S. ISHIHARA AND M. KONISHI
KODAI MATH. J.
5 (1982), 30—37

COMPLEX ALMOST CONTACT STRUCTURES IN A COMPLEX CONTACT MANIFOLD

BY SHIGERU ISHIHARA AND MARIKO KONISHI

§ 1. Introduction

Let M be a complex manifold of odd dimension $2m+1$ ($\geqq 3$) covered by an open covering $\mathfrak{A}=\{O_i\}$ consisting of coordinate neighborhoods. If there is a holomorphic 1-form ω_i in each $O_i \in \mathfrak{A}$ in such a way that for any O_i, $O_j \in \mathfrak{A}$

$$\text{(1.1, i)} \qquad \omega_i \wedge (d\omega_i)^m \neq 0 \qquad \text{in} \quad O_i ,$$

$$\text{(1.1, ii)} \qquad \omega_i = f_{ij}\omega_j \qquad \text{in} \quad O_i \cap O_j \neq \phi ,$$

where f_{ij} is a holomorphic function in $O_i \cap O_j$, then the set $\{(\omega_i, O_i) | O_i \in \mathfrak{A}\}$ of local structures is called a complex contact structure and M a complex contact manifold, where ω_i is called the contact form in O_i.

On the other hand, suppose that there are given in each $O_i \in \mathfrak{A}$ a 1-form u_i, a vector field U_i and a tensor field G_i of type (1, 1) satisfying the following condition (1.2, i) and (1.2, ii): for any O_i, $O_j \in \mathfrak{A}$

$$\text{(1.2, i)} \qquad G_i^2 = -I + u_i \otimes U_i + v_i \otimes V_i , \qquad G_i F = -F G_i ,$$

$$u_i \circ G_i = 0 , \qquad u_i(U_i) = 1 ,$$

I and F being respectively the identity tensor field of type (1, 1) and the complex structure of M, where v_i and V_i are defined in O_i respectively by

$$v_i = u_i \circ F , \qquad V_i = -F U_i .$$

In $O_i \cap O_j \neq \phi$ there are functions a and b in such a way that

$$\text{(1.2, ii)} \qquad \begin{array}{ll} u_i = a u_j - b v_j , & G_i = a G_j - b H_j , \\ v_i = b u_j + a v_j , & H_i = b G_j + a H_j , \end{array} \qquad \text{in} \quad O_i \cap O_j ,$$

where H_i is defined in O_i by

$$H_i = F G_i .$$

Then the set $\{(u_i, U_i, G_i, O_i) | O_i \in \mathfrak{A}\}$ of local structures is called a complex

Received June 30, 1980.

almost contact structure and M a complex almost contact manifold.

It is the purpose of the present paper to discuss the relation between complex contact structures and complex almost contact structures and to prove

THEOREM *Let M be a complex contact manifold of odd complex dimension $2m+1$ ($\geqq 3$). Then M admits always a complex almost contact structure of class C^∞.*

This subject has been partially studied in [3] and proved that a complex almost contact manifold admits a complex contact structure if it is normal.

§ 2. Lemmas.

From now on, M is assumed to be a complex contact manifold of complex dimension $2m+1$ ($\geqq 3$) with structure $\{(\omega_i, O_i) \mid O_i \in \mathfrak{A}\}$. Then for any $O_i, O_j, O_k \in \mathfrak{A}$

$$\omega_i = f_{ij}\omega_j \quad \text{in} \quad O_i \cap O_j \neq \phi \tag{2.1}$$

and hence the cocycle condition

$$f_{jk} f_{ki} f_{ij} = 1 \quad \text{in} \quad O_i \cap O_j \cap O_k$$

holds. Then there is a complex line bundle $\tilde{P}$ over M with $\{f_{ij}\}$ as its transition functions. Denote by P a circle bundle over M associated with $\tilde{P}$. Then there is a non-vanishing complex-valued function τ_i in $O_i \in \mathfrak{A}$ such that the function

$$h_{ij} = \tau_i^{-1} f_{ij} \tau_j \quad \text{in} \quad O_i \cap O_j \neq \phi \tag{2.2}$$

is the transition function of P in $O_i \cap O_j$ and satisfies the condition

$$|h_{ij}| = 1 .$$

On putting

$$\pi_i = \tau_i^{-1}\omega_i \quad \text{in} \quad O_i ,$$

we have by using (2.1) and (2.2)

$$\pi_i = h_{ij}\pi_j , \quad |h_{ij}| = 1 \quad \text{in} \quad O_i \cap O_j . \tag{2.3}$$

Let σ be a connection in the circle bundle P. Then there is a local real 1-form σ_i in each $O_i \in \mathfrak{A}$ such that

$$\sqrt{-1}\,\sigma_i = \sqrt{-1}\,\sigma_j + \frac{dh_{ij}}{h_{ij}} \quad \text{in} \quad O_i \cap O_j . \tag{2.4}$$

The condition $\omega_i \wedge (d\omega_i)^m \neq 0$ implies

$$\pi_i \wedge (d\pi_i)^m \neq 0 \quad \text{in} \quad O_i \tag{2.5}$$

and hence

$$\pi_i \wedge \Omega_i{}^m \neq 0 \quad \text{in} \quad O_i, \tag{2.6}$$

where Ω_i is defined by

$$\Omega_i = d\pi_i - \sqrt{-1}\,\sigma_i \wedge \pi_i \quad \text{in} \quad O_i. \tag{2.7}$$

Thus Ω_i is a local 2-form of rank $2m$ in each O_i and

$$\Omega_i = h_{ij}\Omega_j \quad \text{in} \quad O_i \cap O_j \tag{2.8}$$

holds because of (2.3) and (2.4). Thus we have

LEMMA 2.1. *There is in each $O_i \in \mathfrak{A}$ a 2-form Ω_i of rank $2m$ and of class C^∞ satisfying* (2.8).

We now put

$$u_i = \frac{1}{2}(\pi_i + \bar{\pi}_i), \qquad v_i = \frac{1}{2\sqrt{-1}}(\pi_i - \bar{\pi}_i) \tag{2.9}$$

which are local real 1-forms in each $O_i \in \mathfrak{A}$. Then we obtain

$$v_i = u_i \circ F, \tag{2.10}$$

F being the complex structure of M. Using (2.5), we get

$$\begin{aligned} u_i &= a u_j - b v_j, \\ v_i &= b u_j + a v_j, \end{aligned} \quad \text{in} \quad O_i \cap O_j, \tag{2.11}$$

where $h_{ij} = a + \sqrt{-1}\,b$. The relation (2.5) implies

$$(\pi_i + \bar{\pi}_i) \wedge (d\pi_i + d\bar{\pi}_i)^m \neq 0, \qquad (\pi_i - \bar{\pi}_i) \wedge (d\pi_i - d\bar{\pi}_i)^m \neq 0,$$

or equivalently

$$u_i \wedge v_i \wedge (du_i)^{2m} \neq 0, \qquad u_i \wedge v_i \wedge (dv_i)^{2m} \neq 0. \tag{2.12}$$

On the other hand, we obtain from (2.7) and (2.9)

$$\begin{aligned} \frac{1}{2}(\Omega_i + \bar{\Omega}_i) &= du_i - \sigma_i \wedge v_i, \\ \frac{1}{2\sqrt{-1}}(\Omega_i - \bar{\Omega}_i) &= dv_i + \sigma_i \wedge u_i. \end{aligned} \tag{2.13}$$

Therefore (2.12) and (2.13) imply

$$u_i \wedge v_i \wedge (\hat{G}_i)^{2m} \neq 0, \qquad u_i \wedge v_i \wedge (\hat{H}_i)^{2m} \neq 0, \tag{2.14}$$

where

$$\hat{G}_i = du_i - \sigma_i \wedge v_i, \qquad \hat{H}_i = dv_i + \sigma_i \wedge u_i. \tag{2.15}$$

Thus (2.8), (2.11), (2.12) and (2.13) imply

LEMMA 2.2. *There are in each $O_i \in \mathfrak{A}$ skew-symmetric local tensor fields $\hat{G}_i$ and $\hat{H}_i$ of rank $4m$ such that*

$$\hat{H}_i(X, Y)=\hat{G}_i(FX, Y), \quad \hat{G}_i(X, Y)=-\hat{H}_i(FX, Y) \qquad \text{in} \quad O_i \tag{2.16}$$

for any vector fields X and Y and

$$\hat{G}_i=a\hat{G}_j-b\hat{H}_j, \quad \hat{H}_i=b\hat{G}_j+a\hat{H}_j, \qquad \text{in} \quad O_i \cap O_j, \tag{2.17}$$

where $h_{ij}=a+\sqrt{-1}\,b$.

We now state two more lemmas which are essential in the proof of our theorem (cf. [1], [2], [4]). Let $O(n)$ be the orthogonal group acting on n variables, $H(n)$ be the space consisting of all positive definite symmetric (n, n)-matrices and $GL(n, \boldsymbol{R})$ be the general linear group acting on n variables.

LEMMA 2.3. *Any real non-singular (n, n)-matrix ρ can be written in one and only one way as the product $\rho=\alpha\beta$ with $\alpha \in O(n)$ and $\beta \in H(n)$. The mapping $\phi: GL(n, \boldsymbol{R}) \to O(n)\times H(n)$ defined by this decomposition gives a homeomorphism.*

Remark. Since $O(n)$ and $H(n)$ are real analytic submanifolds of $GL(n, \boldsymbol{R})$, $\phi: GL(n, \boldsymbol{R}) \to O(n)\times H(n)$ is a real analytic homeomorphism, i.e. any $\rho \in GL(n, \boldsymbol{R})$ can be decomposed analytically in one and only one way as $\rho=\alpha\beta$, where $\alpha \in O(n)$ and $\beta \in H(n)$, (See Hatakeyama [2] for example).

LEMMA 2.4. *Let $\mathfrak{A}=\{O_i\}$ be an open covering of M by coordinate neighborhoods. Suppose that there is in each $O_i \in \mathfrak{A}$ a local tensor fields α_i of type (0.2) and of class C^∞. Choose a field of orthonormal frames in each $O_i \in \mathfrak{A}$ and let γ_{ij} be the transformations of these fields of orthonormal frames in $O_i \cap O_j$. Then $\{\alpha_i\}$ defines globally a tensor field of class C^∞ in M if and only if $\gamma_{ij}\alpha_j {}^t\gamma_{ij} =\alpha_i$ holds in $O_i \cap O_j$ for any O_i, $O_j \in \mathfrak{A}$.*

§ 3. Proof of theorem

Let $\hat{G}_i$ and $\hat{H}_i$ be the skew-symmetric tensors appearing in Lemma 2.2. If we put for any $p \in O_i$ $(\in \mathfrak{A})$

$$D_i(p)=\{Y \in T_p(O_i): \hat{G}_i(Y, X)=0 \text{ for any } X \in T_p(O_i)\},$$

then we get in O_i a local distribution $D_i: p \mapsto D_i(p)$. Lemma 2.2 implies $D_i(p) =D_j(p)$ for any $p \in O_i \cap O_j$. Hence the local distributions D_i defined in O_i determines a global distribution D in M, which is of real dimension 2.

LEMMA 3.1. *There is a unique local basis $\{U_i, V_i\}$ of the distribution D in each $O_i \in \mathfrak{A}$ such that*

$$\left.\begin{aligned} &u_i(U_i)=1, \quad u_i(V_i)=0, \quad v_i(U_i)=0, \quad v_i(V_i)=1, \\ &\hat{G}_i(U_i, X)=\hat{G}_i(V_i, X)=\hat{H}_i(U_i, X)=\hat{H}_i(V_i, X)=0, \end{aligned}\right\} \quad \text{in} \quad O_i \cap O_j$$

for any vector field X *and*

$$U_i=aU_j-bV_j\,,\quad V_i=bU_j+aV_j \qquad in \quad O_i\cap O_j\,.$$

Proof. If we put for any $p\in O_i$ $(\in\mathfrak{A})$

$$A_i(p)=\{Y\in T_p(O_i): v_i(Y)=0\}\,,$$

then we get a distribution $A_i: p\mapsto A_i(p)$ in O_i. The distribution $D\cap A_i$ is 1-dimensional in O_i because of (2.14). Thus there is a unique vector field U_i in O_i such that U_i spans $D\cap A_i$ and $u_i(U_i)=1$, i.e. such that

$$\hat{G}_i(U_i, X)=0\,,\quad u_i(U_i)=1\,,\quad v_i(U_i)=0 \qquad \text{in} \quad O_i$$

for any vector field X. Lemma 2.2 implies

$$\hat{H}_i(U_i, X)=0$$

for any vector field X. Putting in O_i

$$V_i=-FU_i \tag{3.1}$$

and using Lemma 2.2 and (2.10), we get

$$\hat{G}_i(V_i, X)=\hat{H}_i(V_i, X)=0\,,\quad v_i(V_i)=1\,,\quad u_i(V_i)=0$$

for any vector field X. Therefore $\{U_i, V_i\}$ is in O_i a local basis of the distribution D. As a consequence of (2.11), we have

$$U_i=aU_j-bV_j\,,\quad V_i=bU_j+aV_j\,, \qquad \text{in} \quad O_i\cap O_j\,.$$

Thus Lemma 3.1 is proved.

We shall now prove the theorem stated in § 1.

Proof of theorem. Let $\tilde{g}$ be a Hermitian metric in M such that $\tilde{g}(U_i, X)=u_i(X)$ and $\tilde{g}(V_i, X)=v_i(X)$ for any vector field X. Take an orthonormal adapted frame $\{E_1, FE_1, \cdots, E_{2m}, FE_{2m}, U_i, V_i\}$ with respect to g in each $O_i\in\mathfrak{A}$. Then by Lemma 3.1 $\hat{G}_i$ has components of the form

$$\Phi_i=\left(\begin{array}{c:cc} \Phi_i' & \multicolumn{2}{c}{O} \\ \hdashline & 0 & 0 \\ O & & \\ & 0 & 0 \end{array}\right) \tag{3.2}$$

with respect to the frame $\{E_a, FE_a, U_i, V_i\}$ in O_i, where Φ_i' is a nonsingular real skew-symmetric $(4m, 4m)$-matrix. By Lemma 2.3 Φ_i' can be written in the form

$$\Phi_i'=\alpha_i'\cdot\beta_i' \tag{3.3}$$

with $\alpha_i'\in O(4m)$ and $\beta_i'\in H(4m)$. If we put

$$\alpha_\iota=\left(\begin{array}{c:cc} \alpha'_\iota & & O \\ \hdashline & 0 & 0 \\ O & & \\ & 0 & 0 \end{array}\right), \qquad \beta_\iota=\left(\begin{array}{c:cc} \beta'_\iota & & O \\ \hdashline & 0 & 0 \\ O & & \\ & 0 & 0 \end{array}\right), \tag{3.4}$$

then α_ι and β_ι define tensor fields of class C^∞ in O_ι. Since Φ'_ι is skew-symmetric,

$$\beta'_\iota\cdot{}^t\alpha'_\iota=-\alpha'_\iota\cdot\beta'_\iota$$

and hence

$$\beta'_\iota=-\alpha'_\iota\cdot\beta'_\iota\cdot\alpha'_\iota=-\alpha'^2_\iota\cdot{}^t\alpha_\iota\beta'_\iota\alpha'_\iota\,.$$

As is easily seen, $-\alpha'^2_\iota\in O(4m)$, ${}^t\alpha'_\iota\cdot\beta'_\iota\cdot\alpha'_\iota\in H(4m)$. Thus, by the uniqueness of the decomposition, we obtain

$$\alpha'^2_\iota=-I_{4m}\,,$$

$${}^t\alpha'_\iota\cdot\beta'_\iota\cdot\alpha'_i=\beta'_\iota\,, \quad \text{i.e.} \quad \beta'_\iota\cdot\alpha'_\iota=\alpha'_\iota\cdot\beta_i\,,$$

I_{4m} being the unit $(4m, 4m)$-matrix. Consequently, we have

$$\Phi_\iota=\alpha_\iota\cdot\beta_i\,, \tag{3.5}$$

$$\alpha_\iota{}^2=-\left(\begin{array}{c:cc} I_{4m} & & O \\ \hdashline & 0 & 0 \\ O & & \\ & 0 & 0 \end{array}\right)=-I_{4m+2}+\left(\begin{array}{c:cc} O & & O \\ \hdashline & 1 & 0 \\ O & & \\ & 0 & 1 \end{array}\right). \tag{3.6}$$

On the other hand, the complex structure F has components of the form

$$\Gamma=\left(\begin{array}{c:cc} \Gamma' & & O \\ \hdashline & 0 & -1 \\ O & & \\ & -1 & 0 \end{array}\right), \qquad \Gamma'=\begin{pmatrix} 0 & -1 & & & O \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ O & & & 1 & 0 \end{pmatrix}\in O(4m) \tag{3.7}$$

with respect to the adapted frame $\{E_a, FE_a, U_\iota, V_i\}$ in O_ι. Hence Lemma 2.2 implies that $\hat{H}_\iota$ has components of the form

$$\Psi_\iota=\left(\begin{array}{c:cc} \Psi'_\iota & & O \\ \hdashline & 0 & 0 \\ O & & \\ & 0 & 0 \end{array}\right), \qquad \Psi'_\iota=\Gamma'\Phi'_i=-\Phi'_\iota\Gamma'\,. \tag{3.8}$$

Therefore Ψ'_ι can be decomposed as

$$\Psi'_\iota=(\Gamma'\cdot\alpha'_\iota)\cdot\beta'_\iota=\delta'_\iota\cdot\beta'_\iota\,, \qquad \delta'_\iota=\Gamma'\cdot\alpha'_\iota\,. \tag{3.9}$$

Denote by $\gamma_{\iota\jmath}$ the transformation of adapted frames $\{E_a, FE_a, U_\iota, V_i\}$ and $\{E'_a, FE'_a, U_\jmath, V_\jmath\}$ in $O_\iota\cap O_\jmath$. Then Lemmas 2.2 and 2.4 imply

$$\Phi_\iota=\gamma_{\iota\jmath}\cdot(a\Phi_\jmath-b\Psi_\jmath)\cdot{}^t\gamma_{\iota\jmath}\,.$$

Substituting (3.5) and (3.9) into this, we have

$$\begin{aligned}\alpha'_\iota\cdot\beta'_\iota&=\gamma'_{\iota\jmath}\cdot(a\alpha'_\jmath-b\delta'_\jmath)\cdot\beta'_\jmath\cdot{}^t\gamma'_{\iota\jmath}\\&=(\gamma'_{\iota\jmath}\cdot(a\alpha'_\jmath-b\delta'_\jmath)\cdot{}^t\gamma'_{\iota\jmath})\cdot(\gamma'_{\iota\jmath}\cdot\beta'_\jmath\cdot{}^t\gamma'_{\iota\jmath})\,,\end{aligned}$$

where $\gamma'_{\iota\jmath}$ denotes the element of $O(4m)$ such that

$$\gamma_{\iota\jmath}=\left(\begin{array}{c:cc}\gamma'_{\iota\jmath} & & O\\ \hdashline & a & -b\\ O & & \\ & b & a\end{array}\right).$$

Since $\gamma'_{\iota\jmath}\in O(4m)$ and the decomposition is unique, we get

$$\alpha'_\iota=\gamma'_{\iota\jmath}\cdot(a\alpha'_\jmath-b\delta'_\jmath)\cdot{}^t\gamma'_{\iota\jmath}\,, \tag{3.10}$$

$$\beta'_\iota=\gamma'_{\iota\jmath}\cdot\beta'_\jmath\cdot{}^t\gamma'_{\iota\jmath}\,. \tag{3.11}$$

The equation (3.11) shows by means of Lemma 2.4 that $\{\beta_\iota\}$ defines a global tensor field g of class C^∞, which is a Hermitian metric in M.

Denote by G_ι the local tensor field of type (1, 1) having components α_ι with respect to the adapted frame $\{E_a, FE_a, U_\iota, V_\iota\}$ in O_ι. Thus (3.4) implies

$$G_\iota U_\iota=G_\iota V_\iota=0\,,\quad u_\iota\circ G_\iota=v_\iota\circ G_\iota=0 \qquad \text{in} \quad O_\iota \tag{3.12}$$

and (3.6) implies

$$G_\iota{}^2=-I+u_\iota\otimes U_\iota+v_\iota\otimes V_\iota \qquad \text{in} \quad O_\iota\,. \tag{3.13}$$

Furthermore, using (3.3) and (3.4), we have

$$\left.\begin{aligned}&g(G_\iota X, Y)=\hat{G}_\iota(X, Y)\\&g(U_\iota, X)=u_\iota(X)\,,\quad g(V_\iota, X)=v_\iota(X)\end{aligned}\right\} \quad \text{in} \quad O_\iota \tag{3.14}$$

for any vector fields X and Y.

Next, we denote by H_ι the local tensor field of type (1, 1) having components $\Gamma\cdot\alpha_\iota$ with respect to the adapted frame $\{E_a, FE_a, U_\iota, V_\iota\}$ in O_ι. Then

$$H_\iota=FG_\iota \qquad \text{in} \quad O_\iota \tag{3.15}$$

holds. Then (3.10) implies

$$G_\iota=aG_\jmath-bH_\jmath\,,\quad H_\iota=bG_\jmath+aH_\jmath\,, \qquad \text{in} \quad O_\iota\cap O_\jmath \tag{3.16}$$

Summing up Lemma 3.1, (3.12), (3.13), (3.15) and (3.16), we see that $\{(u_\iota, U_\iota, G_\iota, O_\iota) : O_\iota\in\mathfrak{A}\}$ is a complex almost contact structure in M. Thus the Theorem is proved.

BIBLIOGRAPHY

[1] CHEVALLEY., C., Theory of Lie groups I, Princeton Univ. Press, 1946.
[2] HATAKEYAMA, Y., On the existence of Riemannian metrics associated with 2-form of rank $2r$, Tôhoku Math. J. 14 (1962), 162-166.
[3] ISHIHARA, S. and M. Konishi, Real contact 3-structure and complex contact structure, Southeast Asian Bulletin of Math., 3 (1979), 151-161.
[4] SHIBUYA, Y., On the existence of a complex almost contact structure, Kodai Math. J. 1 (1978) 197-204.

TOKYO INSTITUTE OF TECHNOLOGY