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ON THE RATIONAL HOMOTOPY OF MAP(HP^m, HP^n)

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Let KP^n be a n-dimensional real, complex or quartenion projective space for $K=R$, C or H, respectively. For each pair of topological spaces (X, A), let $i_A: A\to X$ be the inclusion map and $\mathrm{Map}(A, X)$ denotes the path-component containing i_A of the space of all continuous maps from A to X with the compact open topology. Further let $\mathrm{Aut}(X)=\mathrm{Map}(X, X)$. Then $\mathrm{Aut}(X)$ becomes a homotopy associative H-space with the multiplication induced from the composition of maps, and it has its classifying space $\mathrm{BAut}(X)$.

D. Sullivan studied the rational homotopy of $\mathrm{BAut}(X)$ from the point of view of minimal models, and he also computed the group $\pi_*(\mathrm{BAut}(CP^n))\otimes Q$ in [3].

S. Sasao investigated the homotopy group of $\mathrm{Map}(CP^m, CP^n)$ for $m\leqq n$, and determined the rational homotopy of it in [2].

Furthermore, recently W. Meier and R. Strebel also computed the rational homotopy group $\pi_*(\mathrm{Aut}(RP^n))\otimes Q$ in [1].

Then the purpose of this paper is to determine the rational homotopy group $\pi_*(\mathrm{Map}(HP^m, HP^n))\otimes Q$ for $m\leqq n$, by using the analogous method given in [2].

§ 1. Definitions and Results.

Let R, C and H be the real, the complex and quaternion number field, respectively. For each field $K=R$, C or H, let KP^n denote the projective space of dimension n, defined by

$$KP^n=K^{n+1}-(0, 0, \cdots, 0)/\sim$$

where

$$k(z_0, z_1, z_2, \cdots, z_n)\sim(z_0, z_1, z_2, \cdots, z_n) \qquad \text{for } k\in K-\{0\}.$$

If $m\leqq n$, we define the inclusion map $i_m: HP^m\to HP^n$ by

$$i_m([z_0, z_1, \cdots, z_m])=[z_0, z_1, \cdots, z_m, 0, 0, \cdots, 0],$$

as usual.

Let $\mathrm{map}(HP^m, HP^n)$ (resp. $\mathrm{map}_0(HP^m, HP^n)$) be the space of all continuous maps (resp. of all base-point preserving maps) from HP^m to HP^n with the compact open topology. In general, the spaces $\mathrm{map}(HP^m, HP^n)$ and $\mathrm{map}_0(HP^m, HP^n)$

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have infinitely many path-components and we are mainly concerned with the component containing the inclusion map i_m. So we denote these components by $\mathrm{Map}(HP^m, HP^n)$ and $\mathrm{Map}_0(HP^m, HP^n)$ respectively.

Let $Sp(n)$ be the n-th sympletic group defined by

$$Sp(n)=\{A\in \mathrm{Mat}(n, H): A^*A=AA^*=E_n\},$$

where $\mathrm{Mat}(n, K)$ denotes the ring of (n, n)-matrices with coefficients of K for $K=R$, C or H.

The group $Sp(n+1)$ acts on the space HP^n as usual, and the subgroup of $Sp(n+1)$, which fixes the subspace HP^m, contains the subgroup $Z_2\times Sp(n-m)$. Then we have a map

$$s_m: Z_2\times Sp(n-m)\backslash Sp(n+1) \longrightarrow \mathrm{Map}(HP^m, HP^n)$$

defined by

$$s_m(A)([z_0, z_1, \cdots, z_m])=[z_0, z_1, \cdots, z_m, 0, 0, \cdots, 0]A$$

$$\text{for } [z_0, z_1, \cdots, z_m]\in HP^m \text{ and } A\in Sp(n+1).$$

Then the main object of this note is to show the following results:

THEOREM 1. *The induced homomorphism*

$$s_{m*}\otimes 1: \pi_i(Z_2\times Sp(n-m)\backslash Sp(n+1))\otimes Q \longrightarrow \pi_i(\mathrm{Map}(HP^m, HP^n))\otimes Q$$

is an isomorphism for $1\leqq m\leqq n$ *and all* $i>4$.

In particular, for the case $m=n$, we also obtain

COROLLARY 2. *The induced homomorphism*

$$s_{n*}\otimes 1: \pi_i(Z_2\backslash Sp(n+1))\otimes Q \longrightarrow \pi_i(\mathrm{Aut}(HP^n))\otimes Q$$

is an isomorphism for $n\geqq 1$ *and all* $i>4$, *where* $\mathrm{Aut}(HP^n)$ *denotes the space of self-homotopy equivalences of* HP^n *which is homotopic to* id_{HP^n}.

Thus, by using the above results and Sullivan's technique for $1\leqq i\leqq 4$, we can establish

COROLLARY 3. (a) $\pi_i(\mathrm{Aut}(HP^n))\otimes Q=0$ *for* $1\leqq i\leqq 6$.

(b) *For* $7\leqq i\leqq 4n+3$,

$$\pi_i(\mathrm{Aut}(HP^n))\otimes Q=\begin{cases} Q & \textit{if } i\equiv 3, 7 \pmod 8 \\ 0 & \textit{otherwise} \end{cases}$$

(c) $\pi_i(\mathrm{Aut}(HP^n))\otimes Q=0$ *for* $i\geqq 4n+4$.

This paper is organized as follows.

In §2 we recall several lemmas and commutative diagram needed for the proof of Theorem 1. In §3 we give the proof of Proposition 2.5, and in §4 the proof of Theorem 1 is completed. In §5 we consider the problem from the view-point of Sullivan's minimal model theory.

Finally in this section, the author would like to take this opportunity to thank Professors S. Sasao, A. Kono and H. Shiga for many valuable suggestions.

§2. Preliminaries.

Let $1 \leqq m \leqq n$ and $j: HP^{m-1} \to HP^m$ be the inclusion map. Then it induces a fibration $j^{\#}: \mathrm{Map}(HP^m, HP^n) \to \mathrm{Map}(HP^{m-1}, HP^n)$ and we denote by F_m its fibre, which is defined by

$$F_m = \{f \in \mathrm{Map}(HP^m, HP^n) : f \circ j = \iota_{m-1}\}.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
Z_2 \times Sp(n-m) \backslash Z_2 \times Sp(n-m+1) \arrow[r, "s'_m"] \arrow[d] & F_m \arrow[d] \\
Z_2 \times Sp(n-m) \backslash Sp(n+1) \arrow[r, "s_m"] \arrow[d] & \mathrm{Map}(HP^m, HP^n) \arrow[d] \\
Z_2 \times Sp(n-m+1) \backslash Sp(n+1) \arrow[r, "s_{m-1}"] & \mathrm{Map}(HP^{m-1}, HP^n)
\end{tikzcd}$$

Fig. (2.1)

We define the pairing $P: F_m \times \Omega^{4m} HP^n \to F_m$ by

$$P(f, \omega) = \nabla \circ (f \vee \omega) \circ \mu' \qquad \text{for } (f, \omega) \in F_m \times \Omega^{4m} HP^n,$$

where the map $\mu': HP^m = HP^{m-1} \cup_\eta e^{4m} \to HP^m \vee S^{4m}$ is a co-action map and $\nabla: HP^n \vee HP^n \to HP^n$ is a folding map. In particular, we also define the map $P': \Omega^{4m} HP^n \to F_m$ by

$$P'(\omega) = P(i_m, \omega) \qquad \text{for } \omega \in \Omega^{4m} HP^n.$$

Here we note the following result which is obtained by the standard arguments.

LEMMA 2.2. *The map $P': \Omega^{4m} HP^n \to F_m$ is a weak homotopy equivalence.*

Then it follows from the above lemma that we have a map $s'': S^{4n-4m+3} \to \Omega^{4m} HP^n$ which makes the diagram

$$\begin{tikzcd} S^{4n-4m+3} \arrow[r, "s'_m"] \arrow[dr, "s''"'] & F_m \\ & \Omega^{4m}HP^n \arrow[u, "P'", "\simeq"'] \end{tikzcd}$$

Fig. (2.3)

homotopy commutative.

LEMMA 2.4. *Let* $\eta: S^{4n+3}\to HP^n$ *be the Hopf fibering. Then* $\pi_i(HP^n)\cong \eta_*\pi_i(S^{4n+3})\oplus\pi_{i-1}(S^3)$.

Proof. Consider the fibration $S^3 \xrightarrow{i} S^{4n+3} \xrightarrow{\eta} HP^n$. Since the inclusion map i is null-homotopic, we have a short exact sequence

$$0 \longrightarrow \pi_i(S^{4n+3}) \xrightarrow{\eta_*} \pi_i(HP^n) \xrightarrow{\Delta} \pi_{i-1}(S^3) \longrightarrow 0\,.$$

Since $HP^1=S^4$, there is an inclusion map $j: S^4\to HP^n$. Then we define a homomorphism $\gamma: \pi_{i-1}(S^3)\to\pi_i(HP^n)$ by the composite

$$\pi_{i-1}(S^3) \xrightarrow{E} \pi_i(S^4) \xrightarrow{j_*} \pi_i(HP^n)$$

where E is the suspension homomorphism. Then it is easy to see that γ is a splitting homomorphism of the above short exact sequence. This completes the proof. Q.E.D.

PROPOSITION 2.5. *For* $i=4n-4m+3$, *the induced homomorphism*

$$s''_*: \pi_i(S^{4n-4m+3}) \longrightarrow \pi_i(\Omega^{4m}HP^n)\cong\pi_{4n+3}(HP^n)$$

satisfies $\mathrm{Im}\,(s''_*)=Z\{\eta\}$.

The proof of (2.5) will be given in § 3. In particular, it follows from (2.5) that we also have

COROLLARY 2.6. *Let* $\mathrm{ad}(s''): S^{4n+3}\to HP^n$ *be the adioint map of* s''. *Then the map* $\mathrm{ad}(s'')$ *and* η *are homotopic.*

§ 3. Proof of Proposition 2.5.

First, we remark the inclusion map

$$S^{4n-4m+3} \longrightarrow Z_2\times Sp(n-m)\backslash Sp(n+1)$$

is given by

$$Z=(z_0, z_1, \cdots, z_{n-m}) \longmapsto \text{the coset of } A(Z)=\begin{bmatrix} E_m & & 0 \\ & z_0\ z_1\ z_2 \cdots z_{n-m} & \\ 0 & * \ * \ * & \end{bmatrix}$$

where $S^{4n-4m+3}=\{Z=(z_0, z_1, \cdots, z_{n-m})\in H^{n-m+1};\ \sum_{i=0}^{n-m}|z_i|^2=1\}$. Thus the map $s'_m : S^{4n-4m+3}\to\text{Map}(HP^m, HP^n)$ may be regarded as the map

$$\mu : HP^m\times S^{4n-4m+3} \longrightarrow HP^n$$

which is defined by

$$\begin{aligned} \mu(W, Z)&=[w_0, w_1, \cdots, w_m, 0, 0, \cdots, 0]\cdot A(Z) \\ &=[w_0, w_1, \cdots, w_{m-1}, w_m z_0, w_m z_1, \cdots, w_m z_{n-m}] \\ &\qquad \text{for } Z=(z_0, z_1, \cdots, z_{n-m})\in S^{4n-4m+3} \\ &\qquad \text{and } W=[w_0, w_1, \cdots, w_m]\in HP^m . \end{aligned}$$

Similarly, the constant map $S^{4n-4m+3}\to F_m$ is given by

$$\nu : HP^m\times S^{4n-4m+3} \longrightarrow HP^n$$

which is defined by

$$\nu(W, Z)=[w_0, w_1, \cdots, w_m, 0, 0, \cdots, 0] .$$

Hence two maps μ and ν agree on $(HP^{m-1}\times S^{4n-4m+3})\cup HP^m$, and we wish to know the difference element between them. For this purpose, it is convenient to replace the pair (HP^m, HP^{m-1}) with the pair (D^{4m}, S^{4m-1}) by using a characteristic map of the cell e^{4m}, and so we embed $D^{4m}\times S^{4n-4m+3}$ into S^{4n+3} in the usual way. Here we identify

$$S^{4n+3}=D^{4m}\times S^{4n-4m+3}\cup S^{4m-1}\times D^{4n-4m+4} .$$

Let the variables

$$W=(w_0, w_1, \cdots, w_{m-1})$$

and

$$Z=(z_0, z_1, \cdots, z_{n-m})$$

run over S^{4m-1} and $S^{4n-4m+3}$, respectively. Then the point

$$((\sin\theta)W, (\cos\theta)Z)$$

runs over S^{4n+3} for $0\leqq\theta\leqq\pi/2$, and $D^{4m}\times S^{4n-4M+3}$ may be regarded as the subset of points with the condition $0\leqq\theta\leqq\pi/4$.

Therefore the characteristic map of the cell e^{4m},

$$\lambda : (D^{4m}, S^{4m-1}) \longrightarrow (HP^m, HP^{m-1})$$

is given by

$$\lambda((\sin\theta)W)=[((\sin 2\theta)W,\ \cos 2\theta)] \qquad \text{for } 0\leqq\theta\leqq\pi/4.$$

Now two maps

$$\mu' : D^{4m}\times S^{4n-4m+3} \longrightarrow HP^n$$

and

$$\nu' : D^{4m}\times S^{4n-4m+3} \longrightarrow HP^n$$

are defined by

$$\mu'((\sin\theta)W,\ (\cos\theta)Z)=[(\sin 2\theta)W,\ (\cos\theta)Z]$$

and

$$\nu'((\sin\theta)W,\ (\cos\theta)Z)=[(\sin 2\theta)W,\ (\cos\theta)(1,0,0,\cdots,0)]$$
$$\text{for } 0\leqq\theta\leqq\pi/4,$$

respectively.

Then the two maps μ' and ν' agree on $(S^{4m-1}\times S^{4n-4m+3})\cup D^{4m}$, and we wish to know the difference element between them. For this aim, we define the extensions μ'' of μ' and ν'' of ν' over $S^{4m-1}\times D^{4n-4m+4}$ by setting

$$\mu''((\sin\theta)W,\ (\cos\theta)Z)=\nu''((\sin\theta)W,\ (\cos\theta)Z)$$
$$=[W,\ 0] \qquad \text{for } \pi/4\leqq\theta\leqq\pi/2.$$

The two maps μ'' and ν'' now agree on $(S^{4m-1}\times D^{4n-4m+4})\cup D^{4m}$, which is a contractible space. Therefore their difference element σ is equal to

$$[\mu'']-[\nu'']\in\pi_{4n+3}(HP^n).$$

However, the map ν'' clearly factors through D^{4m} and so $[\nu'']=0$. Hence $\sigma=[\mu'']\in\pi_{4n+3}(HP^n)$. On the other hand, the map μ'' can be lifted to the map

$$\tilde{\mu} : S^{4n+3} \longrightarrow S^{4n+3}$$

defined by

$$\tilde{\mu}((\sin\theta)W,\ (\cos\theta)Z)=((\sin\Phi(\theta))W,\ (\cos\Phi(\theta))Z),$$

where the function $\Phi=\Phi(\theta)$ is given by the equation

$$\Phi(\theta)=\begin{cases} 2\theta & \text{if } 0\leqq\theta\leqq\pi/4 \\ \pi/2 & \text{if } \pi/4\leqq\theta\leqq\pi/2. \end{cases}$$

Since the function $\Phi(\theta)$ is homotopic to the identity function keeping 0 and $\pi/2$, the required difference element σ is equal to

$$[\eta]\in\pi_{4n+3}(HP^n).$$

This completes the proof of Proposition 2.5. Q.E.D.

§4. Proof of the Main Result.

In this section we will show Theorem 1.

Proof of Theorem 1. We will show Theorem 1 by the induction over m, keeping n fixed.

Suppose the assertion is true for $m-1$ and that $2\leqq m\leqq n$. Let

$$E^{4m}: \pi_i(S^{4n-4m+3}) \longrightarrow \pi_{i+4m}(S^{4n+3})$$

denote the iterated suspension homomorphism. Then it follows from Corollary 2.6 that the diagram

$$\begin{tikzcd}
\pi_i(S^{4n-4m+3}) \arrow[r, "s'_{m*}"] \arrow[d, "E^{4m}"'] & \pi_i(F_m) \\
\pi_{i+4m}(S^{4n+3}) \arrow[r, "\eta_*"] & \pi_{i+4m}(HP^n)\cong\pi_i(\Omega^{4m}HP^n) \arrow[u, "\cong"', "P'_*"]
\end{tikzcd}$$

Fig. (4.1)

is commutative. Since

$$E^{4m}\otimes 1: \pi_i(S^{4n-4m+3})\otimes Q \longrightarrow \pi_{i+4m}(S^{4n+3})\otimes Q$$

and

$$\eta_*\otimes 1: \pi_{i+4m}(S^{4n+3})\otimes Q \longrightarrow \pi_{i+4m}(HP^n)\otimes Q$$

are both isomorphic for all i (Lemma 2.4), the induced homomorphism

$$s'_{m*}\otimes 1: \pi_i(S^{4n-4m+3})\otimes Q \longrightarrow \pi_i(F_m)\otimes Q$$

is also isomorphic for all i. Hence the assertion easily follows from the commutative diagram (2.1) and Five Lemma. Thus it is sufficient only to prove the case $m=1$.

Here we recall the following lemma.

LEMMA 4.2.

$$\pi_i(Z_2\times Sp(n-1)\backslash Sp(n+1))\otimes Q=\begin{cases} Q & \text{if } i=4n-1 \text{ or } 4n+3 \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Consider the fibring

$$S^{4n-1}\cong Sp(n-1)\backslash Sp(n) \longrightarrow Sp(n-1)\backslash Sp(n+1) \longrightarrow Sp(n)\backslash Sp(n+1)\cong S^{4n+3}.$$

Then the assertion easily follows from the homotopy exact sequence of the above fibring. Q. E. D.

Let $m=1$ and consider the commutative diagram of fibrings:

$$\begin{array}{ccccc} S^{4n-1} & \longrightarrow & Z_2\times Sp(n-1)\backslash Sp(n+1) & \longrightarrow & S^{4n+3} \\ \downarrow{\scriptstyle s_1'} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle \eta} \\ F_1\cong\Omega^4HP^n & \longrightarrow & \mathrm{Map}(HP^1, HP^n) & \xrightarrow{ev} & HP^n \end{array}$$

where η is the Hopf fibring and ev denotes the evaluation map defined by

$$\mathrm{ev}(f)=f(x_0) \qquad \text{for } f\in\mathrm{Map}(HP^1, HP^n).$$

Here x_0 is a base point of HP^n. Then if follows from the homotopy exact sequence of the above diagram that the induced homomorphism

$$s_{1*}\otimes 1: \pi_i(Z_2\times Sp(n-1)\backslash Sp(n+1))\otimes Q \longrightarrow \pi_i(\mathrm{Map}(HP^1, HP^n))\otimes Q$$

is clearly isomorphic if $i>4$. This completes the proof of Theorem 1. Q.E.D.

§5. The Model of BAut(HP^n).

In this section we consider the rest of the proof of Corollary 3. It follows from Theorem 1 and Lemma 4.2 that it suffices only to consider $\pi_i(\mathrm{Aut}(HP^n))\otimes Q$ for $i\leqq 4$. However, we discuss about Corollary 3 in the more general situation. For this aim, we recall Sullivan's minimal model theory.

DEFINITION 5.1. Let A^* be a non-negative anti-commutative associative differential graded algebra over Q, where we denote its differential by

$$d: A^k \longrightarrow A^{k+1} \qquad (k\geqq 0).$$

Then we define a non-negative differential graded Lie algebra over Q associated to A^*, which is denoted by $L(A^*)$, as follows:

(a) $L(A^*)=\sum_{k\geqq 0} L(A^*)_k$, where for $k>0$, let $L(A^*)_k$ denote the Q-vector space consisting of all derivations of A^* decreasing dimension k and $L(A^*)_0$ be the Q-vector space consisting of all degree zero derivations, commuting d.

(b) Lie bracket $[\,,]: L(A^*)_k\times L(A^*)_j\to L(A^*)_{k+j}$ is given by

$$[\phi, \psi]=\phi\circ\psi-(-1)^{kj}\psi\circ\phi \qquad \text{for } (\phi, \psi)\in L(A^*)_k\times L(A^*)_j.$$

(c) Differential of degree -1, $\delta: L(A^*)_k\to L(A^*)_{k-1}$, is defined by

$$\delta(\phi)=d\circ\phi-(-1)^k\phi\circ d \qquad \text{for } \phi\in L(A^*)_k.$$

THEOREM 5.2. (D. Sullivan) *Let X be a simply-connected finite CW complex, and $M(X)^*$ be its minimal model. Then $L(M(X)^*)$ is a Quillen's type model of* BAut(X). *Thus*

$$H_*(\{L(M(X)^*), \delta\}) \cong \pi_*(\Omega\,\mathrm{BAut}(X)) \otimes Q.$$

Proof. See [3]. Q.E.D.

We apply the above result to the case $X=HP^n$. Since $H^*(HP^n, Q) \cong Q[X]/(X^{n+1})$ for $\dim X=4$, the minimal model of HP^n is equal to

$$\Lambda(x, y) \quad \text{for } x^{n+1}=dy, \quad (\dim x=4, \dim y=4n+3).$$

Let (a, b) denote the derivation of A^* taking a to b and anihilating the other generators. Then it is easy to see the following

LEMMA 5.3.

$$L(M(HP^n)^*)_k = \begin{cases} \{(y, x^j)\} & \text{if } k=4n+3-4j \ (0 \leqq j \leqq n) \\ \{(x, 1)\} & \text{if } k=4 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we note

LEMMA 5.4.

$$\delta(x, 1) = \pm(n+1)(y, x^n).$$

Proof.

$$\begin{aligned} \delta(x, 1)(x) &= d((x, 1)(x)) \pm (x, 1)(dx) \\ &= d(1) \pm (x, 1)(0) \\ &= 0 = \pm(n+1)(y, x^n)(x). \end{aligned}$$

Similarly,

$$\begin{aligned} \delta(x, 1)(y) &= d((x, 1)(y)) \pm (x, 1)(dy) \\ &= d(0) \pm (x, 1)(x^{n+1}) \\ &= \pm(n+1)x^n = \pm(n+1)(y, x^n)(y). \end{aligned}$$

Hence, $\delta(x, 1) = \pm(n+1)(y, x^n)$. This completes the proof. Q.E.D.

Then it follows from (5.3) and (5.4) that

$$H_i(\{L(M(HP^n)^*), \delta\}) = \begin{cases} Q & \text{if } i=7, 11, 15, \cdots, 4n+3. \\ 0 & \text{otherwise.} \end{cases}$$

Since $\pi_i(\Omega\,\mathrm{BAut}(HP^n)) \cong \pi_i(\mathrm{Aut}(HP^n))$, we obtain

$$\pi_i(\mathrm{Aut}(HP^n)) \otimes Q \cong \begin{cases} Q & \text{if } i=7, 11, 15, \cdots, 4n+3. \\ 0 & \text{otherwise.} \end{cases}$$

Therefore Corollary 3 was established.

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