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THE q-ANALOGUE OF THE p-ADIC GAMMA FUNCTION

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Introduction.

The p-adic gamma function $\Gamma_p(x)$ was defined and studied by Morita [9] and the p-adic log-gamma function $G_p(x)$ was defined and studied by Diamond [3]. The Morita's gamma function $\Gamma_p(x)$ is defined by

$$\Gamma_p(x)=\lim_{\substack{n\to x\\ \text{in } \boldsymbol{Z}_p}}(-1)^n \prod_{0<j<n}{}^{*} j \qquad \text{for} \quad x\in \boldsymbol{Z}_p,$$

where n runs over positive integers and $\prod^*$ means that indices j divisivle by p are omitted. The Diamond's log-gamma function $G_p(x)$ and $G_p^*(x)$ are defined by

$$G_p(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}(x+j)\{\log(x+j)-1\} \qquad \text{for} \quad x\in \boldsymbol{C}_p-\boldsymbol{Z}_p$$

and

$$G_p^*(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}{}^{*}(x+j)\{\log(x+j)-1\} \qquad \text{for} \quad x\in \boldsymbol{C}_p-\boldsymbol{Z}_p^*,$$

where log is the Iwasawa p-adic logarithm [5], $\boldsymbol{C}_p$ denotes the completion of the algebraic closure of the p-adic number field $\boldsymbol{Q}_p$ and $\sum^*$ means that indices j divisible by p are omitted in the summation.

Then $G_p(x)$ and $G_p^*(x)$ have the following two connections with $\Gamma_p(x)$.

THEOREM (Diamond [3], Ferrero-Greenberg [4]).

(1) $$\log \Gamma_p(x)=G_p^*(x) \qquad \textit{for} \quad x\in p\boldsymbol{Z}_p.$$

(2) $$\log \Gamma_p(x)=\sum_{\substack{0\leq i<p\\ x+i\in \boldsymbol{Z}_p^*}} G_p\left(\frac{x+i}{p}\right) \qquad \textit{for} \quad x\in \boldsymbol{Z}_p.$$

A generalized p-adic gamma function $\Gamma_{p,q}(x)$, depending on a parameter $q\in \boldsymbol{C}_p$ with $|q-1|_p<1$ and $q\neq 1$, was defined and studied by Koblitz [7], [8]. We recall that the Koblitz' function $\Gamma_{p,q}(x)$ is defined by

$$\Gamma_{p,q}(x)=\lim_{\substack{n\to x\\ \text{in } \boldsymbol{Z}_p}}(-1)^n \prod_{0<j<n}{}^{*} \frac{1-q^j}{1-q} \qquad \text{for} \quad x\in \boldsymbol{Z}_p,$$

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where n runs over positive integers.

As for the log-gamma functions $G_p(x)$ and $G_p^*(x)$ Koblitz defined only the p-adic psi-function $\psi_{p,q}(x)$ and $\psi_{p,q}^*(x)$, which are analogues of the derivatives

$$\psi_p(x)=\frac{d}{dx}G_p(x) \quad \text{and} \quad \psi_p^*(x)=\frac{d}{dx}G_p^*(x).$$

For $q\in \boldsymbol{C}_p$ such that $|q-1|_p<1$ and $\log(q)\neq 0$, let

$$r(q)=\frac{|p|_p^{1/(p-1)}}{|\log(q)|_p}.$$

Let $d(x)=\min_{u\in \boldsymbol{Z}_p}|x-u|_p$ and $d^*(x)=\min_{u\in \boldsymbol{Z}_p^*}|x-u|_p$ for $x\in \boldsymbol{C}_p$.

Let $D(q)=\{x\in \boldsymbol{C}_p \mid 0<d(x)<r(q)\}$ and

$D^*(q)=\{x\in \boldsymbol{C}_p \mid 0<d^*(x)<r(q)\}$.

Putting

$$\psi_{p,q}(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\log\frac{1-q^{x+j}}{1-q} \qquad \text{for} \quad x\in D(q)$$

and

$$\psi_{p,q}^*(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}^*\log\frac{1-q^{x+j}}{1-q} \qquad \text{for} \quad x\in D^*(q),$$

Koblitz [7] gave the following

THEOREM.

(1) $$\frac{d}{dx}\log\Gamma_{p,q}(x)=\psi_{p,q}^*(x) \qquad \text{for} \quad x\in p\boldsymbol{Z}_p.$$

(2) $$\frac{d}{dx}\log\Gamma_{p,q}(x)=\frac{1}{p}\sum_{\substack{0\leq i<p\\ x+i\in \boldsymbol{Z}_p^*}}\psi_{p,q^p}\Big(\frac{x+i}{p}\Big)+\Big(1-\frac{1}{p}\Big)\log\frac{1-q^p}{1-q} \quad \textit{for} \quad x\in \boldsymbol{Z}_p.$$

The purpose of this paper is to construct and study natural analogues $G_{p,q}(x)$ and $G_{p,q}^*(x)$ of the p-adic log-gamma functions $G_p(x)$ and $G_p^*(x)$, which have connections with $\Gamma_{p,q}(x)$.

Let $l_2(z)$ be the p-adic dilogarithm defined and studied by Coleman [2]. For a positive integer n, let $\tilde{n}=[(n-1)/p]+1$ where [ ] means Gauss symbol. Then the map ~ extends to a continuous function on $\boldsymbol{Z}_p$ with values in $\boldsymbol{Z}_p$ (See [7]). Since $l_2(z)$ is locally analytic on $\boldsymbol{C}_p-\{1\}$. Using Diamond's theorem [3], we may define analogues $G_{p,q}(x)$ and $G_{p,q}^*(x)$ of the log-gamma functions $G_p(x)$ and $G_p^*(x)$ by

$$G_{p,q}(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\Big\{-\frac{1}{\log(q)}l_2(q^{x+j})-(x+j)\log(1-q)\Big\}$$

for $x\in D(q)$ and

$$G_{p,q}^{*}(x)=\lim_{n\to\infty}\frac{1}{p^{n}}\sum_{0\leqq j<p^{n}}^{*}\left\{-\frac{1}{\log(q)}l_{2}(q^{x+j})-(x+j)\log(1-q)\right\}$$

for $x\in D^{*}(q)$. Then we obtain

THEOREM (3.1). *Suppose* $|q-1|_{p}<|p|_{p}^{1/(p-1)}$.

(1) $$\log\Gamma_{p,q}(x)=G_{p,q}^{*}(x)+\frac{p-1}{24}\log(q) \quad \textit{for} \quad x\in p\boldsymbol{Z}_{p}.$$

(2) $$\log\Gamma_{p,q}(x)=\sum_{\substack{0\leqq i<p\\ x+i\in\boldsymbol{Z}_{p}^{*}}}\left\{G_{p,q^{p}}\left(\frac{x+i}{p}\right)+\frac{\log(q)}{24}\right\}+(x-\tilde{x})\log\frac{1-q^{p}}{1-q}$$

for $x\in\boldsymbol{Z}_{p}$.

Remark. By the definition of $\tilde{}$ we have

$$\frac{d}{dx}(\tilde{x})=\lim_{n\to\infty}\frac{(x+p^{n})^{\sim}-x^{\sim}}{p^{n}}=\frac{1}{p} \quad \text{for} \quad x\in\boldsymbol{Z}_{p},$$

Differentiating in the equations of our theorem in the above sense we have the equations of the Koblitz' theorem.

For $G_{p,q}(x)$ and $G_{p,q}^{*}(x)$ we have the difference equations (2.3), the multiplication-theorem (2.6) and the reflection formula (2.4).

Remark. It is possible to define and study "twisted" functions of our p-adic gamma functions.

Notation and definition.

Let $\boldsymbol{Q}$ be the rational number field and let $\boldsymbol{Z}$ be the integer ring. Let p be an odd prime. Let $\boldsymbol{Q}_{p}$, $\boldsymbol{Z}_{p}$ and $\boldsymbol{C}_{p}$ be the p-adic number field, the p-adic integer ring and the p-adic completion of the algebraic closure of $\boldsymbol{Q}_{p}$. Let $|x|_{p}$ be the absolute value of $x\in\boldsymbol{C}_{p}^{*}$ such that $|p|_{p}=p^{-1}$.

Let $\log(u)=\log_{p}(u)$ be the Iwasawa p-adic logarithm [5] on $u\in\boldsymbol{C}_{p}^{*}$. Then we have

$$\log(u)=\sum_{n\geqq 1}(-1)^{n-1}\frac{1}{n}(u-1)^{n} \quad \text{for} \quad |u-1|_{p}<1.$$

Let $\exp(u)=\exp_{p}(u)$ be the p-adic exponential function defined by

$$\exp(u)=\sum_{n\geqq 0}\frac{1}{n!}u^{n} \quad \text{for} \quad |u|_{p}<|p|_{p}^{1/(p-1)}.$$

Let $l_{2}(u)$ be the p-adic dilogarithm [2] on $u\in\boldsymbol{C}_{p}$ with $u\neq 1$. Then we have

$$l_{2}(u)=\sum_{n\geqq 1}\frac{1}{n^{2}}u^{n} \quad \text{for} \quad |u|_{p}<1.$$

We assume hereafter that $q\in \boldsymbol{C}_p$ with $|q-1|_p<|p|_p^{1/(p-1)}$ and $q\neq 1$. Then we have [7]

$$r(q)=\frac{|p|_p^{1/(p-1)}}{|\log(q)|_p}=\frac{|p|_p^{1/(p-1)}}{|1-q|_p}>1,$$

$$D(q)=\{x\in \boldsymbol{C}_p-\boldsymbol{Z}_p \mid |x|_p<r(q)\}$$

and

$$D^*(q)=\{x\in \boldsymbol{C}_p-\boldsymbol{Z}_p^* \mid |x|_p<r(q)\}.$$

Let $q^u=\exp(u\cdot\log(q))$ for $|u|_p<r(q)$. Then

$$\log(q^u)=u\cdot\log(q) \qquad \text{for} \quad |u|_p<r(q).$$

1. Definition of $G_{p,q}(x)$ and $G_{p,q}^*(x)$.

Let

$$L_{2,q}(u)=-\frac{1}{\log(q)}l_2(q^u)-u\cdot\log(1-q) \qquad \text{for} \quad |u|_p<r(q),\ u\neq 0.$$

where $l_2(x)$ is the p-adic dilogarithm [2] and log is the p-adic logarithm normalized by $\log(p)=0$ [5].

Using the functional equation

$$l_2(x)+l_2(1-x)=\log(x)\log(1-x) \qquad \text{for} \quad x\neq 0 \text{ and } x\neq 1,$$

we have

$$L_{2,q}(u)=\frac{1}{\log(q)}l_2(1-q^u)+u\cdot\log\frac{1-q^u}{1-q} \qquad \text{for} \quad |u|_p<r(q),\ u\neq 0.$$

Since

$$\begin{aligned} l_2(1-q^u)&=1-q^u+(1/4)(1-q^u)^2+\cdots \\ &=-u\cdot\log(q)+(\log(q))^2\{-u^2/2+u^2/4+\cdots\}. \end{aligned}$$

We have

$$\lim_{q\to 1} L_{2,q}(u)=-u+u\cdot\log(u).$$

LEMMA (1.1). (1) *$L_{2,q}(u)$ is locally analytic on $u\in \boldsymbol{C}_p$ with $|u|_p<r(q)$ and $u\neq 0$.*

(2) $$\frac{d}{du}L_{2,q}(u)=\log\frac{1-q^u}{1-q}.$$

(3) $$L_{2,q}(u)+L_{2,q}(-u)=\frac{1}{2}u^2\cdot\log(q).$$

(4) $$L_{2,q}(u)+L_{2,q^{-1}}(-u)=-u\cdot\log(q).$$

Proof. Since $l_2(x)$ is locally analytic on $x\neq 1$ and q^u is analytic on $|u|_p<$

$r(q)$. $l_2(q^u)$ is locally analytic on $|u|_p<r(q)$, $u\neq 0$. Thus $L_{2,q}(u)$ is locally analytic on $|u|_p<r(q)$, $u\neq 0$.

Since

$$\frac{d}{dx}l_2(x)=-\frac{\log(1-x)}{x} \qquad \text{for} \quad x\neq 1 \text{ and } x\neq 0.$$

Differetiating $L_{2,q}(u)$ gives the equation (2).

Using

$$l_2(x)+l_2(1/x)=-\frac{1}{2}(\log(x))^2,$$

we have the equation (3).

A simple calculation gives the equation (4).

Remark. If we define $L_{2,q}(u)$ as the function on the right hand side above then Lemma 1.1 can be proved without using Coleman [2].

We use the following lemma due to Diamond [3] to construct our p-adic functions.

LEMMA (1.2). *Let D be a subset of $\boldsymbol{C}_p$ such that $D+\boldsymbol{Z}_p w$ contained in D for some $w\in\boldsymbol{C}_p$ with $w\neq 0$. Let b be a positive integer and let $f(x)$ be a locally analytic function on $D\cap(\boldsymbol{C}_p-\{0\})$. Define*

$$F(x)=\lim_{n\to\infty}\frac{1}{bp^n}\sum_{0\leq j<bp^n}f(x+jw) \qquad \textit{for} \quad x\in D\cap(\boldsymbol{C}_p-\boldsymbol{Z}_p w)$$

and

$$F^*(x)=\lim_{n\to\infty}\frac{1}{bp^n}\sum_{0\leq j<bp^n}^{*}f(x+jw) \qquad \textit{for} \quad x\in D\cap(\boldsymbol{C}_p-\boldsymbol{Z}_p^* w).$$

Then

(1) *the limits exist, which are independent of b,*

(2) *$F(x)$ is locally analytic on $D\cap(\boldsymbol{C}_p-\boldsymbol{Z}_p w)$ and*

(3) *$F^*(x)$ is locally analytic on $D\cap(\boldsymbol{C}_p-\boldsymbol{Z}_p^* w)$.*

(See Corollary of Theorem 2 of [3].)

DEFINITION (1.3). We define analogues $G_{p,q}(x)$ and $G_{p,q}^*(x)$ of the Diamond's p-adic log-gamma functions $G_p(x)$ and $G_p^*(x)$ by

$$(1) \qquad G_{p,q}(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}L_{2,q}(x+j) \qquad \text{for} \quad x\in D(q)$$

and

$$(2) \qquad G_{p,q}^*(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}^{*}L_{2,q}(x+j) \qquad \text{for} \quad x\in D^*(q)$$

Then by Lemma (1.1) and Lemma (1.2) $G_{p,q}(x)$ and $G_{p,q}^*(x)$ are well-defined. And $G_{p,q}(x)$ is locally analytic on $D(q)$ and $G_{p,q}^*(x)$ is locally analytic on $D^*(q)$.

2. Properties of $G_{p,q}(x)$ and $G^*_{p,q}(x)$.

In this section we investigate some properties of $G_{p,q}(x)$ and $G^*_{p,q}(x)$. There is a relation between $G_{p,q}(x)$ and $G^*_{p,q}(x)$.

PROPOSITION (2.1). *Let $B_1(x)=x-(1/2)$ the 1-st Bernoulli polynomial. Then*

$$G^*_{p,q}(x)=G_{p,q}(x)-G_{p,q^p}\left(\frac{x}{p}\right)-B_1\left(\frac{x}{p}\right)\log\frac{1-q^p}{1-q}$$

for $x\in D(q)$.

Proof. Since $|q-1|_p<|p|_p^{1/(p-1)}$ and $r(q^p)=r(q)/|p|_p$. If $x\in D(q)$ then $x/p\in D(q^p)$. Thus we have

$$\begin{aligned}
&G_{p,q}(x)-G_{p,q^p}\left(\frac{x}{p}\right)\\
&=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\le j<p^n}L_{2,q}(x+j)-\lim_{n\to\infty}\frac{1}{p^{n-1}}\sum_{0\le j<p^{n-1}}L_{2,q^p}\left(\frac{x}{p}+j\right)\\
&=\lim_{n\to\infty}\frac{1}{p^n}\left\{\sum_{0\le j<p^n}L_{2,q}(x+j)\right.\\
&\quad\left.-\sum_{0\le j<p^{n-1}}L_{2,q}(x+jp)+\sum_{0\le j<p^{n-1}}(x+jp)\log\frac{1-q^p}{1-q}\right\}\\
&=\lim_{n\to\infty}\frac{1}{p^n}\sum^*_{0\le j<p^n}L_{2,q}(x+j)+\lim_{n\to\infty}\frac{1}{p^{n-1}}\sum_{0\le j\le p^{n-1}}\left(\frac{x}{p}+j\right)\log\frac{1-q^p}{1-q}\\
&=G^*_{p,q}(x)+B_1\left(\frac{x}{p}\right)\log\frac{1-q^p}{1-q}.
\end{aligned}$$

Because

$$B_1(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\le j<p^n}(x+j).$$

Thus we complete the proof.

Remark. Koblitz [7] obtained that

$$\psi^*_{p,q}(x)=\psi_{p,q}(x)-\frac{1}{p}\psi_{p,q^p}\left(\frac{x}{p}\right)-\frac{1}{p}\log\frac{1-q^p}{1-q}$$

for $x\in D(q)$.

As for the difference equation for $\Gamma_{p,q}(x)$ Koblitz [7] obtained the following

THEOREM (2.2).

$$\Gamma_{p,q}(x+1)/\Gamma_{p,q}(x)=\begin{cases}-\dfrac{1-q^x}{1-q} & \text{if} \quad x\in \boldsymbol{Z}_p^*,\\ -1 & \text{if} \quad x\in p\boldsymbol{Z}_p.\end{cases}$$

We have a difference equation for our functions $G_{p,q}(x)$ and $G_{p,q}^*(x)$.

THEOREM (2.3).

(1) $$G_{p,q}(x+1)-G_{p,q}(x)=\log\frac{1-q^x}{1-q} \qquad \textit{for} \quad x\in D(q).$$

(2) $$G_{p,q}^*(x+p)-G_{p,q}^*(x)=\sum_{0<i<p}\log\frac{1-q^{x+i}}{1-q} \qquad \textit{for} \quad x\in D^*(q).$$

Remark. Koblitz [7] obtained that

$$\psi_{p,q}(x+1)-\psi_{p,q}(x)=-\frac{q^x}{1-q^x}\log(q)\left(=\frac{d}{dx}\log\frac{1-q^x}{1-q}\right) \qquad \text{for} \quad x\in D(q).$$

Note that by (2) of Lemma (1.1) we have

$$\frac{1}{p^n}\{L_{2,q}(x+p^n)-L_{2,q}(x)\}=\log\frac{1-q^x}{1-q}+o_x(p^n),$$

where $o_x(p^n)\to 0(n\to\infty)$.

Proof of Theorem (2.3). By the definition we have

(1) $$\begin{aligned}&G_{p,q}(x+1)-G_{p,q}(x)\\ &=\lim_{n\to\infty}\frac{1}{p^n}\{\sum_{0\leq j<p^n}L_{2,q}(x+1+j)-\sum_{0\leq j<p^n}L_{2,q}(x+j)\}\\ &=\lim_{n\to\infty}\frac{1}{p^n}\{L_{2,q}(x+p^n)-L_{2,q}(x)\}\\ &=\log\frac{1-q^x}{1-q} \qquad \text{for} \quad x\in D(q)\end{aligned}$$

and

(2) $$\begin{aligned}&G_{p,q}^*(x+p)-G_{p,q}^*(x)\\ &=\lim_{n\to\infty}\frac{1}{p^n}\{\sum_{0\leq j<p^n}^*L_{2,q}(x+p+j)-\sum_{0\leq j<p^n}^*L_{2,q}(x+j)\}\\ &=\lim_{n\to\infty}\sum_{0<i<p}\frac{1}{p^n}\{L_{2,q}(x+i+p^n)-L_{2,q}(x+i)\}\\ &=\sum_{0<i<p}\log\frac{1-q^{x+i}}{1-q} \qquad \text{for} \quad x\in D^*(q)\end{aligned}$$

We have the reflection formula for our functions $G_{p,q}(x)$. Let $B_2(x)=x^2-$

$x+(1/6)$ the 2-nd Bernoulli polynomial.

THEOREM (2.4).

(1) $$G_{p,q}(x)+G_{p,q}(1-x)=\frac{1}{2}B_2(x)\log(q) \quad \textit{for} \quad x\in D(q).$$

(2) $$G_{p,q}(x)+G_{p,q^{-1}}(1-x)=-B_1(x)\log(q) \quad \textit{for} \quad x\in D(q).$$

Remark. Koblitz [7] obtained that

$$\psi_{p,q}(x)-\psi_{p,q^{-1}}(1-x)=-\log(q)=-\frac{d}{dx}B_1(x)\log(q).$$

Proof. (1) Using the definition and replacing j by p^n-j-1, we have

$$\begin{aligned}&G_{p,q}(x)+G_{p,q}(1-x)\\&=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\{L_{2,q}(x+j)+L_{2,q}(1-x+j)\}\\&=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\{L_{2,q}(x+j)+L_{2,q}(-x-j)\\&\quad+L_{2,q}(-x-j+p^n)-L_{2,q}(-x-j)\} \quad (*).\end{aligned}$$

Since (1) and (2) of Lemma (1.1), we have

$$\frac{1}{p^n}\{L_{2,q}(-x-j+p^n)-L_{2,q}(-x-j)\}=\log\frac{1-q^{-x-j}}{1-q}+o_{-x-j}(p^n),$$

where $o_{-x-j}(p^n)\to 0(n\to\infty)$.

Using this formula and (3) of Lemma (1.1) we have

$$\begin{aligned}(*)&=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\{L_{2,q}(x+j)+L_{2,q}(-x-j)\}\\&\quad+\lim_{n\to\infty}p^n\cdot\frac{1}{p^n}\sum_{0\leq j<p^n}\log\frac{1-q^{-x-j}}{1-q}+\lim_{n\to\infty}\sum_{0\leq j<p^n}o_{-x-j}(p^n)\\&=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\frac{1}{2}(x+j)^2\log(q)\\&\quad+\lim_{n\to\infty}p^n\cdot\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\log\frac{1-q^{-x-j}}{1-q}+\lim_{n\to\infty}\sum_{0\leq j<p^n}o_{-x-j}(p^n)\\&=\frac{1}{2}B_2(x)\log(q).\end{aligned}$$

(2) Using the definition and replacing j by p^n-j-1, we have

$$G_{p,q}(x)+G_{p,q^{-1}}(1-x)$$
$$=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\{L_{2,q}(x+j)+L_{2,q^{-1}}(1-x+j)\}$$
$$=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\{L_{2,q}(x+j)+L_{2,q^{-1}}(-x-j)$$
$$+L_{2,q^{-1}}(-x-j+p^n)-L_{2,q^{-1}}(-x-j)\} \quad (**).$$

Since (1) and (2) of Lemma (1.1), we have

$$\frac{1}{p^n}\{L_{2,q^{-1}}(-x-j+p^n)-L_{2,q^{-1}}(-x-j)\}=\log\frac{1-q^{x+j}}{1-q^{-1}}+o_{x+j}(p^n),$$

where $o_{x+j}(p^n)\to 0(n\to\infty)$.

Then, using this formula and (4) of Lemma (1.1), we have

$$(**)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\{L_{2,q}(x+j)+L_{2,q^{-1}}(-x-j)\}$$
$$+\lim_{n\to\infty}p^n\cdot\frac{1}{p^n}\sum_{0\leq j<p^n}\log\frac{1-q^{x+j}}{1-q^{-1}}+\lim_{n\to\infty}\sum_{0\leq j<p^n}o_{x+j}(p^n)$$
$$=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\{-(x+j)\log(q)\}$$
$$+\lim_{n\to\infty}p^n\cdot\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq j<p^n}\log\frac{1-q^{x+j}}{1-q^{-1}}+\lim_{n\to\infty}\sum_{0\leq j<p^n}o_{x+j}(p^n)$$
$$=-B_1(x)\log(q).$$

The proof is completed.

We have the following corollary, which will be used in the proof of the theorem for the connection of $G_{p,q}(x)$ and $G^*_{p,q}(x)$ with $\Gamma_{p,q}(x)$.

COROLLARY (2.5).

$$\sum_{0<i<p}G_{p,qp}\left(\frac{i}{p}\right)=-\frac{p-1}{24}\log(q).$$

Proof. We have

$$\sum_{0<i<p}G_{p,qp}\left(\frac{i}{p}\right)=\sum_{1\leq i\leq(p-1)/2}\left\{G_{p,qp}\left(\frac{i}{p}\right)+G_{p,qp}\left(1-\frac{i}{p}\right)\right\}$$
$$=\sum_{1\leq i\leq(p-1)/2}\frac{1}{2}B_2\left(\frac{i}{p}\right)\log(q)$$
$$=-\frac{p-1}{24}\log(q).$$

For our $G_{p,q}(x)$ and $G^*_{p,q}(x)$ we have the following multiplication-theorem.

THEOREM (2.6). *Let m be a positive integer. Then we have*

$$(1)\qquad G_{p,q}(x)=\sum_{0\leq i<m}G_{p,q^m}\left(\frac{x+i}{m}\right)+B_1(x)\log\frac{1-q^m}{1-q}\qquad \textit{for}\quad x\in D(q),$$

and when $m\equiv 0(\mathrm{mod}\, p)$ we have

$$(2)\qquad G^*_{p,q}(x)=\sum^*_{0\leq i<m}G_{p,q^m}\left(\frac{x+i}{m}\right)+\left(1-\frac{1}{p}\right)x\cdot\log\frac{1-q^m}{1-q}\qquad \textit{for}\quad x\in D^*(q).$$

Remark. Koblitz [7] obtained that

$$\psi_{p,q}(x)=\frac{1}{m}\sum_{0\leq i<m}\psi_{p,q^m}\left(\frac{x+i}{m}\right)+\log\frac{1-q^m}{1-q}\qquad \text{for}\quad x\in D(q).$$

Proof. (1) By the definition we have for $x\in D(q)$

$$\begin{aligned}&\sum_{0\leq i<m}G_{p,q^m}\left(\frac{x+i}{m}\right)\\&=\lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leq i<m}\sum_{0\leq j<p^n}\left\{-\frac{1}{\log(q^m)}l_2(q^{x+i+jm})-\left(\frac{x+i}{m}+j\right)\log(1-q^m)\right\}\\&=\lim_{n\to\infty}\frac{1}{mp^n}\sum_{0\leq j<mp^n}\left\{L_{2,q}(x+j)-(x+j)\log\frac{1-q^m}{1-q}\right\}\\&=G_{p,q}(x)-B_1(x)\log\frac{1-q^m}{1-q}.\end{aligned}$$

(2) By the definition we have for $x\in D^*(q)$

$$\begin{aligned}&\sum_{0\leq i<m}G_{p,q^m}\left(\frac{x+i}{m}\right)\\&=\lim_{n\to\infty}\frac{1}{p^n}\sum^*_{0\leq i<m}\sum_{0\leq j<p^n}\left\{-\frac{1}{\log(q^m)}l_2(q^{x+i+jm})-\left(\frac{x+i}{m}+j\right)\log(1-q^m)\right\}\\&=\lim_{n\to\infty}\frac{1}{mp^n}\sum^*_{0\leq j<mp^n}\left\{L_{2,q}(x+j)-(x+j)\log\frac{1-q^m}{1-q}\right\}\\&=G^*_{p,q}(x)-\left(B_1(x)-B_1\left(\frac{x}{p}\right)\right)\log\frac{1-q^m}{1-q}\\&=G^*_{p,q}(x)-\left(1-\frac{1}{p}\right)x\cdot\log\frac{1-q^m}{1-q}.\end{aligned}$$

Letting $m=p$ we have the following

COROLLARY (2.7).

$$G^*_{p,q}(x)=\sum_{0<i<p}G_{p,q^p}\left(\frac{x+i}{p}\right)+\left(1-\frac{1}{p}\right)x\cdot\log\frac{1-q^p}{1-q}\qquad \textit{for}\quad x\in D^*(q).$$

Letting $x=0$ and using Corollary (2.5) we have the following

COROLLARY (2.8).

$$G_{p,q}^{*}(0)=\sum_{0<i<p} G_{p,q^p}\left(\frac{i}{p}\right)=-\frac{p-1}{24}\log(q).$$

3. Connections with $\Gamma_{p,q}(x)$.

In this section we study the connections of $G_{p,q}(x)$ and $G_{p,q}^{*}(x)$ with $\Gamma_{p,q}(x)$. By the definition of $\Gamma_{p,q}(x)$ [7] we have

$$\Gamma_{p,q}(1)=-1.$$

Then the difference equation of Theorem (2.2) follows

$$\Gamma_{p,q}(0)=1.$$

Thus we have

(i) $$\log \Gamma_{p,q}(0)=0.$$

By Theorem (2.2) we have

(ii) $$\log \Gamma_{p,q}(x+1)-\log \Gamma_{p,q}(x)=\begin{cases}\log \dfrac{1-q^x}{1-q} & \text{if } x\in \boldsymbol{Z}_p^{*}, \\ 0 & \text{if } x\in p\boldsymbol{Z}_p.\end{cases}$$

Using Corollary (2.5) we have

(iii) $$\sum_{0<i<p}\left\{G_{p,q^p}\left(\frac{i}{p}\right)-\frac{\log(q)}{24}\right\}=0$$

Then the connections of $G_{p,q}(x)$ and $G_{p,q}^{*}(x)$ with $\Gamma_{p,q}(x)$ are the following

THEOREM (3.1).

(1) $$\log \Gamma_{p,q}(x)=G_{p,q}^{*}(x)+\frac{p-1}{24}\log(q) \qquad \textit{for} \quad x\in p\boldsymbol{Z}_p.$$

(2) $$\log \Gamma_{p,q}(x)$$
$$=\sum_{\substack{0\leq i<p \\ x+i\in \boldsymbol{Z}_p^{*}}}\left\{G_{p,q^p}\left(\frac{x+i}{p}\right)+\frac{\log(q)}{24}\right\}+(x-\tilde{x})\log\frac{1-q^p}{1-q} \qquad \textit{for} \quad x\in \boldsymbol{Z}_p.$$

Remark. Koblitz [7] obtained that

(1) $$\frac{d}{dx}\log \Gamma_{p,q}(x)=\psi_{p,q}^{*}(x) \qquad \text{for} \quad x\in p\boldsymbol{Z}_p,$$

(2) $$\frac{d}{dx}\log \Gamma_{p,q}(x)=\frac{1}{p}\sum_{\substack{0\leq i<p \\ x+i\in \boldsymbol{Z}_p^{*}}}\psi_{p,q^p}\left(\frac{x+i}{p}\right)+\left(1-\frac{1}{p}\right)\log\frac{1-q^p}{1-q} \qquad \text{for} \quad x\in \boldsymbol{Z}_p.$$

Proof of Theorem (3.1). Since both sides of (2) are continuous in $x\in\boldsymbol{Z}_p$, it suffices to prove (2) for x=any non-negative integer n.

Let A_n denote the left side of (2) for $x=n$, and let B_n denote the right side of (2) for $x=n$. We prove $A_n=B_n$ by induction n.

Note that by (i) and (iii) we have $A_0=B_0$. Suppose that $A_n=B_n$.

By (ii) we have

$$A_{n+1}-A_n=\begin{cases}\log\dfrac{1-q^n}{1-q} & \text{if } n\not\equiv 0 \pmod p,\\ 0 & \text{if } n\equiv 0 \pmod p.\end{cases}$$

We claim that

$$B_{n+1}-B_n=\begin{cases}\log\dfrac{1-q^n}{1-q} & \text{if } n\not\equiv 0 \pmod p,\\ 0 & \text{if } n\equiv 0 \pmod p.\end{cases}$$

In fact we have

$$\begin{aligned}&B_{n+1}-B_n\\ &=\sum_{\substack{0\le i<p\\ n+1+i\not\equiv 0 \pmod p}}\left\{G_{p,q^p}\left(\frac{n+1+i}{p}\right)+\frac{\log(q)}{24}\right\}\\ &\quad+\sum_{\substack{0\le i<p\\ n+i\not\equiv 0 \pmod p}}\left\{G_{p,q^p}\left(\frac{n+i}{p}\right)+\frac{\log(q)}{24}\right\}\\ &\quad+\{(n+1)-(n+1)^{\sim}-n+\tilde{n}\}\log\frac{1-q^p}{1-q}\\ &=\sum_{\substack{0\le i<p\\ n+1+i\not\equiv 0 \pmod p}}G_{p,q^p}\left(\frac{n+1+i}{p}\right)+\sum_{\substack{0\le i<p\\ n+i\not\equiv 0 \pmod p}}G_{p,q^p}\left(\frac{n+i}{p}\right)\\ &\quad+\{[(n-1)/p]+1-[n/p]\}\log\frac{1-q^p}{1-q}\cdots\cdots(*).\end{aligned}$$

(a) Case of $n\not\equiv 0 \pmod p$. We have

$$\begin{aligned}(*)&=G_{p,q^p}\left(\frac{n+p}{p}\right)-G_{p,q^p}\left(\frac{n}{p}\right)+\log\frac{1-q^p}{1-q}\\ &=\log\frac{1-q^n}{1-q^p}+\log\frac{1-q^p}{1-q}=\log\frac{1-q^n}{1-q}.\end{aligned}$$

(b) Case of $n\equiv 0 \pmod p$. We have

$$(*)=0.$$

By (a) and (b) we have

$$A_{n+1}-A_n=B_{n+1}-B_n,$$

and so we have

$$A_{n+1}=B_{n+1}.$$

This completes the induction.

To prove (1) we use Corollary (2.7) and the following formula

$$x-\tilde{x}=\left(1-\frac{1}{p}\right)\cdot x \qquad \text{for} \quad x\in p\boldsymbol{Z}_p .$$

By (2) for $x\in p\boldsymbol{Z}_p$, we have

$$\begin{aligned}
&\log \Gamma_{p,q}(x)\\
&=\sum_{0\leq i<p}\left\{G_{p,q^p}\left(\frac{x+i}{p}\right)+\frac{\log(q)}{24}\right\}+(x-\tilde{x})\log\frac{1-q^p}{1-q}\\
&=\sum_{0\leq i<p}G_{p,q^p}\left(\frac{x+i}{p}\right)+\left(1-\frac{1}{p}\right)x\cdot\log\frac{1-q^p}{1-q}+\frac{p-1}{24}\log(q)\\
&=G^*_{p,q}(x)+\frac{p-1}{24}\log(q).
\end{aligned}$$

The proof is completed.

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