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ASYMPTOTIC BEHAVIOR OF PERIODIC SOLUTIONS IN BANACH SPACE

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1. Introduction.

We consider the following problem:

$$\frac{du(t)}{dt}+Au(t)\ni f(t), \quad t\in(0, \infty),$$
$$u(0)=x, \tag{1}$$

where A is an m-accretive operator in Banach space X and $f\in L^1_{\mathrm{loc}}(0, \infty; X)$ is T-periodic. Let $\{C_t\}_{t\geqq 0}$ be a nonempty closed convex subset of a Banach space and let $U=\{U(t, s): 0\geqq s\geqq t\}$ be a nonexpansive operator constrained in $\{C_t\}$, i.e., U is a family of mappings $U(t, s): C_s\to C_t$ such that

$$U(t, s)U(s, r)=U(t, r), \quad U(r, r)=I,$$
$$|U(t, s)x-U(t, s)y|\leqq|x-y|$$

for all $0\leqq r\leqq s\leqq t$ and $x, y\in C_s$. Such an evolution operator U is said to be T-periodic ($T>0$) if

$$C_{t+T}=C_t \quad \text{and} \quad U(t+T, s+T)=U(t, s)$$

for all $0\leqq s\leqq t$. Then, a function $u: [0, \infty)\to X$ is an almost semitrajectory of U if

$$\lim_{s\to\infty}\sup_{t\geqq s}|u(t)-U(t, s)u(s)|=0.$$

In what follows, let $U=\{U(t, s): 0\leqq s\leqq t\}$ be a T-periodic nonexpansive evolution operator constrained in $\{C_t\}$ and we take $u(t)=U(t, 0)u(0)$ for $t\geqq 0$. We shall denote $u(nT+t)$ by $u_n(t)$.

If $F(U_t)=\{x: U(T+t, t)x=x \text{ for } 0\leqq t\leqq T\}$ is nonempty, then we can take $z\in F(U_t)$, and we see that

$$\lim_{n\to\infty}|u_n(t)-z|=\rho(t)$$

exists. It is well known [1] that (1) has a unique integral solution $U(t; s, x)$

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whenever $x \in \overline{D(A)}$ and by setting $U(t, s)x = U(t; s, x)$, we see that $\{U(t, s): 0 \leqq s \leqq t\}$ forms a T-periodic nonexpansive evolution operation operator constrained in $\{C_t\}$.

The present paper is concerned with the asymptotic behavior of the T-periodic integral solution of (1). We prove that if u is an almost semitrajectory of U and $u_n(t) = u(nT+t)$, then the closed convex set

$$\bigcap_k \overline{co}\,\{u_n(t): n \leqq k\} \cap F(U_t)$$

consists of at most one point, where $\overline{co}\{u_n(t): n \geqq k\}$ is the closed convex hull of $\{u_n(t): n \geqq k\}$. This result is applied to study the problem of weak convergence of the sequence $\{u_n(t): n \geqq 0\}$. We also prove that if P is the metric projection of X onto $F(U_t)$, then the strong $\lim_{n\to\infty} Pu_n(t)$ exists. Our proofs employ the methods of Lau-Takahashi [6] and W. Takahashi-J. Y. Park [7].

2. Lemmas.

LEMMA 1. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and u is an almost semitrajectory of U. Let*

$$F(U_t) \neq \phi,\ y \in F(U_t),\ 0 < \alpha \leqq \beta < 1 \quad \textit{and} \quad r = \lim_{n\to\infty} |u_n(t) - y|\,.$$

Then, for any $\varepsilon > 0$, there exists $n_0 \geqq 0$ such that

$$|U(mT+t, t)(\lambda u_n(t) + \delta(1-\lambda)y) - (\lambda U(mT+t, t)u_n(t) + (1-\lambda)y)| < \varepsilon$$

for all $n \geqq n_0$, $m \geqq 0$ and $\lambda \in R$ with $\alpha \leqq \lambda \leqq \beta$.

Proof. Let $r > 0$. Then we can choose $d > 0$ so small that

$$(r+d)\Big(1 - c\delta\Big(\frac{\varepsilon}{r+d}\Big)\Big) = r_0 < r$$

where δ is the modulus of convexity of the norm and

$$c = \min\{2\lambda(1-\lambda): \alpha \leqq \lambda \leqq \beta\}\,.$$

Let $a > 0$ with $r_0 + 2a < r$. Then we can choose $n_0 \geqq 0$ such that

$$|u_n(t) - y| \geqq r - a \quad \text{and} \quad |u_{m+n}(t) - U(mT+t, t)u_n(t)| < a$$

for all $n \geqq n_0$ and $m \geqq 0$ because u is an almost semitrajectory of U. Suppose that

$$|U(mT+t, t)(\lambda u_n(t) + (1-\lambda)y) - (\lambda U(mT+t, t)u_n(t) + (1-\lambda)y| \geqq \varepsilon$$

for some $n \geqq n_0$, $m \geqq 0$ and $\lambda \in R$ with $\alpha \leqq \lambda \leqq \beta$. Put $u = (1-\lambda)(U(mT+t, t)z - y)$ and $v = \lambda(U(mT+t, t)u_n(t) - U(mT+t, t)z)$, where $z = \lambda u_n(t) + (1-\lambda)y$. Then $|u| \leqq \lambda(1-\lambda)|u_n(t) - y|$ and $|v| \leqq \lambda |u_n(t) - z| = \lambda(1-\lambda)|u_n(t) - y|$. We also have that

$$|u-v|=|U(mT+t,\ t)z-(\lambda U(mT+t,\ t)u_n(t)+(1-\lambda)y|\geqq\varepsilon$$

and

$$\lambda u+(1-\lambda)v=\lambda(1-\lambda)(U(mT+t,\ t)u_n(t)-y).$$

So by using the Lemma in [5], we have

$$\begin{aligned}\lambda(1-\lambda)|U(mT+t,\ t)u_n(t)-y| &= |\lambda u+(1-\lambda)v|\\ &\leqq\lambda(1-\lambda)|u_n(t)-y|\Big(1-2\lambda(1-\lambda)\delta\Big(\frac{\varepsilon}{|u_n(t)-y|}\Big)\Big)\\ &\leqq\lambda(1-\lambda)(r+d)\Big(1-c\delta\Big(\frac{\varepsilon}{r+d}\Big)\Big)\\ &=\lambda(1-\lambda)r_0\end{aligned}$$

and hence $|U(mT+t,\ t)u_n(t)-y|\leqq r_0$. This implies

$$\begin{aligned}|u_{n+m}(t)-y| &\leqq |u_{n+m}(t)-U(mT+t,\ t)u_n(t)|+|U(mT+t,\ t)u_n(t)-y|\\ &\leqq a+r_0<r-a\,.\end{aligned}$$

On the other hand, $|u_n(t)-y|\geqq r-a$ for all $n\geqq n_0$, this is a contradiction. In the case when $r=0$, let $y\in F(U_t)$ and $\lambda\in R$ with $0\leqq\lambda\leqq 1$,

$$\begin{aligned}&|U(mT+t,\ t)(\lambda u_n(t)+(1-\lambda)y)-(\lambda U(mT+t,\ t)u_n(t)+(1-\lambda)y)|\\ &\leqq\lambda|U(mT+t,\ t)(\lambda u_n(t)+(1-\lambda)y)-U(mT+t,\ t)u_n(t)|\\ &\quad+(1-\lambda)|U(mT+t,\ t)(\lambda u_n(t)+(1-\lambda)y)-y|\\ &\leqq\lambda|\lambda u_n(t)+(1-\lambda)y-u_n(t)|+(1-\lambda)|\lambda u_n(t)+(1-\lambda)y-y|\\ &\leqq 2\lambda(1-\lambda)|u_n(t)-y|\,.\end{aligned}$$

So, we obtain the desired result.

Let x and y be element of X, then we denote by $[x,\ y]$ the set $\{\lambda x+(1-\lambda)y : 0\leqq\lambda\leqq 1\}$.

LEMMA 2 [6]. *Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm and $\{x_\alpha\}$ a bounded set in C. Let $z\in\bigcap_\beta\overline{co}\{x_\alpha : \alpha\geqq\beta\}$, $y\in C$ and $\{y_\alpha\}$ a net of element in C with $y_\alpha\in[y,\ x_\alpha]$ and*

$$|y_\alpha-z|=\min\{|u-z| : u\in[y,\ x_\alpha]\}.$$

If $y_\alpha\to y$, then $y=z$.

LEMMA 3. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and u is an almost semitrajectory of U. Let $F(U_t)\neq\phi$,*

$$z\in\bigcap_k\overline{co}\{u_n(t) : n\geqq k\}\cap F(U_t)$$

and $y \in F(U_t)$. *Then, for any* $\varepsilon>0$, *there is* $n_0 \geqq 0$ *such that*

$$\langle u_n(t)-y,\ J(y-z)\rangle \leqq \varepsilon |y-z|$$

for all $n \geqq n_0$.

Proof. Let $z \in \bigcap_k \overline{co}\{u_n(t): n \geqq k\} \cap F(U_t)$, $y \in F(U_t)$ and $\varepsilon>0$. If $y=z$, this lemma is obvious. So, let $y \neq z$. For any $n \geqq 0$, define a unique element y_n such that $y_n \in [y, u_n(t)]$ and $|y_n - z| = \min\{|u-z| : u \in [y, u_n(t)]\}$.

Then, since $y \neq z$, by Lemma 2 we have $y_n \not\to y$. There exists $c>0$ such that for any $n \geqq 0$ there is $n' \geqq n$ with $|y_{n'}-y| \geqq c$. Setting

$$y_{n'} = a_{n'} u_{n'}(t) + (1-a_{n'})y, \qquad 0 \leqq a_{n'} \leqq 1.$$

We also obtain $c_0>0$ so small that $a_{n'} \geqq c_0$. In fact, since

$$\begin{aligned} c \leqq |y_{n'}-y| &= a_{n'} |u_{n'}(t)-y| \\ &\leqq a_{n'} |U(t, 0)x-y|, \end{aligned}$$

we may put $c_0 = c/|U(t, 0)x-y|$. Since the limit of $|u_n(t)-y|$ exists, putting $k = \lim_{n\to\infty} |u_n(t)-y|$, we have $k>0$. If not, we have $u_n(t) \to y$ and hence $y_n \to y$, which contradictions $y_n \not\to y$. Let r be a positive number such that $\varepsilon > r$ and $k > 2r$. Choose $a>0$ so small that

$$(R+a)\left(1-\delta\left(\frac{c_0 r}{R+a}\right)\right) < R,$$

where δ is the modulus of convexity of the norm and $R=|z-y|$. By Lemma 1, there exists $n_0 \geqq 0$ such that

$$|U(mT+t, t)(c_0 u_n(t)+(1-c_0)y)-(c_0 U(mT+t, t)u_n(t)+(1-c_0)y| < a \tag{2}$$

for all $n \geqq n_0$ and $m \geqq 0$. Fix $n' \geqq 0$ with $n' \geqq n_0$ and $|u_{m+n'}(t)-y| \geqq 2r$ and $|u_{m+n'}(t)-U(mT+t, t)u_{n'}(t)| < r$ for all $m \geqq 0$. Then since

$$c_0 u_{n'}(t)+(1-c_0)y \in [y,\ a_{n'} u_{n'}(t)+(1-a_{n'})y] = [y, y_{n'}].$$

Hence

$$\begin{aligned} |c_0 u_{n'}(t)+(1-c_0)y-z| &\leqq \max\{|z-y|, |z-y_{n'}|\} \\ &= |z-y| = r. \end{aligned}$$

By using (2), we obtain

$$\begin{aligned} &|c_0 U(mT+t, t)u_{n'}(t)+(1-c_0)y-z| \\ &\quad \leqq |U(mT+t, t)(c_0 u_{n'}(t)+(1-c_0)y)-z| + |c_0 U(mT+t, t)u_{n'}(t) \\ &\qquad +(1-c_0)y-U(mT+t, t)(c_0 u_{n'}(t)+(1-c_0)y| \end{aligned}$$

$$\leqq |U(mT+t,\ t)(c_0u_{n'}(t)+(1-c_0)y)-z|+a$$
$$\leqq |c_0u_{n'}(t)+(1-c_0)y-z|+a$$
$$\leqq R+a$$

for all $m\geqq 0$. On the other hand, since $|y-z|=R<R+a$ and

$$|c_0U(mT+t,\ t)u_{n'}(t)+(1-c_0)y-y|$$
$$=c_0|U(mT+t,\ t)u_{n'}(t)-y|$$
$$\geqq c_0(|u_{m+n'}(t)-y|-|u_{m+n'}(t)-U(mT+t,\ t)u_{n'}(t)|)$$
$$\geqq c_0(|u_{m+n'}(t)-y|-r)$$
$$\geqq c_0r$$

for all $m\geqq 0$. By uniform convexity, we have

$$\left|\frac{1}{2}((c_0U(mT+t,\ t)u_{n'}(t)+(1-c_0)y-z)+(y-z))\right|$$
$$\leqq (R+a)\Big(1-\delta\Big(\frac{c_0r}{R+a}\Big)\Big)<R$$

for all $m\geqq 0$, and hence

$$\left|\frac{c_0}{2}U(mT+t,\ t)u_{n'}(t)+\Big(1-\frac{c_0}{2}\Big)y-z\right|<R$$

for all $m\geqq 0$. This implies that if $u_m=(c_0/2)U(mT+t,\ t)u_{n'}(t)+(1-(c_0/2))y$, then $|u_m+\alpha(y-u_m)-z|\geqq|y-z|$ for all $\alpha\geqq 1$. By Theorem 2.5 in [4], we have

$$\langle u_m+\alpha(y-u_m)-y,\ J(y-z)\rangle\geqq 0$$

and hence $\langle u_m-y,\ J(y-z)\rangle\leqq 0$. Then $\langle U(mT+t,\ t)u_{n'}(t)-y,\ J(y-z)\rangle\leqq 0$. Therefore

$$\langle u_{m+n'}(t)-y,\ J(y-z)\rangle$$
$$\leqq |u_{m+n'}(t)-U(mT+t,\ t)u_{n'}(t)||y-z|$$
$$+\langle U(mT+t,\ t)u_{n'}(t)-y,\ J(y-z)\rangle$$
$$\leqq \varepsilon|y-z|$$

for all $m\geqq 0$. This completes the proof.

3. Theorems.

THEOREM 1. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and u be an almost semitrajectory of U. If $F(U_t)\neq\phi$, then* *for any* $n \in N$, *the set*

$$\bigcap_k \overline{co}\{u_n(t) : n \geqq k\} \cap F(U_t)$$

consists of at most one point.

Proof. For any $n \in N$, let $y, z \in \bigcap_k \overline{co}\{u_n(t) : n \geqq k\} \cap F(U_t)$. Then, since $((y+z)/2) \in F(U_t)$, it follows from Lemma 3 that for any $\varepsilon > 0$, there exists $n_0 \geqq 0$ such that

$$\left\langle u_n(t) - \frac{y+z}{2}, J\left(\frac{y+z}{2} - z\right)\right\rangle$$
$$\leqq \varepsilon \left| \frac{y+z}{2} - z \right|$$

for all $n \geqq n_0$. Since $y \in \overline{co}\{u_n(t) : n \geqq k\}$, we have

$$\left\langle y - \frac{y+z}{2}, J\left(\frac{y+z}{2} - z\right)\right\rangle \leqq \varepsilon \left| \frac{y+z}{2} - z \right|$$

and hence $\langle y-z, J(y-z)\rangle \leqq 2|y-z|$. Thus $|y-z| \leqq 2\varepsilon$. Since ε is arbitrary, consequently $y=z$.

THEOREM 2. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and u be an almost semitrajectory of U. If $F(U_t) \neq \phi$ and $\omega(u_n(t)) \subset F(U_t)$, then the sequence $\{u_n(t) : n \in N\}$ converges weakly to some $z \in F(U_t)$, where $\omega(u_n(t)) = \{y \in X : u_{n_i}(t) \rightharpoonup y$ with $n_i \to \infty$ as $n \to \infty\}$.*

Proof. Since $F(U_t) \neq \phi$. $\{u_n(t) : n \in N\}$ bounded. So, the sequence $\{u_n(t)\}$ must contain a subsequence $\{u_{n_i}(t)\}$ of $\{u_n(t)\}$ which converges weakly to some $z \in C_t = D(A)$. Since $\omega(u_n(t)) \subset F(U_t)$ and $z \in \bigcap_k \overline{co}\{u_n(t) : n \geqq k\}$, we obtain

$$z \in \bigcap_k \overline{co}\{u_n(t) : n \geqq k\} \cap F(U_t).$$

Therefore, it follows from Theorem 1 that $\{u_n(t) : n \in N\}$ converges weakly to $z \in F(U_t)$.

THEOREM 3. *Let X be a uniformly convex Banach space and $F(U_t) \neq \phi$. Let P be the metric projection of X onto $F(U_t)$. Then the strong $\lim_{n\to\infty} u_n(t)$ exists and $\lim_{n\to\infty} Pu_n(t) = z_0$, where z_0 is a unique element of $F(U_t)$ such that*

$$\lim_{n\to\infty} |u_n(t) - z_0| = \min\{\lim_{n\to\infty} |u_n(t) - z| : z \in F(U_t)\}.$$

Proof. Since $F(U_t) \neq \phi$, we know that $\{u_n(t) : n \in N\}$ is bounded and $\lim_{n\to\infty} |u_n(t) - z| = \rho(z)$ exists for each $z \in F(U_t)$. Let $R = \min\{\rho(z) : z \in F(U_t)\}$. Then, since ρ is convex and continuous on $F(U_t)$ and $\rho(z)\to\infty$ as $z\to\infty$, there exists $z_0\in F(U_t)$ such that $\rho(z_0)=R$; see [2: p 79]. On the other hand, since $|u_n(t)-Pu_n(t)|\leqq|u_n(t)-y|$ for all $n\in N$ and $y\in F(U_t)$, we have

$$\lim_{n\to\infty}|u_n(t)-Pu_n(t)|\leqq R.$$

Suppose that $\lim_{n\to\infty}|u_n(t)-Pu_n(t)|<R$. Then we can choose $\varepsilon>0$ and $n_0\geqq 0$ such that

$$|u_n(t)-Pu_n(t)|<R-\varepsilon \qquad \text{for all} \quad n\geqq n_0.$$

We observe that

$$|u_{n+1}(t)-Pu_{n+1}(t)|\leqq|u_n(t)-Pu_n(t)| \qquad \text{for all} \quad n\geqq 0.$$

Thus, there exists $n_0\geqq 0$ such that

$$\begin{aligned}|u_{n+1}(t)-Pu_{n+1}(t)|&\leqq|u_n(t)-Pu_n(t)|\\&<R-\varepsilon\end{aligned}$$

for all $n\geqq n_0$. Thus $\lim_{n\to\infty}|u_n(t)-Pu_n(t)|<R$. This is a contradiction. So we conclude that

$$\lim_{n\to\infty}|u_n(t)-Pu_n(t)|=R.$$

We claim that $\lim_{n\to\infty}Pu_n(t)=z_0$. If not, then we have $|Pu_n(t)-z_0|\geqq\varepsilon$ for some $\varepsilon>0$ and $n\to\infty$. Let δ denote the modulus of convexity of X. There is a positive a such that

$$(R+a)\Big(1-\delta\Big(\frac{a}{R+a}\Big)\Big)=R_1<R.$$

We also have $|u_n(t)-Pu_n(t)|\leqq R+a$ and $|u_n(t)-z_0|\leqq R+a$ for all large enough n. Therefore

$$\begin{aligned}\left|u_n(t)-\frac{Pu_n(t)+z_0}{2}\right|&\leqq(R+a)\Big(1-\delta\Big(\frac{\varepsilon}{R+a}\Big)\Big)\\&=R_1<R-\varepsilon.\end{aligned}$$

Since the points $w_n=(Pu_n(t)+z_0)/2$ belong to $F(U_0)$, also, there is $n_0\geqq 0$ such that

$$\begin{aligned}|u_{n+1}(t)-w_{n+1}(t)|&\leqq|u_n(t)-w_n(t)|\\&<R-\varepsilon<R\end{aligned}$$

for all $n\geqq n_0$. Thus we obtain $\rho(w_n)<R$. This is a contradiction. Therefore $\lim_{n\to\infty}Pu_n(t)=z_0$. Consequently, it follows that an element $z_0\in F(U_0)$ with $\rho(z_0)=\min\{\rho(z): z\in F(U_t)\}$ is unique.

4. Remarks.

Fix $t\in[0, T]$, let $G=\{0, 1, 2, \cdots\}$ and $S(n)=U_t^n$, $n\in G$ where $U_t=U(T+t, t)$: $\overline{D(A)}\to\overline{D(A)}$. Then, $\{S(n): n\in G\}$ is nonexpansive semigroup on $\overline{D(A)}$ and $F(S)=F(U_t)\neq\phi$.

Next, we define $u(n)=u_n(t)=U(nT+t, 0)x$, fix $t\in[0, T]$, then $u: G\to X$ is an almost-orbit of $\{S(n)\}$.

In fact,

$$\begin{aligned} u(n)&=u_n(t)\\ &=U(nT+t, 0)x\\ &=U(nT+t, t)U(t, 0)x\\ &=U(nT+t, t)z,\ z=U(t, 0)x\\ &=U_t^n z=S(n)z. \end{aligned}$$

Thus we have

$$\begin{aligned} |u(n+m)-S(n)u(m)|&=|S(n+m)z-S(n)S(m)z|\\ &=|S(n+m)z-S(n+m)z|=0. \end{aligned}$$

Hence u: $G\to X$ is an almost-orbit of $\{S(n)\}$. Therefore, by [5]

THEOREM 1-3.

$$\bigcap_{m\geqq 0}\overline{co}\{u(n): n\geqq m\}\cap F(U_t)$$

consists of at most one point.

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