem 10.2 from [4]:

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THEOREM. *Let $g: C^2 \to C$ be a polynomial function. Then there exists a polynomial automorphism $P=(P_1, P_2): C^2 \to C^2$ such that $g=P_1$ if and only if g has no critical values and $L_\infty(g)>-1$.*

1.4. In the next Section we will prove the following

PROPOSITION. $L_\infty(f_{n,q})=-\dfrac{n}{q}$.

In particular, if $n \geqq q$, then $L_\infty(f_{n,q}) \leqq -1$. Using Remark 1.2, our Proposition shows that Theorem 1.3 can not be extended to the case of a polynomial function $g: C^m \to C$, when $m \geqq 3$.

1.5. It is proved in [6] and [4] that a polynomial function $g: C^2 \to C$ has $L_\infty(g) \neq -1$. Proposition 1.4 shows this is no longer true for polynomial functions $C^m \to C$, when $m \geqq 3$.

1.6. If $g: C^m \to C$ is a polynomial function, we call $t_0 \in C$ a *typical* value of g if there exists $U \subseteqq C$, an open neigbourhood of t_0, such that the restricition $g: g^{-1}(U) \to U$ is a C^∞ trivial fibration. If t_0 is not a typical value of g, then t_0 is called an *atypical* value of g. In general, the *bifurcation set*, B_g, of atypical values of g, contains, besides the set $\sum_g$ of critical values of g, some extra values, the so-called "critical values coming from infinity". For example, if $g(x, y)=x^2y+x$, then $\sum_g=\emptyset$ and $B_g=\{0\}$.

Several classes of polynomials without "critical values coming from infinity" are considered in literature, see for instance [1], [8], [9], [13], [11]. We recall now three of them. In the next Section we will use the polynomials $f_{n,q}$ to show that these classes are distinct.

For a polynomial $g: C^m \to C$, we denote:

$$M(g) := \{x \in C^m \mid \exists \lambda \in C \text{ such that } \operatorname{grad} g(x)=\lambda \cdot x\}.$$

Geometrically, a point $x \in M(g)$ if and only if either x is a critical point of g, or x is not a critical point of g, but the hypersurface $g^{-1}(g(x))$ does not intersect transversally, at x, the sphere $\{z \in C^m \mid \|z\|=\|x\|\}$.

A polynomial $g: C^m \to C$ is called *M-tame* if for any sequence $\{z^k\} \subseteqq M(g)$ such that $\lim_{k\to\infty}\|z^k\|=\infty$, we have $\lim_{k\to\infty}|g(z^k)|=\infty$. See [12] for properties of M-tame polynomials.

A polynomial $g: C^m \to C$ is called *quasitame* if for any sequence $\{z^k\} \subseteqq C^m$ such that $\lim_{k\to\infty}\|z^k\|=\infty$ and $\lim_{k\to\infty} \operatorname{grad} g(z^k)=0$, we have $\lim_{k\to\infty}|g(z^k)-\langle z^k, \operatorname{grad} g(z^k)\rangle|=\infty$. Here, $\langle \cdot, \cdot \rangle$ denotes the Hermitian product on C^m. See [8], [9] for properties of quasitame polynomials.

Follwing [13], [14], we will say that a polynomial $g: C^m \to C$ satisfies Malgrange's condition for $t_0 \in C$ if, for $\|x\|$ large enough and for $g(x)$ close to t_0, there exists $\delta>0$ such that $\|x\| \cdot \|\operatorname{grad} g(x)\| \geqq \delta$. Equivalent formulation is:

There exists no sequence $\{z^k\}\subseteq \boldsymbol{C}^m$ such that $\lim_{k\to\infty}\|z^k\|=\infty$, $\lim_{k\to\infty} g(z^k)=t_0$ and $\lim_{k\to\infty}\|z^k\|\cdot\|\operatorname{grad} g(z^k)\|=0$.

The next result seems to be well-known, see [12], [10], [15]. Its proof can be easily obtained, by contradiction.

1.7. PROPOSITION. *For $m\geqq 2$, let $g: \boldsymbol{C}^m\to\boldsymbol{C}$ be a polynomial function.*

(a) *If g is quasitame, then g satisfies Malgrange's condition for ang $t_0\in\boldsymbol{C}$.*

(b) *If g satisfies Malgrange's condition for any $t_0\in\boldsymbol{C}$, then g is M-tame.*

We will show that these implications can not be reversed, if $m\geqq 3$ (for (a), see also [2]). More precisely, we have:

1.8. PROPOSITION. (a) *For any $n, q\in\boldsymbol{N}\setminus\{0\}$, the polynomial $f_{n,q}$ is M-tame, but not quasitame.*

(b) *The polynomial $f_{n,q}$ satisfies Malgrange's condition for any $t_0\in\boldsymbol{C}$, if and only if $n\leqq q$.*

Thus, if $f=f_{n,q}$ for some $n>q$, then, by [14], the family $\bar{f}$ of projective closures of fibres of f has nontrivial vanishing cycles, despite Remark 1.2. Also, such an f is not t-regular at infinity, in the sense of [15], since by [14], the t-regularity at infinity is equivalent to Malgrange's condition for any $t_0\in\boldsymbol{C}$.

2. Proofs

2.1. Let $g: \boldsymbol{C}^m\to\boldsymbol{C}$ be a polynomial function with only isolated singularities. For an analytic curve $p: (0,\varepsilon)\to\boldsymbol{C}^m$ such that $\lim_{t\to 0}\|p(t)\|=\infty$, we consider the expansions in Laurent series:

$$p(t)=at^{\alpha}+a_1t^{\alpha+1}+\cdots,\qquad \text{with } \alpha<0 \text{ and } a\neq 0 \tag{1}$$

$$\operatorname{grad} g(p(t))=bt^{\beta}+b_1t^{\beta+1}+\cdots,\qquad \text{with } \beta\neq 0 \text{ and } b\neq 0 \tag{2}$$

and we denote $L(g;p):=\operatorname{ord}(\operatorname{grad} g(p(t)))/\operatorname{ord}(p(t))=\beta/\alpha$. Here, ord denotes the order of series. It follows, for example from (the proof of) Proposition 1 in [5], that

$$L_\infty(g)=\inf\left\{ L(g;p) \,\middle|\, \begin{matrix} p:(0,\varepsilon)\to\boldsymbol{C}^m \text{ is an analytic curve} \\ \text{such that (1) and (2) are fulfilled} \end{matrix}\right\}. \tag{3}$$

2.2. *Proof of Proposition* 1.4. Consider the curve $\Psi:(0,1)\to\boldsymbol{C}^m$ defined by: $\Psi(t):=(t^{-q}, t^n, 0)$. Then $L(f_{n,q};\Psi)=-(n/q)$, hence $L_\infty(f_{n,q})\leqq -(n/q)<0$.

Let now consider an arbitrary analytic curve $p:(0,\varepsilon)\to\boldsymbol{C}^m$, $p(t)=(x(t), y(t), z(t))$, such that $\lim_{t\to 0}\|p(t)\|=\infty$.

If $\lim_{t\to 0}\|\operatorname{grad} f_{n,q}(p(t))\|\neq 0$, then $\operatorname{ord}(\operatorname{grad} f_{n,q}(p(t)))\leqq 0$. Hence, $L(f_{n,q};p)\geqq 0>L(f_{n,q};\Psi)$.

Suppose now that $\lim_{t\to 0}\|\operatorname{grad} f_{n,q}(p(t))\|=0$. Then $\lim_{t\to 0} y(t)=0$ and

(4) $\lim_{t\to 0}(x(t))^n\cdot(y(t))^q$ is a root of the equation $1-(6n+3)T^2+(6n+2)T^3=0$.

Hence

$$y(t)\not\equiv 0,\quad \lim_{t\to 0} y(t)=0 \quad\text{and}\quad \lim_{t\to 0}\|x(t)\|=\infty. \tag{5}$$

Therefore, we have:

$$\operatorname{ord}(y(t))\geqq\operatorname{ord}(\operatorname{grad} f_{n,q}(p(t)))>0 \quad\text{and}\quad \operatorname{ord}(p(t))\leqq\operatorname{ord}(x(t))<0 \tag{6}$$

It follows, using (4) and (6), that

$$L(f_{n,q};\ p)=\frac{\operatorname{ord}(\operatorname{grad} f_{n,q}(p(t)))}{\operatorname{ord}(p(t))}\geqq\frac{\operatorname{ord}(y(t))}{\operatorname{ord}(p(t))}\geqq\frac{\operatorname{ord}(y(t))}{\operatorname{ord}(x(t))}=-\frac{n}{q}.$$

Proposition 1.4 is proved. □

2.3. *Proof of Proposition* 1.8. Part (b) follows from (the proof of) Proposition 1.4.

If $n,\ q\in\boldsymbol{N}\setminus\{0\}$ are fixed, then the curve $\Psi(t):=(t^{-q},\ t^n,\ 0)$ can be used to show that $f_{n,q}$ is not quasitame.

Suppose now that $f=f_{n,q}$ is not M-tame. Using Curve Selection Lemma at infinity, see [12], one can find an analytic curve $p:(0,\varepsilon)\to M(f)$, $p(t)=(x(t),\ y(t),\ z(t))$, such that $\lim_{t\to 0} f(p(t))\in\boldsymbol{C}$. This implies that $\lim_{t\to 0}\operatorname{grad} f(p(t))=0$, hence relations (4) and (5) hold. The condition $p(t)\in M(f)$ means that

$$\left(\overline{\frac{\partial f}{\partial x}(p(t))},\ \overline{\frac{\partial f}{\partial y}(p(t))},\ \overline{\frac{\partial f}{\partial z}(p(t))}\right)=\lambda(t)\cdot(x(t),\ y(t),\ z(t)) \tag{7}$$

for some suitable analytic curve $\lambda:(0,\varepsilon)\to\boldsymbol{C}$. It follows that none of the components of $p(t)$ or of $\operatorname{grad} f(p(t))$ is identically zero.

If $A:=\operatorname{ord}(x(t))$, $B:=\operatorname{ord}(y(t))$ and $C:=\operatorname{ord}((\partial f/\partial x)(p(t)))$, then $A<0$, $B>0$, $C>0$, and relations (4) and (7) give us

$$\operatorname{ord}(\lambda(t))=C-A,\quad nA+qB=0 \quad\text{and}\quad \operatorname{ord}(z(t))=B+A-C. \tag{8}$$

Since $f=y(\partial f/\partial y)+x(1+(6q-3)x^{2n}y^{2q}-(6q-2)x^{3n}y^{3q})$, it is easy to see that

$\lim_{t\to 0}(x(t))^n\cdot(y(t))^q$ is a root of the equation $1+(6q-3)T^2-(6q-2)T^3=0$.

Thus, using (4), it follows that $\lim_{t\to 0}(x(t))^n\cdot(y(t))^q=1$. Hence we can assume that

$$x(t)=t^A,\quad y(t)=t^B+t^{B+D}\cdot\rho(t),\quad\text{with}\quad D>0\quad\text{and}\quad\rho(0)\neq 0$$

(since $(x(t))^n\cdot(y(t))^q\equiv 1$ implies that $(\partial f/\partial x)(p(t))\equiv 0$). By a direct computation, we find that

$$\lambda(t)=\frac{1}{x(t)}\cdot\frac{\partial f}{\partial x}(p(t))=6nq\cdot\overline{\rho(0)}\cdot t^{D-A}+\text{higher terms},$$

hence, by (8), $D=C$. Next, the second component in (7) gives us

$$z(t)=6qx(t)^{2n+1}y(t)^{2q-1}(1-x(t)^n y(t)^q)+\overline{\lambda(t)y(t)}$$
$$=-6q^2\cdot\rho(0)\cdot t^{A-B+D}+\text{higher terms}.$$

Hence, by (8), we have $B=C=D$. Finally, the third component in (7) gives us

$$t^B+t^{B+D}\cdot\overline{\rho(t)}=6nq\cdot\overline{\rho(0)}\cdot t^{D-A}\cdot(-6q^2\cdot\rho(0)\cdot t^{A-B+D})+\text{higher terms}.$$

Comparing the ledings coefficients, we obtain

$$1=-36nq^3\cdot|\rho(0)|^2,$$

which is impossible. Thus, Proposition 1.8 is proved. □

2.4. *Remark.* (i) If $n\leqq q$, it is possible to prove that $f_{n,q}$ is M-tame just looking at the order of various Laurent expansions.

(ii) It is not difficult to see that for the polynomials $f_{n,q}$, the Newton nondegeneracy condition fails on a face of dimension 1. Using this, one can construct other similar examples.

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