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# SELF-MAPS OF SPHERE BUNDLES II

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## § 1. Introduction.

Let $E$ be an oriented orthogonal $q$-sphere bundle over a connected finite $CW$-complex $B$. A fibre-preserving map $f: E \to E$ is said to have degree $m$ when its restriction to some fibre is a map of degree $m$ in the familiar sense; because $B$ is path-connected it makes no difference which fibre we choose. Given $E$ and an integer $m$ is there a fibre-preserving map $f: E \to E$ of degree $m$? This question was put to me in 1971 by I. M. James, and in [2] there are some answers in fairly general situations. In the present paper I consider in more detail the special case where $B$ is a sphere $S^{r+1}$. We first make some simple observations.

The identity map has degree 1, and when $q$ is even $E$ always admits a fibre-preserving map of degree $-1$; this is because the antipodal map $a: S^q \to S^q$ commutes with the action of the group $SO(q+1)$ of rotations in $\boldsymbol{R}^{q+1}$ and therefore extends to a fibre-preserving map: it would be interesting to know what happens when $E$ is a general oriented $q$-spherical fibration with $q$ even. If $E$ admits fibre-preserving maps of degrees $m, n$ then their composite is a fibre-preserving map of degree $mn$. Apart from this, nothing is very obvious.

Let $\pi: E \to B$ be the projection. Then when $E$ has a cross-section $s$ the composite $s\pi: E \to E$ is a fibre-preserving map of degree 0. In [2] the converse is proved, namely that if $E$ admits a fibre-preserving map of degree 0 then $E$ has a cross-section. (It is not the case that every fibre-preserving map $f: E \to E$ of degree 0 is homotopic through fibre-preserving maps to one of the form $s\pi$ for some cross-section $s$, but if $B$ is covered by $k$ contractible open subsets then $f^k$ is homotopic through fibre-preserving maps to $s\pi$ for some cross-section $s$.) Some of the main results of [2] describe the structure of the set $A(E)$ of integers $m$ such that $E$ admits a fibre-preserving map of degree $m$. In the present paper we prove some results that allow us to estimate $A(E)$ when $B = S^{r+1}$.

If $E^*$ is a fibre bundle over $S^{r+1}$ with fibre $F^*$ let $o(E^*)$ be the obstruction to a cross-section of $E^*$, as defined in § 2 below. From now on let $B = S^{r+1}$. In § 2 we show that a necessary condition for there to be a fibre-preserving map $E \to E$ of degree $m$ is that $\phi_m o(E) = o(E)$. Here $\phi_m: \pi_r S^q \to \pi_r S^q$ is induced by a map of degree $m$ on $S^q$.

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THEOREM 1. *Let $q$ be odd. If there is a fibre-preserving map $E\rightarrow E$ of degree $m$ then*

(i) $\phi_{1-m}o(E)=0$ *and*

(ii) $(m(m-1)/2)[\iota_{q+1}, \iota_{q+1}]\circ\Sigma_*^{q+1}o(E)=0$.

Here $[\,,]$ is the J.H.C. Whitehead product, and $\Sigma_*$ is the suspension homomorphism. Theorem 1 is proved as (4.1), (4.3) in § 4.

Our methods can also be used to give conditions sufficient for the existence of a fibre-preserving map $E\rightarrow E$ of degree $m$. In [1] Part II, § 5 some calculations are carried out and, for example, necessary and sufficient conditions are given when $q$ is odd and $r\leqq q+2$.

In § 5 of the present paper we consider the special cases where $q=1, 3, 7$, and prove that the necessary conditions given in Theorem 1 are then sufficient. When $q=1$ condition (ii) of Theorem 1 is satisfied trivially. When $q=3$ we obtain as (5.3), (5.5)

COROLLARY 1. *Let $q=3$. Then there is a fibre-preserving map $E\rightarrow E$ of degree $m\Leftrightarrow$both*

(i) $(m-1)o(E)=0$ *and*

(ii) $(m(m-1)/2)a_4\circ\Sigma_*^4 o(E)=0$

where $a_4\in\pi_7 S^4$ is described explicitly in § 5. In an interesting paper [5] Seiya Sasao considers a related problem, and in § 6 of the present paper we compare Corollary 1 with Sasao's results.

When $q$ is even it may be the case that $E$ has a cross-section and yet there exist no fibre-preserving maps $E\rightarrow E$ of some degrees $m$: this cannot happen when $q$ is odd. When $E$ has a cross-section we can write $E$ as the fibre suspension of an oriented orthogonal $q-1$-sphere bundle $E'$, and in § 7 we prove

THEOREM 2. *Let $q$ be even and suppose that $E$ has a cross-section. Then there is a fibre-preserving map $E\rightarrow E$ of degree $m\Leftrightarrow$*

$$(m(m-1)/2)[\iota_q, \iota_q]\circ\Sigma_*^q o(E')\in[m\iota_q, \pi_{r+1}S^q].$$

By [9] 3.59 we have $[\iota_q, \iota_q]\circ\Sigma_*^q o(E')=[\iota_q, \Sigma_* o(E')]$ and so we have

COROLLARY 2. *Let $q$ be even and suppose that $E$ has a cross-section. Then there are fibre-preserving maps $E\rightarrow E$ of all odd degrees, and of all degrees $m\equiv 0$* mod 4.

## § 2. The Obstruction to a Fibre-Preserving Map—Generalities.

It is known that a fibre-preserving map of fibre bundles corresponds naturally to a cross-section of a bundle whose fibre is a function space. This point of view is traceable to I.M. James and was taken in [2] to prove results about the structure of $A(E)$; it also turns out to be helpful when doing calculations.

Let $G_m^q$ be the function space of maps $f: S^q \to S^q$ of degree $m$, with the compact-open topology. We define a left action $*$ of the group $SO(q+1)$ of rotations on $G_m^q$ by $(A*f)(x)=A\cdot(f(A^{-1}\cdot x))$. Here $\cdot$ is the standard action of $SO(q+1)$ on $S^q$. Let $E$ be an oriented orthogonal $q$-sphere bundle over $B$, and let $P$ be its associated principal $SO(q+1)$-bundle. Let $E_m$ be the bundle $PG_m^q$ associated with $P$ and with fibre $G_m^q$.

There is a natural one to one correspondence between fibre-preserving maps $E \to E$ of degree $m$ and cross-sections of $E_m$, and we look for obstructions to a cross-section of $E_m$. When $B$ is a sphere $S^{r+1}$ there is only one obstruction which can be defined in a familiar way, but its calculation in particular cases is not so easy.

Indeed, let $E^*$ be any fibre bundle over $S^{r+1}$ with fibre $F^*$. Then the homotopy exact sequence of the fibering takes the form

$$\cdots \to \pi_{t+1}S^{r+1} \longrightarrow \pi_t F^* \longrightarrow \pi_t E^* \longrightarrow \pi_t S^{r+1} \longrightarrow \pi_{t-1}F^* \to \cdots.$$

Let $\iota_{r+1}$ be the generator of $\pi_{r+1}S^{r+1}$ represented by the identity map, and let $o(E^*)$ be the image in $\pi_r F^*$ of $\iota_{r+1}$. Then $E^*$ has a cross-section if and only if $o(E^*)$ is the trivial element of $\pi_r F^*$, and so $o(E^*)$ may be regarded as the obstruction to a cross-section of $E^*$. So when $B=S^{r+1}$ the obstruction to a fibre-preserving map $E \to E$ of degree $m$ is $o(E_m) \in \pi_r G_m^q$. We want to calculate this obstruction in terms of standard invariants of $E$, for example $o(E)$. We first point out

(2.1) A necessary condition for there to be a fibre-preserving map $E \to E$ of degree $m$ is that $\phi_m o(E)=o(E)$.

Here $\phi_m: \pi_r S^q \to \pi_r S^q$ is induced by composition with a map $S^q \to S^q$ of degree $m$. We note that $\phi_m$ is not in general multiplication by $m$, although this is the case if $r<2q-1$. (For clarification of this point see [9] Theorem 5.15.)

To prove (2.1) observe that a fibre-preserving map $f: E \to E$ of degree $m$ produces the following commuting diagram, where the rows are the homotopy exact sequence of $E$.

$$\begin{array}{ccccccc}
\cdots \to & \pi_{t+1}S^{r+1} & \longrightarrow & \pi_t S^q & \longrightarrow & \pi_t E & \to \cdots \\
& \downarrow 1 & & \downarrow \phi_m & & \downarrow f_* & \\
\cdots \to & \pi_{t+1}S^{r+1} & \longrightarrow & \pi_t S^q & \longrightarrow & \pi_t E & \to \cdots.
\end{array}$$

## §3. Odd Values of $q$.

Let $k_m: S^q \to G_n^q$ be the map defined in [2] §2 where $n=m$, 1, 0 according as $q$ is odd, $q$ is even and $m$ is odd, or $q$ and $m$ are both even. (Given $x, y \in S^q$ let $\theta$ be the distance along some geodesic from $x$ to $y$. On this geodesic and at distance $m\theta$ we have $k_m(x)(y)$. Then $k_m$ is equivariant with respect to the standard left action of $SO(q+1)$ on $S^q$ and the left action $*$ on $G_n^q$.

As in § 2, $P$ is the principal $SO(q+1)$-bundle over $B$ associated with $E$, and because it is equivariant $k_m$ extends from fibres to a fibre-preserving map $P(k_m): E \to E_n$; for the rest of this section we take $q$ to be odd, so that $n=m$. Then if $E$ has a cross-section $s$ the composite $P(k_m)s$ is a cross-section of $E_m$, and therefore there is a fibre-preserving map $E \to E$ of degree $m$.

Now let $B=S^{r+1}$. When $E$ does not have a cross-section the condition that $o(E_m)$ should be zero translates into a condition on $o(E)$ as follows. Consider the commuting diagram of group homomorphisms

$$\begin{array}{ccccccccccc}
\cdots \to & \pi_{t+1}S^{r+1} & \longrightarrow & \pi_t S^q & \longrightarrow & \pi_t E & \longrightarrow & \pi_t S^{r+1} & \longrightarrow & \pi_{t-1}S^q & \to \cdots \\
& \Vert & & \downarrow {\scriptstyle k_{m*}} & & \downarrow & & \Vert & & \downarrow {\scriptstyle k_{m*}} & \\
\cdots \to & \pi_{t+1}S^{r+1} & \longrightarrow & \pi_t G_m^q & \longrightarrow & \pi_t E_m & \longrightarrow & \pi_t S^{r+1} & \longrightarrow & \pi_{t-1}G_m^q & \to \cdots
\end{array}$$

where the unlabelled vertical arrow denotes the homomorphism induced by $P(k_m)$. Taking $t=r+1$ we obtain $o(E_m)=k_{m*}o(E)$. In practice we usually know $o(E)$; but the homomorphism $k_{m*}$ is a less accessible object, in part because the computation of $\pi_r G_m^q$ is complicated by the appearance of Whitehead products [8].

## § 4. Homotopy Groups of Function Spaces.

Let $(1, 0, 0, \cdots, 0) \in S^q$ be chosen as the basepoint and let $F_m^q$ be the subspace of $G_m^q$ consisting of the basepoint-preserving maps of degree $m$. The evaluation map $e: G_m^q \to S^q$, given by $e(f)=f(1, 0, 0, \cdots, 0)$, is a fibration with fibre $F_m^q$. Let $I_m: \pi_t F_m^q \to \pi_{t+q} S^q$ be the Hurewicz isomorphism [8]. Then the homotopy exact sequence of $e$ gives us an exact sequence

$$\cdots \to \pi_{t+1}S^q \xrightarrow{P_{t+1}} \pi_{t+q}S^q \longrightarrow \pi_t G_m^q \xrightarrow{e_*} \pi_t S^q \xrightarrow{P_t} \pi_{t+q-1}S^q \to \cdots$$

of homomorphisms of abelian groups. The unlabelled arrow denotes the homomorphism $i_* I_m^{-1}$ where $i$ is the inclusion of $F_m^q$ in $G_m^q$. According to [8], [10], $P_t(\theta)=\pm[m\iota_q, \theta]$ where $[\ ,\ ]$ is the Whitehead product.

For the rest of § 4 let $q$ be odd, and let $E$ be an oriented orthogonal $q$-sphere bundle over $S^{r+1}$. The composite $ek_m: S^q \to S^q$ is a map of degree $1-m$, and composition with this defines the homomorphism $\phi_{1-m}: \pi_r S^q \to \pi_r S^q$. Consequently

(4.1) A necessary condition for $E$ to admit a fibre-preserving map $E \to E$ of degree $m$ is that $\phi_{1-m}(o(E))=0$.

(When $r<2q-1$ this is equivalent to (2.1), and (2.1) has been proved whether $q$

is odd or even.)

Let $\jmath: G_m^q \to F_m^{q+1}$ be the inclusion defined by suspending maps of degree $m$.

LEMMA (4.2). *The homomorphism*

$$I_m j_* k_{m*}: \pi_r S^q \longrightarrow \pi_r G_m^q \longrightarrow \pi_r F_m^{q+1} \longrightarrow \pi_{r+q+1} S^{q+1}$$

*is given by*

$$\beta \longmapsto \pm(m(m-1)/2)[\iota_{q+1}, \iota_{q+1}] \circ \Sigma_*^{q+1} \beta\,.$$

*Here* $\Sigma_*: \pi_r S^q \to \pi_{r+1} S^{q+1}$ *is the suspension homomorphism.*

To prove (4.2) we note that $I_m j_* k_{m*} \beta$ is the Hopf construction of the adjoint $k'_m: S^q \times S^q \to S^q$ of $k_m$, preceded by $\Sigma_*^{q+1}\beta$. It therefore suffices to show that $I_m j_* k_{m*} \iota_q = \pm(m(m-1)/2)[\iota_{q+1}, \iota_{q+1}]$. But when $m=-1$ [6] § 23.5 tells us that $I_{-1} j_* k_{-1*} \iota_q$ is the Hopf construction of a generator $\varphi$ of the kernel of the homomorphism $\pi_q SO(q+1) \to \pi_q SO(q+2)$ induced by the usual inclusion $i'$.

Now $\Sigma_* I_{-1} j_* k_{-1*} \iota_q = \Sigma_* J\varphi = J i'_* \varphi = 0$ where $J$ denotes the Whitehead homomorphisms. By [4] Theorem 7.7 $I_{-1} j_* k_{-1*} \iota_q = n[\iota_{q+1}, \iota_{q+1}]$ for some integer $n$.

But $k'_{-1}$ has type $(2, -1)$ and so, by [9] 3.70, $I_{-1} j_* k_{-1*} \iota_q$ has Hopf invariant $\pm 2$. But, by [9] Theorem 5.31, $[\iota_{q+1}, \iota_{q+1}]$ has Hopf invariant $\pm 2$. So $n = \pm 1$. This proves (4.2) in the special case where $m=-1$.

From the definition of $k_m$ we find that $k'_m(x, y) = k'_{1-m}(y, x) = k'_{m-1}(y, k'_{-1}(y, x))$ and so $I_m \jmath_* k_{m*} \iota_q$ is the Hopf construction of the composite

$$S^q \times S^q \xrightarrow{\text{switch}} S^q \times S^q \xrightarrow{1 \times k'_{-1}} S^q \times S^q \xrightarrow{k'_{m-1}} S^q$$

which, according to [3] Theorem 2.19, differs by a multiple of a Whitehead product $[\iota_{q+1}, \iota_{q+1}]$ from the negative of the Hopf construction of just

$$S^q \times S^q \xrightarrow{1 \times k'_{-1}} S^q \times S^q \xrightarrow{k'_{m-1}} S^q$$

namely $-I_{1-m} \jmath_* c_*(k_{-1*}\iota_q, k_{m-1*}\iota_q)$ where $c: G_{-1}^q \times G_{m-1}^q \to G_{1-m}^q$ is defined by taking composites. But this last expression equals $I_{m-1} j_* k_{m-1*} \iota_q - \phi_{m-1} I_{-1} j_* k_{-1*} \iota_q$ and so, inductively, $\Sigma_* I_m j_* k_{m*} \iota_q = 0$.

We can now argue as in the case where $m=-1$, noting that $k'_m$ is a map of type $(1-m, m)$. This proves (4.2).

Recall again that $o(E_m) = k_{m*} o(E)$. Then in addition to (4.1) we have

(4.3) A necessary condition for $E$ to admit a fibre-preserving map $E \to E$ of degree $m$ is that $(m(m-1)/2)[\iota_{q+1}, \iota_{q+1}] \circ \Sigma_*^{q+1} o(E) = 0$.

We emphasise that here $q$ is odd. In the particular cases where $q=1, 3, 7$ it is possible to say even more.

## § 5. Necessary and Sufficient Conditions When $q=1, 3, 7$.

When $q=1$ questions about fibre-preserving maps are easy to answer, because $\pi_{i+1}S^1$ is trivial for $i>0$. Consequently $e: G_m^1 \to S^1$ is a weak homotopy equivalence. Let $E$ be an oriented 1-sphere bundle over a connected finite $CW$-complex $B$, and let $\chi(E) \in H^2(B; \boldsymbol{Z})$ be the Euler characteristic of $E$, namely the obstruction to a cross-section of $E$. Then because $ek_m$ is a map of degree $1-m$ it follows that there is a single obstruction to a cross-section of $E_m$, namely $(1-m)\chi(E)$. We have

(5.1) When $q=1$ $E$ admits a fibre-preserving map $E \to E$ of degree $m$ if and only if $(1-m)\chi(E)=0$.

The situation when $q \neq 1$ is nontrivial however, and for the remainder of § 5 we suppose that $B=S^{r+1}$ where $r \leqq 3q-1$, and that $q$ is 3 or 7. We prove

(5.2) The necessary conditions (4.1), (4.3) are also sufficient for $E$ to admit a fibre-preserving map $E \to E$ of degree $m$.

*Proof of* (5.2): If (4.1) holds then we know that $k_{m*}o(E)=i_* I_m^{-1}\gamma$ for some $\gamma \in \pi_{r+q}S^q$. If (4.3) also holds then $I_m j_* i_* I_m^{-1}\gamma=0$. But $I_m j_* i_* I_m^{-1}$ is $\sum_*$ and so we know that $\sum_*\gamma=0$.

However, according to [4] Theorem 7.7 the kernel of $\sum_*: \pi_r S^q \to \pi_{r+1}S^{q+1}$ is the image of the homomorphism

$$\pi_{r-q+1}S^q \longrightarrow \pi_r S^q$$

$$\delta \longmapsto [\delta, \iota_q].$$

Since $q$ is 3 or 7, $S^q$ is an $H$-space and therefore $\sum_*$ is injective. So $\gamma=0$, and therefore $o(E_m)=0$. This proves (5.2).

The conditions (4.1), (4.3) can be simplified somewhat in the special cases considered here, namely when $q=3, 7$. Firstly, because $S^q$ is an $H$-space, (4.1) is equivalent to

(5.3) $$(m-1)o(E)=0.$$

We next analyse (4.3) but restrict ourselves to the case where $q=3$. We recall from [7] Lemma 4.3 that

(5.4) $$[\iota_4, \iota_4]=2\nu_4-a_4.$$

Here $\nu_4 \in \pi_7 S^4$ is the Hopf class, namely the Hopf construction of quaternionic multiplication restricted to $S^3 \times S^3$, and $a_4=\sum_* a_3$ where $a_3 \in \pi_6 S^3$ is the Hopf construction of

$$g: S^3 \times S^2 \longrightarrow S^2$$

given by

$$(x, y) \longmapsto xyx^{-1}.$$

(We are thinking of $S^2$ as the space of unit quaternions with vanishing real part, and multiplication on $S^3$ is again quaternionic.) According to [7] Theorem 7.2, $a_3$ has order 12 and generates $\pi_6 S^3$. Also $\pi_7 S^4 \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/12\boldsymbol{Z}$ where the summands are generated by $\nu_4$ and $a_4$.

In view of (5.4), (5.3) a necessary and sufficient condition for $E$ to admit a fibre-preserving map $E \to E$ of degree $m$ is (5.3) together with

$$(m(m-1)/2)a_4 \circ \textstyle\sum_*^4 o(E) = 0. \tag{5.5}$$

A similar analysis can be carried out in the case where $q=7$.

## § 6. Comparison With a Result of Seiya Sasao.

It is interesting to compare (5.5) with Example 1 of [5]. In Example 1 Sasao takes $q=3$ and $E$ is a principal $S^3$-bundle over $S^{r+1}$, whereas (5.5) is applicable whether $E$ is a principal $S^3$-bundle or not. On the other hand, (5.5) applies when $r \leqq 5$, whereas Sasao makes no such requirement. Sasao proves that a map of degree $m$ on fibres $S^3 \to S^3$ extends to a map $E \to E$ (not necessarily fibre-preserving) if and only if

$$(m(m-1)/2)a_3 \circ \textstyle\sum_*^3 o(E) \in o(E) \circ (\pi_{r+3} S^r). \tag{6.1}$$

When $r \leqq 5$ $\sum_* : \pi_{r+3} S^3 \to \pi_{r+4} S^4$ is injective, and (6.1) is then equivalent to

$$(m(m-1)/2)a_4 \circ \textstyle\sum_*^4 o(E) \in \sum_* (o(E) \circ (\pi_{r+3} S^r). \tag{6.2}$$

So Sasao's necessary and sufficient condition is less restrictive than (5.5) alone, at least when $r \leqq 5$. ((5.5) and (5.3) must together imply Sasao's condition, because a fibre-preserving map $E \to E$ of degree $m$ extends a map $S^3 \to S^3$ of degree $m$ on fibres.)

## § 7. When $q$ is Even.

Let $q$ be even. We shall see that $E$ may have a cross-section and yet fail to admit fibre-preserving maps $E \to E$ of all degrees. It follows that there is no $SO(q+1)$-equivariant map from $S^q$ to $G_m^q$, whereas when $q$ was odd we had the map $k_m$.

By (2.1), if there is a fibre-preserving map $E \to E$ of even degree $m$ then $\phi_m o(E) = o(E)$, and so if $r < 2q-1$ we have $(m-1)o(E)=0$. But $q$ is even and so there is a fibre-preserving map $E \to E$ of degree $-1$. Therefore when there is a fibre-preserving map $E \to E$ of even degree, and $r < 2q-1$, we have $o(E)=0$, namely $E$ has a cross-section. From now on we consider only bundles $E$ which have cross-sections, namely bundles $E$ which are unreduced fibre suspensions of orthogonal $q-1$-sphere bundles $E'$; we do not require $r<2q-1$.

Let $P'$ be the principal $SO(q)$-bundle associated with $E'$ and let $SO(q)$ act on $S^q$ by the suspension of the action on $S^{q-1}$. Then $k_m: S^{q-1}\to G_m^{q-1}$, $j: G_m^{q-1}\to F_m^q$, $i: F_m^q\to G_m^q$ are $SO(q)$-equivariant and therefore $jk_m$ extends from fibres to a fibre-preserving map $P'(jk_m): E'\to E_m$. It follows that $i_*j_*k_{m*}o(E')$ is the obstruction $o(E_m)\in\pi_r G_m^q$ to a fibre-preserving map of degree $m$ from $E$ to itself.

(7.1) There is a fibre-preserving map $E\to E$ of degree $m\Leftrightarrow$

$$(m(m-1)/2)[\iota_q, \iota_q]\circ\Sigma_*^q o(E')\in[m\iota_q, \pi_{r+1}S^q].$$

To prove (7.1) note that $i_*j_*k_{m*}o(E')=i_*I_m^{-1}(I_m j_*k_{m*})o(E')$. Now (4.2) says that

$$I_m j_* k_{m*}: \pi_r S^{q-1} \longrightarrow \pi_r G_m^{q-1} \longrightarrow \pi_r F_m^q \longrightarrow \pi_{r+q}S^q$$

is given by

$$\beta \longmapsto \pm(m(m-1)/2)[\iota_q, \iota_q]\circ\Sigma_*^q\beta.$$

On the other hand, exactness of the sequence of homomorphisms in §4 tells us that the kernel of $i_*I_m^{-1}$ is $[m\iota_q, \pi_{r+1}S^q]$, and this proves (7.1).

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