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AN ESTIMATE FOR THE MEAN CURVATURE OF COMPLETE SUBMANIFOLDS IN A TUBE

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1. Introduction.

Let $f: M \rightarrow E^n$ be an isometric immersion of a compact Riemannian manifold M into the Euclidean space E^n. If $f(M)$ is contained in a ball of radius λ, then the mean curvature vector field H of the immersion f satisfies the following inequality:

$$\sup|H| \geqq 1/\lambda .$$

Recently, generalizing the above inequality, Jorge and Xavier [4], and Jorge and Koutroufiotis [2] proved the following theorem.

THEOREM A. *Let M be a complete Riemannian manifold whose scalar curvature is bounded below and let B_λ be a closed normal ball of radius λ in a Riemannian manifold N. Set b for the supremum of the sectional curvature of N in B_λ. Let $f: M \rightarrow B_\lambda \subset N$ be an isometric immersion. Then the mean curvature vector field H of the immersion f satisfies the following:*

(1) $\sup|H| \geqq \sqrt{b}/\tan(\lambda\sqrt{b})$, if $b>0$ and $\lambda<\pi/2\sqrt{b}$,

(2) $\sup|H| \geqq 1/\lambda$, if $b=0$,

(3) $\sup|H| \geqq \sqrt{-b}/\tanh(\lambda\sqrt{-b})$, if $b<0$.

In this paper we show a natural extension of the inequalities in Theorem A considering a tube instead of a ball.

2. Statement of results.

Let $f: M \rightarrow N$ be an isometric immersion of an m-dimensional complete Riemannian manifold M whose scalar curvature is bounded below into an n-dimensional Riemannian manifold N whose sectional curvature K_N satisfies $-\infty<\inf K_N$ and $K_N \leqq b$. For $n>p\geqq 1$, let P be a p-dimensional embedded submanifold in N and let $TP^\perp$ be the normal bundle of P. We denote $\tau(P, \lambda)$ the tube of radius λ about P in N (i.e. $\{\xi \in TP^\perp : |\xi| \leqq \lambda\}$ is mapped diffeomorphically onto $\tau(P, \lambda)$ through the exponential map). We define μ by

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$$\mu=\sup\{\mu(\xi): \xi\in TP^{\perp},\ |\xi|=1\},$$

where $\mu(\xi)$ denotes the maximum eigenvalue of the shape operator A_ξ. Then our main result is the following.

THEOREM B. *In the notations above suppose that $f(M)$ is contained in a tube about P. Let λ be the infimum of the radius of tubes about P which contain $f(M)$. If $p\leqq m$, and $0<\lambda$, then the mean curvature vector field H of the immersion f satisfies the following:*

(1) $\sup|H|\geqq -\dfrac{p}{m}\left(\dfrac{\mu\sqrt{b}+b\cdot\tan(\lambda\sqrt{b})}{\sqrt{b}-\mu\cdot\tan(\lambda\sqrt{b})}\right)+\dfrac{m-p}{m}\left(\dfrac{\sqrt{b}}{\tan(\lambda\sqrt{b})}\right)$,

if $b>0$, $\lambda<\pi/2\sqrt{b}$ *and* $\mu<\sqrt{b}/\tan(\lambda\sqrt{b})$,

(2) $\sup|H|\geqq -\dfrac{p}{m}\left(\dfrac{\mu}{1-\mu\lambda}\right)+\dfrac{m-p}{m}\left(\dfrac{1}{\lambda}\right)$, *if* $b=0$ *and* $\mu<1/\lambda$,

(3) $\sup|H|\geqq -\dfrac{p}{m}\left(\dfrac{\mu\sqrt{-b}+b\cdot\tanh(\lambda\sqrt{-b})}{\sqrt{-b}-\mu\cdot\tanh(\lambda\sqrt{-b})}\right)+\dfrac{m-p}{m}\left(\dfrac{\sqrt{-b}}{\tanh(\lambda\sqrt{-b})}\right)$,

if $b<0$ *and* $\mu<\sqrt{-b}/\tanh(\lambda\sqrt{-b})$.

Applying Theorem B to minimal immersions, we have immediately:

THEOREM C. *Let $f: M\to N$ be an isometric immersion of an m-dimensional complete Riemannian manifold M whose scalar curvature is bounded below into an n-dimensional Riemannian manifold N whose sectional curvature K_N satisfies $-\infty<\inf K_N$ and $K_N\leqq b$. For $n>p\geqq 1$, let P be a p-dimensional embedded submanifold in N and let $\tau(P,\lambda)$ be the tube of radius λ about P. Suppose that f is minimal and P is totally geodesic. Then the following holds:*

(1) $f(M)\not\subset\tau(P,\lambda)$, *if* $b>0$, $\lambda<\pi/2\sqrt{b}$ *and* $p\{1+\tan^2(\lambda\sqrt{b})\}<m$,

(2) $f(M)\not\subset\tau(P,\lambda)$, *if* $b=0$ *and* $p<m$,

(3) $f(M)\subset P$, *if* $b<0$, $p\leqq m$ *and* $f(M)\subset\tau(P,\lambda)$.

Remark. Let P be a linear subspace of E^3. It is interesting to study complete minimal surfaces in a tube $\tau(P,\lambda)$. For $\dim P\leqq 1$, Theorems A and C imply that $\tau(P,\lambda)$ contains no complete minimal surface whose Gaussian curvature is bounded. For $\dim P=2$, Jorge and Xavier [3] proved that there exists a complete non-flat minimal surface in $\tau(P,\lambda)$.

3. Preliminaries.

For $n>p\geqq 1$, let N be an n-dimensional Riemannian manifold and let P be a p-dimensional embedded submanifold in N. The Riemannian metric, Riemannian connection and curvature tensor of N are denoted by $\langle\ ,\ \rangle$, ∇ and R respectively. Let $\sigma: [0,\lambda]\to N$ be a geodesic parametrized by arclength such that $\sigma(0)\in P$ and $\dot\sigma(0)\in T_{\sigma(0)}P^{\perp}$, where $T_{\sigma(0)}P^{\perp}$ denotes the normal space to P at $\sigma(0)$.

Let $L(P, \sigma)$ denote the vector space of all piecewise smooth vector fields along σ whose initial value is tangent to P. The index form for the pair (P, σ) is a symmetric bilinear form $I: L(P, \sigma)\times L(P, \sigma)\to \boldsymbol{R}$ defined by

$$I(V, W)=-\langle A_{\dot\sigma(0)}V(0), W(0)\rangle+\int_0^\lambda \{\langle \nabla_{\dot\sigma}V, \nabla_{\dot\sigma}W\rangle+\langle R(\dot\sigma, V)\dot\sigma, W\rangle\}\,dt,$$

where $A_{\dot\sigma(0)}$ denotes the shape operator for $\dot\sigma(0)$. A Jacobi field $J\in L(P, \sigma)$ is called a P-Jacobi field if it satisfies the following condition:

$$A_{\dot\sigma(0)}J(0)+(\nabla_{\dot\sigma}J)(0)\in T_{\sigma(0)}P^{\perp}.$$

For $0<t_0\leqq\lambda$, $\sigma(t_0)$ is called a focal point of the pair (P, σ) if there exists a nonzero P-Jacobi field J along σ such that $J(t_0)=0$.

LEMMA 3.1 ([1, p. 228]). *Suppose that there is no focal point of the pair (P, σ). Then for each $V\in L(P, \sigma)$ there exists a unique P-Jacobi field J along σ such that $J(\lambda)=V(\lambda)$. Furthermore $I(J, J)\leqq I(V, V)$ and equality holds only if $J=V$.*

For $(b, \mu, t)\in R^3$, we define $g_i(b, \mu, t)$ as follows:

$$g_0(b, \mu, t)=t,$$

$$g_1(b, \mu, t)=\begin{cases}\cos(t\sqrt{b})-\mu\cdot\sin(t\sqrt{b})/\sqrt{b} & \text{if } b>0,\\ 1-\mu t & \text{if } b=0,\\ \cosh(t\sqrt{-b})-\mu\cdot\sinh(t\sqrt{-b})/\sqrt{-b} & \text{if } b<0,\end{cases}$$

$$g_2(b, \mu, t)=\begin{cases}\sin(t\sqrt{b})/\sqrt{b} & \text{if } b>0,\\ t & \text{if } b=0,\\ \sinh(t\sqrt{-b})/\sqrt{-b} & \text{if } b<0.\end{cases}$$

Let $\{E_0, E_1, \cdots, E_{n-1}\}$ be a parallel orthonormal frame field along σ such that $E_0=\dot\sigma$ and $E_j(0)$ is tangent to P for $1\leqq j\leqq p$. Then we have the following.

LEMMA 3.2. *Let J be a P-Jacobi field along σ and let $f_j=\langle J, E_j\rangle$. Suppose that N has constant sectional curvature b and the shape operator $A_{\dot\sigma(0)}$ has a unique eigenvalue μ. Then f_j satisfies the following:*

$$f_j(t)=\begin{cases}f_0'(0)g_0(b, \mu, t) & \text{if } j=0,\\ f_j(0)g_1(b, \mu, t) & \text{if } 1\leqq j\leqq p,\\ f_j'(0)g_2(b, \mu, t) & \text{if } p+1\leqq j\leqq n-1.\end{cases}$$

LEMMA 3.3. *Suppose that N has constant sectional curvature b and the shape* *operator* $A_{\dot\sigma(0)}$ *has a unique eigenvalue* μ. *Then there is no focal point of the pair* (P, σ) *if one of the following holds:*

(3.1) $b>0$, $\lambda<\pi/2\sqrt{b}$ *and* $\mu<\sqrt{b}/\tan(\lambda\sqrt{b})$,

(3.2) $b=0$ *and* $\mu<1/\lambda$,

(3.3) $b<0$ *and* $\mu<\sqrt{-b}/\tanh(\lambda\sqrt{-b})$.

Remark. Lemma 3.2 implies Lemma 3.3. In Lemma 3.3, if $\mu\geqq 0$, then the nonexistence of focal points of the pair (P, σ) implies one of (3.1)-(3.3).

For (b, μ, λ) which satisfies one of (3.1)-(3.3), we define $h_i(b, \mu, \lambda)$ as follows:

$$h_0(b, \mu, \lambda)=1/\lambda,$$

$$h_1(b, \mu, \lambda)=\begin{cases} -\dfrac{\mu\sqrt{b}+b\cdot\tan(\lambda\sqrt{b})}{\sqrt{b}-\mu\cdot\tan(\lambda\sqrt{b})} & \text{if (3.1) holds,} \\ -\dfrac{\mu}{1-\mu\lambda} & \text{if (3.2) holds,} \\ -\dfrac{\mu\sqrt{-b}+b\cdot\tanh(\lambda\sqrt{-b})}{\sqrt{-b}-\mu\cdot\tanh(\lambda\sqrt{-b})} & \text{if (3.3) holds,} \end{cases}$$

$$h_2(b, \mu, \lambda)=\begin{cases} \sqrt{b}/\tan(\lambda\sqrt{b}) & \text{if (3.1) holds,} \\ 1/\lambda & \text{if (3.2) holds,} \\ \sqrt{-b}/\tanh(\lambda\sqrt{-b}) & \text{if (3.3) holds.} \end{cases}$$

For the pair (P, σ), let $V^i(P, \sigma)$ be the subspace of $T_{\sigma(\lambda)}N$ defined by

$$V^0(P, \sigma)=\operatorname{span}\{E_0(\lambda)\},$$

$$V^1(P, \sigma)=\operatorname{span}\{E_1(\lambda), \cdots, E_p(\lambda)\},$$

$$V^2(P, \sigma)=\operatorname{span}\{E_{p+1}(\lambda), \cdots, E_{n-1}(\lambda)\}.$$

LEMMA 3.4. *Under the same assumptions as in Lemma* 3.3, *suppose that one of* (3.1)-(3.3) *holds. Let* J *be a P-Jacobi field along* σ. *Then*

$$I(J, J)=\sum_{i=0}^{2} h_i(b, \mu, \lambda)\,|\,V^i(P, \sigma)\text{-component of } J(\lambda)|^2.$$

Proof. Let $f_j=\langle J, E_j\rangle$. Then $I(J, J)=\langle(\nabla_{\dot\sigma}J)(\lambda), J(\lambda)\rangle=\sum_{j=0}^{n-1} f_j'(\lambda)f_j(\lambda)$. By Lemma 3.2 we have

$$f_j'(\lambda)=\begin{cases} f_0(\lambda)h_0(b, \mu, \lambda) & \text{if } j=0, \\ f_j(\lambda)h_1(b, \mu, \lambda) & \text{if } 1\leqq j\leqq p, \\ f_j(\lambda)h_2(b, \mu, \lambda) & \text{if } p+1\leqq j\leqq n-1. \end{cases}$$

Hence $I(J, J)=h_0(b, \mu, \lambda)f_0^2(\lambda)+h_1(b, \mu, \lambda)\sum_{j=1}^{p} f_j^2(\lambda)+h_2(b, \mu, \lambda)\sum_{p<j} f_j^2(\lambda)$. q. e. d.

LEMMA 3.5. *Suppose that the sectional curvature K_N of N satisfies $K_N \leqq b$ and the maximum eigenvalue of $A_{\dot\sigma(0)}$ is not larger than μ. If one of* (3.1)-(3.3) *holds for (b, μ, λ), then each $V \in L(P, \sigma)$ satisfies the following*:

$$I(V, V) \geqq \sum_{i=0}^{2} h_i(b, \mu, \lambda)\,|\,V^i(P, \sigma)\text{-component of } V(\lambda)|^2.$$

Proof. Let $N(b)$ denote the n-dimensional complete simply connected Riemannian manifold of constant sectional curvature b and let $\tau: [0, \lambda] \to N(b)$ be a geodesic parametrized by arclength. We construct a p-dimensional embedded submanifold $\tilde P$ in $N(b)$ such that $\tau(0) \in \tilde P$, $\dot\tau(0) \in T_{\tau(0)}\tilde P^{\perp}$ and the shape operator $A_{\dot\tau(0)}$ has a unique eigenvalue μ. Let $\{\tilde E_0, \cdots, \tilde E_{n-1}\}$ be a parallel orthonormal frame field along τ such that $\tilde E_0 = \dot\tau$ and $\tilde E_j(0)$ is tangent to $\tilde P$ for $1 \leqq j \leqq p$. We define $\tilde V$ in $L(\tilde P, \tau)$ by $\tilde V = \sum_{j=0}^{n-1} \langle V, E_j \rangle \tilde E_j$. Since $K_N \leqq b$ and the maximum eigenvalue of $A_{\dot\sigma(0)}$ is not larger than μ, we have $I(V, V) \geqq \tilde I(\tilde V, \tilde V)$, where $\tilde I$ denotes the index form for the pair $(\tilde P, \tau)$. By Lemmas 3.1, 3.3 and 3.4 we have

$$\tilde I(\tilde V, \tilde V) \geqq \sum_{i=0}^{2} h_i(b, \mu, \lambda)\,|\,V^i(\tilde P, \tau)\text{-component of } \tilde V(\lambda)|^2.$$

This implies Lemma 3.5. q. e. d.

4. Proof of Theorem B.

We may assume $\sup|H| < \infty$. Let ρ be the scalar curvature of M and let β be the second fundamental form of the immersion $f: M \to N$. Then by the Gauss equation we have

$$m(m-1)b \geqq \rho - m^2|H|^2 + |\beta|^2,$$

$$\sup|K_N| + 2\sup|\beta|^2 \geqq |K_M|.$$

Since ρ has a lower bound, the above inequalities imply the boundedness of the sectional curvature K_M.

Let $\psi: \tau(P, \lambda) \to P$ be the canonical projection and let $F: M \to \boldsymbol{R}$ be the smooth function defined by

$$F(x) = \frac{1}{2}\{d(f(x), \psi f(x))\}^2,$$

where $d(\ ,\ )$ is the distance function on N. Since M is a complete Riemannian manifold whose sectional curvature is bounded, [5, Theorem A′] implies the existence of a sequence $\{x_k\}_{k=1}^{\infty}$ in M such that

$$|\operatorname{grad} F|(x_k) < 1/k\,, \tag{4.1}$$

(4.2) $$(\nabla^2 F)(X, X)<1/k \quad \text{for all unit vector } X\in T_{x_k}M,$$

(4.3) $$\lim_{k\to\infty} F(x_k)=\sup F,$$

where $\nabla^2 F$ denotes the Hessian of F with respect to the Riemannian metric of M. We set $\lambda_k=d(f(x_k), \phi f(x_k))$. Then (4.3) implies $\lim_{k\to\infty}\lambda_k=\lambda$. Since $\lambda>0$, we may assume $0<\lambda_k\leqq\lambda$ for all k. Let $\sigma_k:[0, \lambda_k]\to N$ be the geodesic parametrized by arclength such that $\sigma_k(0)=\phi f(x_k)$ and $\sigma_k(\lambda_k)=f(x_k)$. Then $\dot\sigma_k(0)$ is perpendicular to P. Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of $T_{x_k}M$ such that $V^1(P, \sigma_k)$-component of f_*e_j vanishes for all $j>p$. We set as follows:

$$h_i(k)=h_i(b, \mu, \lambda_k),$$

$$c_j^i(k)=|V^i(P, \sigma_k)\text{-component of } f_*e_j|^2.$$

LEMMA 4.1. $\sum_{i=0}^{2}\sum_{j=1}^{m} h_i(k)c_j^i(k)\geqq ph_1(k)+(m-p)h_2(k)+h(k)$, *where* $h(k)=-\{p|h_0(k)-h_1(k)|+(m-p)|h_0(k)-h_2(k)|\}/(k\lambda_k)^2$.

Proof. For convenience, put $h_i=h_i(k)$ and $c_j^i=c_j^i(k)$. Since $c_j^1=0$ $(j>p)$, $\sum_{i=0}^{2} c_j^i=1$ and $h_2\geqq h_1$, we see that

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=1}^{m} h_ic_j^i&=\sum_{j=1}^{p}\{h_0c_j^0+h_1(1-c_j^0-c_j^2)+h_2c_j^2\}+\sum_{p<j}\{h_0c_j^0+h_2(1-c_j^0)\},\\ &\geqq\sum_{j=1}^{p}\{h_1+(h_0-h_1)c_j^0\}+\sum_{p<j}\{h_2+(h_0-h_2)c_j^0\}.\end{aligned}$$

Since $\langle \operatorname{grad} F, e_j\rangle=\lambda_k\langle\dot\sigma_k(\lambda_k), f_*e_j\rangle$, (4.1) implies $c_j^0(k)<1/(k\lambda_k)^2$. Hence we have

$$\sum_{i=0}^{2}\sum_{j=1}^{m} h_ic_j^i\geqq p\{h_1-|h_0-h_1|/(k\lambda_k)^2\}+(m-p)\{h_2-|h_0-h_2|/(k\lambda_k)^2\}.$$

q. e. d.

Let I_k be the index form for the pair (P, σ_k). Then a calculation shows that

(4.4) $$\frac{1}{\lambda_k}\nabla^2 F(e_j, e_j)=\langle\beta(e_j, e_j), \sigma_k(\lambda_k)\rangle+I_k(J_j, J_j),$$

where J_j is the P-Jacobi field along σ_k such that $J(\lambda_k)=f_*e_j$. Applying Lemma 3.5 to the pair (P, σ_k), we have

(4.5) $$I_k(J_j, J_j)\geqq\sum_{i=0}^{2} h_i(k)c_j^i(k).$$

Hence (4.2), (4.4), (4.5) and Lemma 4.1 imply

$$m/k\lambda_k\geqq -m\sup|H|+ph_1(k)+(m-p)h_2(k)+h(k).$$

Since $\lim_{k\to\infty} h_i(k)=h_i(b, \mu, \lambda)$ and $\lim_{k\to\infty} h(k)=0$, we have

$$\sup|H| \geqq \frac{p}{m} h_1(b, \mu, \lambda)+\frac{m-p}{m} h_2(b, \mu, \lambda).$$

This completes the proof of Theorem B.

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