M. TANIGUCHI
KODAI MATH. J.
12 (1989), 283–295

A NOTE ON THE SECOND VARIATIONAL FORMULAS OF FUNCTIONALS ON RIEMANN SURFACES

BY MASAHIKO TANIGUCHI

§ 1. Introduction.

Recently, F. Maitani proved the second variational formulas for several fundamental functionals of arbitrary Riemann surfaces when surfaces varies holomorphically ([6, § 3]). His proof based on the classical and fundamental method of orthogonal decomposition of square integrable abelian differentials. But since he considered very general classes of differentials, the key of his argument is not so clear. Hence it seems helpful to give a short and rather elementary proof in case of typical and fundamental functionals, such as the extremal length of the homology class of a given curve and Robin's constant at a given point (which were originally investigated by H. Yamaguchi [9]).

In this paper, the author uses only two very elementary orthogonal decompositions. But we succeed in handling fairly general quasiconformal deformations and giving a general second variational formulas which reduces to Maitani's ones in case of holomorphic families of surfaces.

In the next section, we will state the second variational formulas for the above two functionals under general quasiconformal deformation. Proofs of formulas, which the author intends to be self-contained, will be given in § 4 after preparing more general second variational formulas in § 3. The lines of the proofs are similar to those of Maitani's ones.

§ 2. Statements of main results.

1. In general, for a given family $\{x(t)\in X\colon t\in I,\, I$ is a neighborhood of 0 in $\mathfrak{R}\}$ of elements in a Banach space X with the norm $\|\cdot\|_X$, we say that $x(t)$ is differentiable at $t_0\in I$ if there is an element $y\in X$ such that $\lim_{t\neq 0, t\to 0} 1/t\cdot \|(x(t+t_0)-x(t_0))-ty\|_X=0$, and denote y by $(dx/dt)(t_0)$. When $x(t)$ depends on several real parameters $t=(t_1, \cdots, t_n)$, then, for every j, partial differentiability and the partial derivative $\partial x/\partial t_j$ are defined in a similar way (cf. [5, Ch. 8]).

For a Riemann surface R, we denote by $\Gamma(R)$ the Hilbert space consisting of all real square integrable abelian differentials on R with the inner product

Received April 14, 1988.

$(\alpha, \beta)_R=\iint_R \alpha\wedge{}^*\beta$. Let $\Gamma_c(R)$ and $\Gamma_h(R)$ be subspaces of $\Gamma(R)$ consisting of all closed differentials, and of all harmonic ones, respectively. We denote by $\Gamma_{e0}(R)$ the closure of $\{df: f$ is smooth and has a compact support in $R\}$ in $\Gamma(R)$. Then the following orthogonal decompositions are classically well-known (cf. [3, Ch. V, §2]);

$$\Gamma(R)=\Gamma_c(R)+{}^*\Gamma_{e0}(R), \quad \text{and} \quad \Gamma_c(R)=\Gamma_h(R)+\Gamma_{e0}(R), \tag{1}$$

where we set ${}^*\Gamma_{e0}(R)=\{{}^*df : df\in\Gamma_{e0}(R)\}$. Next for the complex Hilbert space $\Gamma^{\mathfrak{C}}(R)=\{\alpha+i\beta: \alpha, \beta\in\Gamma(R)\}$ with the inner product $(\varphi, \psi)_R^{\mathfrak{C}}=\iint_R \varphi\wedge{}^*\bar{\psi}$, the following orthogonal decomposition is clear;

$$\Gamma^{\mathfrak{C}}(R)=\Gamma^{1,0}(R)+\Gamma^{0,1}(R), \tag{2}$$

where we set $\Gamma^{1,0}(R)=\{\phi\in\Gamma^{\mathfrak{C}}(R): {}^*\phi=-i\phi\}$ and $\Gamma^{0,1}(R)=\{\phi\in\Gamma^{\mathfrak{C}}(R): {}^*\phi=i\phi\}$. Also we set $\Gamma_\chi^{\mathfrak{C}}(R)=\{\alpha+i\beta: \alpha, \beta\in\Gamma_\chi(R)\}$ with $\chi=c$, h or $e0$.

2. Let R be an arbitrary Riemann surface, and $B(R)$ be the complex Banach space consisting of all Beltrami differentials, i.e. all bounded $(-1, 1)$-forms, μ on R with the norm $\|\mu\|_\infty=\text{ess.}\sup_{p\in R}|\mu|(p)$. First we consider a real 1-parameter family $\{\mu(t): t\in I\}$ in $B(R)$, and suppose that

(i) $\mu(0)\equiv 0$ and $\|\mu(t)\|_\infty<1$ for every t, and

(ii) $\mu(t)$ is differentiable at every $t\in I$.

Let f_t be the quasiconformal mapping of $R=R_0$ to another R_t with the complex dilatation $\mu(t)$ for every t. In the sequel, we denote by $\alpha_t\circ f_t$ the pull-back of a differential α_t on R_t by f_t.

3. Fix a closed curve C_0 on R_0 and let C_t be the 1-cycle on R_t determined by the curve $f_t(C_0)$. Let $\sigma_t=\sigma(C_t, R_t)$ be the period reproducer of C_t in $\Gamma_h(R_t)$, i.e. the differential in $\Gamma_h(R_t)$ such that

$$(\alpha, \sigma_t)_{R_t}=\int_{C_t}\alpha \qquad \text{for every } \alpha\in\Gamma_h(R_t),$$

and $\lambda(t)$ be the extremal length of the homology class of C_t. Then Accola's theorem ([1]) implies $\lambda(t)=\|\sigma_t\|_{R_t}^2$, and we have the following

THEOREM 1. *Under the assumptions* (i) *and* (ii), *further suppose that*

(iii) $\dfrac{d\mu}{dt}$ *is differentiable at* $t=0$.

Then $\lambda(t)$ *is twice differentiable at* $t=0$.

Moreover, set $\phi_t=\sigma_t+i\cdot{}^*\sigma_t$ *and* $\Phi(t)=\phi_t\circ f_t-\phi_0$ *for every* t. *Then* $\Phi(t)$ *is differentiable* (*in* $\Gamma(R_0)$) *at* $t=0$ *and*

$$\frac{d^2\lambda}{dt^2}(0)=\mathrm{Re}\iint_{R_0}\Big(\frac{d^2\mu}{dt^2}(0)\phi_0\wedge{}^*\phi_0+\frac{d\Phi}{dt}(0)\wedge{}^*\frac{d\Phi}{dt}(0)\Big). \tag{3}$$

4. Next fix a point p_0 on R_0, and suppose that R_0 admits Green's functions. Fix a simply connected neighborhood U_0 of p_0 in R_0 and a conformal mapping Z_0 of U_0 onto the unit disk $B=\{|z|<1\}$ such that $Z_0(p_0)=0$. Assume that

(iv) $$\mu(t)\equiv 0 \quad \text{on } U_0 \quad \text{for every } t\,.$$

Then $Z_t=Z_0\circ(f_t)^{-1}$ is a conformal mapping of $f_t(U_0)$ onto B such that $Z_t(p_t)=0$, where $p_t=f_t(p_0)$.

Let $g_t(p)=g(p,\, p_t\,;\, R_t)$ be Green's function on R_t with the pole p_t for every t. Then we can define Robin's constant $\gamma(t)$ at p_t on R_t by setting

$$\gamma(t)=\lim_{z\to 0} g_t(Z_t^{-1}(z))+\log|z|$$

for every t, and we have the following

THEOREM 1′. *Under the assumptions* (i) *and* (ii), *further suppose that* $\{\mu(t)\}$ *satisfies* (iii) *and* (iv). *Then* $\gamma(t)$ *is twice differentiable at* $t=0$.

Moreover, set $\phi_t=-{}^*dg_t+i\cdot dg_t$ *and* $\Phi(t)=\phi_t\circ f_t-\phi_0$. *Then* $\Phi(t)$ *is differentiable at* $t=0$ *and*

$$\frac{d^2\gamma}{dt^2}(0)=\frac{-1}{2\pi}\cdot\mathrm{Re}\iint_{R_0}\Big(\frac{d^2\mu}{dt^2}(0)\phi_0\wedge{}^*\phi_0+\frac{d\Phi}{dt}(0)\wedge{}^*\frac{d\Phi}{dt}(0)\Big). \tag{3′}$$

5. Now we consider the case that μ depends on a complex parameter t on a neighborhood U of 0 in $\mathfrak{C}$. Then considering that $\mu(t)$ depends on two real parameters $x=\mathrm{Re}\,t$ and $y=\mathrm{Im}\,t$, we write $\mu(x+iy)$ also by $\mu(x,\,y)$.

Suppose that $\{\mu(t)\}$ satisfies (i) and that

(ii′) $$\frac{\partial\mu}{\partial x} \text{ and } \frac{\partial\mu}{\partial y} \text{ exist at every } (x,\,y) \text{ with } x+iy\in U\,.$$

Let $\{f_t\}$ and $\{R_t\}$ be as in §2-2. Then Theorems 1 and 1′ imply the following

THEOREM 2 (cf. [6, §3]). *Under the assumptions* (i) *and* (ii′), *further suppose that*

(iii′) $$\frac{\partial^2\mu}{\partial x^2} \textit{ and } \frac{\partial^2\mu}{\partial y^2} \textit{ exist at } t=0, \textit{ and}$$

(v) $$\frac{\partial\mu}{\partial\bar{t}}(0)=0, \quad \textit{and} \quad \Delta\mu(0)=0\,.$$

Then

$$\Delta\lambda(0)=4\cdot\left\|\frac{\partial\Phi}{\partial\bar{t}}(0)\right\|^2_{R_0}\geqq 0. \tag{4}$$

Moreover if $\{\mu(t)\}$ *also satisfies* (iv), *then*

$$\Delta\gamma(0)=-\frac{2}{\pi}\cdot\left\|\frac{\partial\Phi}{\partial\bar{t}}(0)\right\|^2_{R_0}\leqq 0, \tag{4'}$$

where we set $\partial/\partial t=1/2((\partial/\partial x)-i\cdot\partial/\partial y)$, $\partial/\partial\bar{t}=1/2((\partial/\partial x)+i\cdot\partial/\partial y)$, and $\Delta=((\partial^2/\partial x^2)+\partial^2/\partial y^2)$ as usual.

Now it is easy to show the following

COROLLARY ([6], [9]). *Suppose that* μ *depends on n complex parameters* $t=(t_1, \cdots, t_n)$ *holomorphically on a neighborhood* U *of* $(0, \cdots, 0)$ *in* $\mathfrak{C}^n$, *and that* (i) *in* § 2-2 *holds. Then* $\lambda(t)$ *is continuous and plurisubharmonic on* U.

Further if $\{\mu\}$ *also satisfies* (iv), *then* $\gamma(t)$ *is continuous and plurisuperharmonic on* U.

Here recall that one of mutually equivalent definitions of holomorphical dependence of μ on t is the following (cf., for instance, [5, 8.9 and 9.10]);

(ii*) μ and all $\dfrac{\partial\mu}{\partial(\mathrm{Re}\, t_j)}, \dfrac{\partial\mu}{\partial(\mathrm{Im}\, t_j)}$ $(j=1, \cdots, n)$

are (exist and) continuous on U, and

(v*) $\dfrac{\partial\mu}{\partial\bar{t}_j}\equiv 0$ on U for every j.

§ 3. A general second variational formula.

For the sake of convenience, we include proofs of all lemmas in this section, though some of them are well-known.

1. Let $\{\mu(t)\}$, $\{f_t\}$ and $\{R_t\}$ be as in § 2-2, and a meromorphic differential ϕ_t on R_t be given for every t. Set $\Phi(t)=\phi_t\circ f_t-\phi_0$, and assume that

(A^0) $\omega(t)=\mathrm{Re}\,\Phi(t)\in\Gamma_c(R_0)$, and $\tau(t)=\mathrm{Im}\,\Phi(t)\in\Gamma_{e0}(R_0)$ for every t, and

(B^0) there is a subsurface S of R_0 such that $\|\phi_0\|_S<+\infty$ and

$\mu(t)\equiv 0$ on R_0-S for every t.

Consider the following elementary orthogonal decomposition of $\Phi(t)$ due to (2) in § 2;

$$\Phi(t)=\Phi^{1,0}(t)+\Phi^{0,1}(t)\,; \tag{1}$$

$$\Phi^{1,0}(t)\in\Gamma^{1,0}(R_0),\qquad \Phi^{0,1}(t)\in\Gamma^{0,1}(R_0)\,.$$

Then the following two lemmas are well-known (cf., for instance, [6], [8] and References of them).

LEMMA 1. *For every t,*

$$\Phi^{0,1}(t)=\mu(t)\cdot(\Phi^{1,0}(t)+\phi_0), \tag{2}$$

and

$$\|\Phi^{1,0}(t)\|_{R_0}=\|\Phi^{0,1}(t)\|_{R_0}\leqq\frac{k_t}{1-k_t}\cdot\|\phi_0\|_S\,, \tag{3}$$

where $k_t=\|\mu(t)\|_\infty<1$.

Moreover $\Phi^{0,1}(t)$ *is differentiable at* $t=0$ *(in* $\Gamma^{0,1}(R_0)$*) and*

$$\frac{d\Phi^{0,1}}{dt}(0)=\frac{d\mu}{dt}(0)\phi_0\,. \tag{4}$$

Proof. First write $\phi_t=a_t(z_t)dz_t$ and $f_t=(z_t)^{-1}\circ F_t\circ z$ with generic local parameters z_t and $z=z_0$ on R_t and R_0, respectively. Then we have

$$\Phi^{1,0}(t)=a_t(F_t(z))\cdot(F_t)_z dz-\phi_0\,, \tag{5}$$

and

$$\Phi^{0,1}(t)=a_t(F_t(z))\cdot(F_t)_{\bar z} d\bar z\,. \tag{6}$$

Hence we have the equation (2).

Next by (A^0) and (1) in § 2, we have

$$N_t\equiv\iint_{R_0}\Phi(t)\wedge\overline{\Phi(t)}=-2i\cdot\iint_{R_0}\omega(t)\wedge\tau(t)$$
$$=2i\cdot(\omega(t),\ {}^*\tau(t))_{R_0}=0\,.$$

On the other hand,

$$N_t=\iint_{R_0}\Phi^{1,0}(t)\wedge\overline{\Phi^{1,0}(t)}+\iint_{R_0}\Phi^{0,1}(t)\wedge\overline{\Phi^{0,1}(t)}$$
$$=-i\cdot\|\Phi^{1,0}(t)\|^2_{R_0}+i\cdot\|\Phi^{0,1}(t)\|^2_{R_0}\,.$$

Hence set $E_t\equiv\|\Phi^{0,1}(t)\|_{R_0}$. Then $E_t=\|\Phi^{1,0}(t)\|_{R_0}$. And since $E_t\leqq k_t\cdot(E_t+\|\phi_0\|_S)$ by (B^0) and (2), we conclude the inequality (3).

Finally since $(\Phi^{0,1}(t)-\Phi^{0,1}(0))/t=(\mu(t)/t)\cdot(\Phi^{1,0}(t)+\phi_0)$ by (2), $\mu(t)/t$ converges to $(d\mu/dt)(0)$ in $B(R_0)$ by (ii), and $\Phi^{1,0}(t)$ converges to 0 in $\Gamma(R_0)$ by (3), we conclude the second assertion and the equation (4). q.e.d.

LEMMA 2. *Suppose that*

(C⁰) $\Phi(t)\wedge{}^*\phi_0$ *is absolutely integrable on* R_0 *for every* t.

And set $I(t)=\iint_{R_0}\Phi(t)\wedge{}^*\phi_0$. *Then* $I(t)$ *is differentiable at* $t=0$ *and*

$$\frac{dI}{dt}(0)=\iint_{R_0}\frac{d\mu}{dt}(0)\phi_0\wedge{}^*\phi_0\,. \tag{7}$$

Proof. Since ${}^*\phi_0=-i\phi_0$, (1), (2) and (B^0) implies that

$$\frac{I(t)-I(0)}{t}=\iint_{R_0}\left[\frac{\Phi(t)-\Phi(0)}{t}\right]\wedge{}^*\phi_0$$
$$=\iint_{S}\left[\frac{\Phi^{0,1}(t)-\Phi^{0,1}(0)}{t}\right]\wedge{}^*\phi_0\,.$$

Hence by the second assertion of Lemma 1, $I(t)$ is differentiable at $t=0$, and (7) holds by (4). q.e.d.

Remark. In the above proof, we have used only the differentiability of $\mu(t)$ only at $t=0$.

Next recalling (A^0), we have

$$2\cdot\Phi^{1,0}(t)=(\omega(t)+i\cdot{}^*\omega(t))+i\cdot(\tau(t)+i\cdot{}^*\tau(t))\,,$$

and

$$2\cdot\Phi^{0,1}(t)=(\omega(t)-i\cdot{}^*\omega(t))+i\cdot(\tau(t)-i\cdot{}^*\tau(t))\,.$$

And we can see the following

LEMMA 3 (cf. [6, Theorem 1]). *In* $\Gamma^{\mathfrak{C}}(R_0)$, $\Phi(t)$ *is differentiable at* $t=0$, *hence so is* $\Phi^{1,0}(t)$.

Also $\omega(t)$ *and* $\tau(t)$ *are differentiable at* $t=0$ *in* $\Gamma(R_0)$.

Proof. By the second assertion of Lemma 1, $\mathrm{Re}\,\Phi^{0,1}(t)=\omega(t)+{}^*\tau(t)$ is differentiable at $t=0$. Since $\omega(t)\in\Gamma_c(R_0)$, ${}^*\tau(t)\in{}^*\Gamma_{e0}(R_0)$, and each projection mapping is bounded linear, $\omega(t)$ and $\tau(t)$ are differentiable at $t=0$, hence so is $\Phi(t)=\omega(t)+i\cdot\tau(t)$. q.e.d.

2. Now take another T in I arbitrarily and replace the center R_0 by R_T. Then the quasiconformal mapping $f_t^T=f_t\circ(f_T)^{-1}$ of R_T to R_t has the complex dilatation $\nu(t-T)=\nu(t-T\,;z_T)\cdot(d\bar{z}_T/dz_T)$ with

$$\nu(t-T\,;z_T)=\left[\frac{\mu(t\,;z)-\mu(T\,;z)}{1-\overline{\mu(T\,;z)}\cdot\mu(t\,;z)}\cdot\frac{(F_T)_z}{(\bar{F}_T)_{\bar{z}}}\right]\circ(F_T)^{-1}\,, \tag{8}$$

where z, z_T and F_T are as in the proof of Lemma 1 (with $t=T$) and we write

$\mu(t)=\mu(t\,;z)\cdot(d\bar{z}/dz)$ (cf. [2, Ch. 1-(10)]).

By (ii), $\mu(t)$ is differentiable at $t=T$ in $B(R_0)$, hence so is $\nu(s)$ at $s\equiv t-T=0$ in $B(R_T)$. Also the equation

$$\frac{d\nu(\cdot\,;z_T)}{ds}(0)=\left[\frac{\dfrac{d\mu(\cdot\,;z)}{dt}(T)}{1-|\mu(T\,;z)|^2}\cdot\frac{(F_T)_z}{(\bar{F}_T)_{\bar{z}}}\right]\circ(F_T)^{-1} \tag{9}$$

holds. So we have the following

LEMMA 4 (cf. [6, Theorem 2]). *Set* $\Phi^T(s)=\phi_{s+T}\circ f^T_{s+T}-\phi_T$, *and assume that*

(A^T) $\quad\operatorname{Re}\Phi^T(s)\in\Gamma_c(R_T)$ and $\operatorname{Im}\Phi^T(s)\in\Gamma_{e0}(R_T)$,

(B^T) $\quad$ *there is a subsurface* S_T *of* R_T *such that* $\|\phi_T\|_{S_T}<+\infty$ *and* $\nu(s)\equiv 0$ *on* R_T-S_T, *and*

(C^T) $\quad\Phi^T(s)\wedge{}^*\phi_T$ *is absolutely integrable on* R_T

for every s *with* $s+T\in I$. *For every such* s, *set*

$$I(s\,;T)=\iint_{R_T}\Phi^T(s)\wedge{}^*\phi_T\,. \tag{10}$$

Then $I(s\,;T)$ *is differentiable at* $s=0$ *and*

$$\begin{aligned}\dot{I}(T)\equiv\frac{dI(\cdot\,;T)}{ds}(0)&=\iint_{R_T}\frac{d\nu}{ds}(0)\phi_T\wedge{}^*\phi_T\\&=\iint_{R_0}\frac{d\mu}{dt}(T)(\Phi^{1,0}(T)+\phi_0)\wedge{}^*(\Phi^{1,0}(T)+\phi_0)\,.\end{aligned} \tag{11}$$

Proof. By the same argument as in the proofs of Lemmas 1 and 2 we can show the first assertion and the first equation of (11). Next change the variable z_T to z, and note that $d\nu/ds(0)\equiv 0$ on R_T-S_T by (B^T). Then we see by (9) that

$$\begin{aligned}&\iint_{R_T}\frac{d\nu}{ds}(0)\phi_T\wedge{}^*\phi_T\\&=\iint_{R_0}\frac{d\mu(\cdot\,;z)}{dt}(T)\cdot(a_T\circ F_T)\cdot(F_T)_z d\bar{z}\wedge(-i\cdot a_T\circ F_T)\cdot(F_T)_z dz\,,\end{aligned} \tag{12}$$

which implies the second equation of (11) by (5). q.e.d.

THEOREM 3 (cf. [6, Theorem 3]). *Suppose that* (A^T), (B^T) *and* (C^T) *holds for every* $T\in I$, *and further assume that* (iii) *in Theorem* 1 *holds. Then* $\dot{I}(T)$ *is differentiable at* $T=0$ *and*

$$\frac{d\dot{I}}{dT}(0)=\iint_{R_0}\Big(\frac{d^2\mu}{dt^2}(0)\phi_0\wedge{}^*\phi_0+\frac{d\Phi}{dt}(0)\wedge{}^*\frac{d\Phi}{dt}(0)\Big). \tag{13}$$

Proof. Since $\|\Phi^{1,0}(T)\|_{R_0}=O(T)$ as T tends to 0 by (ii) and (3), and since $\|(d\mu/dt)(T)-(d\mu/dt)(0)\|_\infty=O(T)$ by (iii), (B^0) and (11) implies that

$$\dot{I}(T)=\iint_S\Big(\frac{d\mu}{dt}(T)\phi_0\wedge{}^*\phi_0-2i\cdot\frac{d\mu}{dt}(0)\phi_0\wedge\Phi^{1,0}(T)\Big)+O(T^2).$$

Hence (iii) and (4) implies

$$\frac{d\dot{I}}{dT}(0)=\iint_S\Big(\frac{d^2\mu}{dt^2}(0)\phi_0\wedge{}^*\phi_0-2i\cdot\frac{d\Phi^{0,1}}{dt}(0)\wedge\frac{d\Phi^{1,0}}{dt}(0)\Big). \tag{14}$$

Here set $\dot{\Phi}^{1,0}=(d\Phi^{1,0}/dt)(0)$, $\dot{\Phi}^{0,1}=(d\Phi^{0,1}/dt)(0)$ and $\dot{\Phi}=(d\Phi/dt)(0)$. Then $\dot{\Phi}^{1,0}\in\Gamma^{1,0}(R_0)$ and $\dot{\Phi}^{0,1}\in\Gamma^{0,1}(R_0)$, and hence we have

$$\begin{aligned} -2i\cdot\iint_S\dot{\Phi}^{0,1}\wedge\dot{\Phi}^{1,0}&=-i\cdot\iint_{R_0}(\dot{\Phi}\wedge\dot{\Phi}^{1,0}+\dot{\Phi}^{0,1}\wedge\dot{\Phi}) \\ &=\iint_{R_0}\dot{\Phi}\wedge({}^*\dot{\Phi}^{1,0}+{}^*\dot{\Phi}^{0,1})=\iint_{R_0}\dot{\Phi}\wedge{}^*\dot{\Phi}. \end{aligned} \tag{15}$$

Thus (13) follows from (14) and (15). q.e.d.

Remark 2. Actually, we have used (A^T) to show that $(\mathrm{Re}\,\Phi^T(t-T), {}^*\mathrm{Im}\,\Phi^T(t-T))=0$ for every t and T. Recall that the behavior conditions such as used by Maitani also assure us the desired orthogonality, by definition.

§4. Proofs of Theorems 1, 2 and Corollary.

Although all lemmas in this section are well-known, we again include direct proofs for the sake of convenience.

1. *Proof of Theorem* 1. Set $\phi_t=\sigma_t+i\cdot{}^*\sigma_t$ and $\Phi^T(t)=\phi_t\circ f_t^T-\phi_T$ for every t and T, where $f_t^T=f_t\circ(f_T)^{-1}$. Also we recall a standard construction of σ_t (cf. [3, V-19]). We may assume without loss of generality that C_0 is simple, and take a doubly connected relatively compact subdomain W of R_0 whose relative boundary consists of two smooth Jordan curves, say c^+ and c^-, homotopic to the given C_0 and to $-C_0$, respectively. For every t, let u_t be a bounded smooth function on $W_t=f_t(W)$ such that $u_t\equiv 1$ and $u_t\equiv 0$ in neighborhoods of $f_t(c^+)$ and $f_t(c^-)$, respectively. Then we can consider du_t as an element of $\Gamma_c(R_t)$, by setting $du_t\equiv 0$ on R_t-W_t. Green's formula gives

$$(\omega,{}^*du_t)_{R_t}=\iint_{W_t}du_t\wedge\omega=\int_{C_t}\omega=(\omega,\sigma_t)_{R_t}$$

for every smooth $\omega\in\Gamma_c(R_t)$, which implies that ${}^*du_t-\sigma_t\in{}^*\Gamma_{e0}(R_t)$.

LEMMA 5. *The family* $\{\Phi^T(s): s+T\in I\}$ *satisfies* (A^T), (B^T) *and* (C^T) *for every* $T\in I$.

Proof. Fix $T\in I$ and take any simply connected domain D on R_T. Then there is a harmonic function v_t on $f_t^T(D)$ such that $dv_t=\sigma_t$ for every t. Since $\mathrm{Re}\,\Phi^T(t-T)=d(v_t\circ f_t^T-v_T)$ on D, it belongs to $\Gamma_c(D)$ (, where and in the sequel, du denotes the distributional total differential of u if u is not smooth). Since a locally closed differential is closed (cf. [7, Proposition 4]), we conclude that $\mathrm{Re}\,\Phi^T(t-T)\in\Gamma_c(R_T)$ for every t.

Next set $\alpha_t=du_t+{}^*\sigma_t$ and $h_t=u_t\circ f_t^T-u_T$ for every t. Then $\alpha_t\in\Gamma_{e0}(R_t)$ as is shown above. Also since h_t has a compact support on W_T, a generalized Green's formula ([7, Proposition 3]) gives that $(dh_t, {}^*\omega)_{R_T}=\iint_{W_T}-dh_t\wedge\omega=\int_{\partial W_T}-h_t\cdot\omega=0$ for every smooth $\omega\in\Gamma_c(R_T)$. Hence $dh_t\in\Gamma_{e0}(R_T)$ for every t.

Similarly, we can show that $\alpha_t\circ f_t^T\in\Gamma_{e0}(R_T)$. (Cf. [7, Theorem 3]. In fact, approximate α_t by dh with smooth functions h with compact supports. Then as above we can see that $d(h\circ f_t^T)\in\Gamma_{e0}(R_T)$, and a simple estimation shows that $\|(dh-\alpha_t)\circ f_t^T\|_{R_T}^2\leqq K_t^T\cdot\|dh-\alpha_t\|_{R_t}^2$ ([7, Theorem 2]), where K_t^T is the maximal dilatation of f_t^T. Hence we can see the assertion.)

Thus we conclude that $\mathrm{Im}\,\Phi^T(t-T)=({}^*\sigma_t)\circ f_t^T-{}^*\sigma_T=\alpha_t\circ f_t^T-\alpha_T-dh_t$ belongs to $\Gamma_{e0}(R_T)$.

Finally since $\phi_T\in\Gamma_h^x(R_T)$, (B^T) holds with $S_T=R_T$ and also (C^T) does.

q. e. d.

LEMMA 6. *For every* t *and* T *in* I,

$$\lambda(t)-\lambda(T)=\mathrm{Re}\iint_{R_T}\Phi^T(t-T)\wedge{}^*\phi_T\,.$$

Proof. Since $\mathrm{Im}\,\Phi^T(t-T)\in\Gamma_{e0}(R_T)$ by Lemma 5, $(\mathrm{Im}\,\Phi^T(t-T), {}^*\sigma_T)_{R_T}=0$. Hence we have

$$\begin{aligned}\mathrm{Re}\iint_{R_T}\Phi^T(t-T)\wedge{}^*\phi_T&=(\mathrm{Re}\,\Phi^T(t-T), \sigma_T)_{R_T}\\&=(\sigma_t\circ f_t^T, \sigma_T)_{R_T}-\|\sigma_T\|_{R_T}^2\,.\end{aligned}$$

Here again by Proposition 3 of [7],

$$\begin{aligned}(\sigma_t\circ f_t^T, \sigma_T)_{R_T}&=(\sigma_t\circ f_t^T, {}^*du_T)_{R_T}=-\iint_{R_T}\sigma_t\circ f_t^T\wedge du_T\\&=-\iint_{R_t}\sigma_t\wedge d(u_T\circ(f_t^T)^{-1})=\int_{C_t}\sigma_t=\|\sigma_t\|_{R_t}^2\,.\end{aligned}$$

Thus Accola's Theorem ([1]) implies the assertion. q. e. d.

Now $(d\lambda/dt)(T)$ exists for every T by Lemma 4 and equals to $\mathrm{Re}\,\dot{I}(T)$, where

$\dot{l}(T)$ is as in Lemma 4. Hence $(d^2\lambda/dt^2)(0)$ exists and (3) in Theorem 1 holds by Theorem 3.

2. *Proof of Theorem* 1′. Set $\phi_t=-{}^*dg_t+i\cdot dg_t$ for every t, and let $\Phi^T(t-T)$ be defined as before. We recall one of standard definitions of Green's functions (cf. [3, Ch. IV, 6F]). For every t, take an exhaustion $\{S_n\}_{n=1}^{+\infty}$ of R_t consisting of regular subregions S_n with smooth boundary (cf. [3, Ch. II, 12D]). Assume that $p_t\in S_1$, and let $g_{t,n}$ be Green's function on S_n with the pole p_t, i.e. the harmonic function on $S_n-\{p_t\}$ with singularity $-\log|Z_t|$ and boundary values 0, for every n (cf. [3, Ch. III, 15A]). Consider $g_{t,n}$ as a continuous function on R_t by setting $g_{t,n}\equiv 0$ on R_t-S_n. Then it is well-known and easily seen that $g_t-g_{t,n}$ converges to 0 locally uniformly on R_t as n tends to $+\infty$, that $\|d(g_t-g_{t,n})\|_{R_t}$ decreases as n tends to $+\infty$, and that $\lim_{n\to+\infty}\|d(g_t-g_{t,n})\|_K=0$ for every compact K in R_t. Also we have the following

LEMMA 7. *The family* $\{\Phi^T(s): s+T\in I\}$ *satisfies* (A^T), (B^T) *and* (C^T) *for every* T.

Proof. Fix T and t. First (iv) implies that $\Phi^T(t-T)$ is holomorphic in a neighborhood of p_T. And the same argument as in the proof of Lemma 5 shows that $\mathrm{Re}\,\Phi^T(t-T)\in\Gamma_c(R_T)$.

Next let $\{g_{t,n}\}_{n=1}^{+\infty}$ and $\{g_{T,n}\}_{n=1}^{+\infty}$ be as above, and set $h_n=g_{t,n}\circ f_t^T-g_{T,n}$ for every n. Then, since $\|dh_n-\mathrm{Im}\,\Phi^T(t-T)\|_{R_T}\leqq\|d(g_{t,n}-g_t)\circ f_t^T\|_{R_T}+\|d(g_{T,n}-g_T)\|_{R_T}$, $\{\|dh_n\|_{R_T}\}_{n=1}^{+\infty}$ is a bounded sequence and $\lim_{n\to+\infty}\|dh_n-\mathrm{Im}\,\Phi^T(t-T)\|_K=0$ for every compact set K. Hence it is easy to see (cf. [4, Hilfssatz 7.4]) that dh_n converges weakly to $\mathrm{Im}\,\Phi^T(t-T)$ in $\Gamma(R_t)$, i.e. $\lim_{n\to+\infty}(dh_n, \omega)_{R_T}=(\mathrm{Im}\,\Phi^T(t-T), \omega)_{R_T}$ for every $\omega\in\Gamma(R_T)$.

Now, since every h_n has a compact support, Proposition 3 of [7] gives as before that $(dh_n, {}^*\omega)_{R_T}=0$ for every smooth $\omega\in\Gamma_c(R_T)$. Hence $(\mathrm{Im}\,\Phi^T(t-T), {}^*\omega)_{R_T}=0$ for every such ω. Thus we can conclude that $\mathrm{Im}\,\Phi^T(t-T)\in\Gamma_{e0}(R_T)$.

Finally, since $\nu(t-T)\equiv 0$ on $f_T(U_0)$ for every t (by (iv) and (8) in §3) and $\|dg_T\|_{R_T-U}$ is finite for every neighborhood U of p_T, (B^T) holds with, for instance, $S_T=R_T-\{p\in R_T: |Z_T(p)|\leqq 1/2\}$. Also since $\Phi^T(t-T)$ is holomorphic on $f_T(U_0)$ and p_T is the unique simple pole of ϕ_T, (C^T) holds. q.e.d.

LEMMA 8. *For every t and T,*

$$\gamma(t)-\gamma(T)=\frac{-1}{2\pi}\cdot\mathrm{Re}\iint_{R_T}\Phi^T(t-T)\wedge{}^*\phi_T\,.$$

Proof (cf. the proof of Lemma 4 of [8]). Fix t and T. Let $\{g_{T,n}\}$ be as before. Let $\chi(p)$ be a smooth function on R_T with a compact support in U_0 such that $\chi(p)\equiv 1$ in a neighborhood of p_T, and set $\tilde{h}_n(p)=(1-\chi(p))\cdot g_{T,n}(p)$ for every n. Then we can show as in the proof of Lemma 7 that $d\tilde{h}_n$ converges weakly to $(1-\chi)\cdot g_T$, and that $(d\tilde{h}_n, {}^*\omega)_{R_T}=0$ for every smooth $\omega\in\Gamma_c(R_T)$. Hence we conclude that $(1-\chi)\cdot g_T\in\Gamma_{e0}(R_T)$. Since $\mathrm{Re}\,\Phi^T(t-T)\in\Gamma_c(R_T)$ by Lemma 7, Green's formula gives

$$\iint_{R_T}\mathrm{Re}\,\Phi^T(t-T)\wedge dg_T=\iint_{R_T}\mathrm{Re}\,\Phi^T(t-T)\wedge d(\chi\cdot g_T)$$

$$=\lim_{\varepsilon\to 0}\int_{\partial U_\varepsilon}g_T\cdot\mathrm{Re}\,\Phi^T(t-T)=0\,,$$

where we set $U_\varepsilon=\{p\in R_T:\ |Z_T(p)|\leqq\varepsilon\}$.

Thus again by Proposition 3 of [7], we conclude that

$$\mathrm{Re}\iint_{R_T}\Phi^T(t-T)\wedge{}^*\phi_T=\iint_{R_T}-\mathrm{Im}\,\Phi^T(t-T)\wedge{}^*dg_T$$

$$=\lim_{\varepsilon\to 0}\Big(\lim_{n\to+\infty}\int_{\partial U_\varepsilon}h_n\cdot{}^*dg_T\Big)$$

$$=\lim_{n\to+\infty}-2\pi\cdot h_n(p_T)=-2\pi(\lambda(t)-\lambda(T))\,. \qquad \text{q.e.d.}$$

Thus Lemma 4 and Theorem 3 give the assertion as in the last part of the proof of Theorem 1.

Remark 3. If we used the notions of Royden's compactification and Dirichlet potentials (cf. [4]), then we could make some of above proofs geometrically clearer. But the author thinks the above proofs are reasonably elementary.

3. *Proof of Theorem* 2. In the sequel of this section, set $F(t)=\lambda(t)$ or $-2\pi\cdot\gamma(t)$.

To prove Theorem 2, consider the 1-parameter families $\{\mu(t, 0): t\in\mathfrak{R}\cap U\}$ and $\{\mu(0, t): t\in\mathfrak{R},\ it\in U\}$ with a real parameter t. Then d/dt for these families correspond to $\partial/\partial x$ and $\partial/\partial y$, respectively. Hence by Theorems 1 and 1′, we have

$$\varDelta F(0)=\mathrm{Re}\iint_{R_0}\varDelta\mu(0)\phi_0\wedge{}^*\phi_0$$

$$+\mathrm{Re}\iint_{R_0}\Big(\frac{\partial\Phi}{\partial x}(0)\wedge{}^*\frac{\partial\Phi}{\partial x}(0)+\frac{\partial\Phi}{\partial y}(0)\wedge{}^*\frac{\partial\Phi}{\partial y}(0)\Big).$$

Since $\varDelta\mu(0)=0$ by (v) and since

$$\iint_{R_0}\frac{\partial\Phi}{\partial x}(0)\wedge\frac{{}^*\partial\Phi}{\partial y}(0)=\Big(\frac{\partial\Phi}{\partial x}(0),\ \overline{\frac{\partial\Phi}{\partial y}(0)}\Big)^{\mathfrak{C}}_{R_0}$$

$$=\Big(\frac{\partial\Phi}{\partial y}(0),\ \overline{\frac{\partial\Phi}{\partial x}(0)}\Big)^{\mathfrak{C}}_{R_0}=\iint_{R_0}\frac{\partial\Phi}{\partial y}(0)\wedge{}^*\frac{\partial\Phi}{\partial x}(0)\,,$$

we have

$$\Delta F(0)=4\cdot\mathrm{Re}\iint_{R_0}\frac{\partial\Phi}{\partial\bar{t}}(0)\wedge{}^*\frac{\partial\Phi}{\partial t}(0).$$

Now set $\dot{\Phi}=(\partial\Phi/\partial\bar{t})(0)$. Since $(\partial\Phi^{0,1}/\partial\bar{t})(0)=(\partial\mu/\partial\bar{t})(0)\cdot\phi_0=0$ by (v) and (4) in Lemma 1, $\dot{\Phi}=(\partial\Phi^{1,0}/\partial\bar{t})(0)$, and hence $^*\dot{\Phi}=-i\cdot\dot{\Phi}$. Since $\dot{\Phi}\in\Gamma_c^{\mathfrak{C}}(R_0)$ by (A^0), $d\dot{\Phi}=0$ and also $d^*\dot{\Phi}=-i\cdot d\dot{\Phi}=0$. Thus $\dot{\Phi}\in\Gamma_h^{\mathfrak{C}}(R_0)$ (, which implies further that $\dot{\Phi}$ is a holomorphic differential).

Since $\Psi=(\partial\Phi/\partial\bar{t})(0)-\overline{(\partial\Phi/\partial t)(0)}=2i\cdot(\partial\,\mathrm{Im}\,\Phi/\partial\bar{t})(0)\in\Gamma_{e0}^{\mathfrak{C}}(R_0)$ by (A^0), $(\dot{\Phi},\Psi)_{R_0}^{\mathfrak{C}}=0$. Hence $(\dot{\Phi},\overline{(\partial\Phi/\partial t)(0)})_{R_0}^{\mathfrak{C}}=(\dot{\Phi},\dot{\Phi})_{R_0}^{\mathfrak{C}}\geqq 0$. Thus we obtain (4) and (4′) in Theorem 2.

4. *Proof of Corollary.* First, continuity of $F(t)$ follows from Lemmas 6 and 8. In fact, fix $T=(T_1,\cdots,T_n)$. Since $\iint_{R_T}\Phi^T(t-T)\wedge{}^*\phi_T=\iint_{R_T}(\Phi^T)^{0,1}(t-T)\wedge{}^*\phi_T$, where $(\Phi^T)^{0,1}(t-T)$ is the projection of $\Phi^T(t-T)$ to $\Gamma^{0,1}(R_T)$, we can see by Lemma 4 that $|F(t)-F(T)|=O(\sum_{j=1}^n|t_j-T_j|)$ as t tends to T.

Next fix a complex line $L=\{at+b: t\in\mathfrak{C},\ a\in\mathfrak{C}^n,\ b\in\mathfrak{C}^n\}$ arbitrarily, and let G be any component of $L\cap U$. Set $\Omega=\{t: at+b\in G\}$ and $\mu^L(t)=\mu(at+b)$ for every $t\in\Omega$. Then since $\mu^L(t)$ depends on t holomorphically, $\{\mu^L(t)\}$ satisfies (ii′) and $\partial\mu^L/\partial\bar{t}\equiv 0$ on Ω. And since $\partial\mu^L/\partial t$ also depends on t holomorphically, $\{\mu^L(t)\}$ satisfies (iii′) and (v). Hence by Theorem 2, $\Delta(F|_L)(0)$ exists and non-negative, where $F|_L(t)=F(at+b)$. Since b can be taken arbitrarily in G, we conclude that $\Delta(F|_L)$ exists and non-negative on the whole Ω, which implies that $F|_L$ is subharmonic on Ω. (In fact, set $F_\varepsilon(t)=F|_L(t)+\varepsilon\cdot|t|^2$ for every positive ε, and let h_ε^V be the harmonic function on V such that $h_\varepsilon^V=F_\varepsilon$ on ∂V for every disk V with $\bar{V}\subset\Omega$. If $\max_{\bar{V}}(h_\varepsilon^V-F_\varepsilon)$ were positive and hence were attained at a point t_0 in V, then $\Delta F_\varepsilon(t_0)$ should be non-positive, which contradicts to the fact that $\Delta F_\varepsilon\geqq 4\cdot\varepsilon>0$. Hence $h_\varepsilon^V\leqq F_\varepsilon$ on V for every V as above. And letting ε tend to 0, we can conclude that $F|_L$ is subharmonic on Ω.) Since L is arbitrary, we conclude the assertion.

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DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY