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KODAI MATH. J.
28 (2005), 551–558

ENTIRE FUNCTIONS THAT SHARE TWO VALUES

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Abstract

In this paper, we find all the possible forms of two nonconstant entire functions f and g that share two values counting multiplicities. As applications, we generalize some known results and confirm a conjecture proposed by Osgood-Yang.

1. Introduction and results

Let f be a meromorphic functions defined on the complex plane **C**. In this paper, we shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$, the counting function of the poles $N(r,f)$, and the proximity function $m(r,f)$ (see, e.g., [3]). As usual, $\overline{N}(r,f)$ is the reduced counting function of the poles of f, i.e., the counting function which count every poles only once ignoring the multiplicities. We denote by $N_{k)}(r,f)$ the counting function of the poles of f of multiplicities $\leq k$, and denote by $N_{(k}(r,f)$ the counting function of the poles of f of multiplicities $\geq k$. The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure. Let f and g be two nonconstant meromorphic functions, and a be a value in **C**. We say that f and g share a IM (CM) provided that $f(z) - a$ and $g(z) - a$ have same zeros ignoring multiplicities (counting multiplicities). It is well known (see [2]) that the Nevanlinna characteristic functions $T(r,f)$ and $T(r,g)$ satisfy the following relation:

$$T(r,f) = T(r,g) + S(r,g)$$

provided that f and g share four values IM. In 1976, Osgood-Yang [8] proved that if f and g are two nonconstant entire functions of finite order, and share two distinct finite values CM, then $T(r,f) \sim T(r,g)$ $(r \to \infty)$. Osgood-Yang conjectured that the restriction for the order in this result can be removed. There have been published many other results related to entire functions sharing two values CM, or meromorphic functions sharing three values CM (see [4, 5, 6] and [9, 10, 11]).

AMS(2000) *Subject Classification*. 30D35.
Keywords and phrases. meromorphic function, entire function, value sharing.
Received January 11, 2005; revised April 22, 2005.

In 1998, the author proved the following result.

Theorem A ([5]). *Let f and g be nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. Suppose additionally that f is not a Möbius transformation of g and that there exists an $a \neq 0, 1, \infty$ and a positive constant c such that*

$$T(r,f) \leq c\overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + S(r,f), \tag{1}$$

then there exist a nonconstant entire function γ, a nonzero constant λ and two integers s, t $(t > 0)$ which are mutually prime, such that

$$f = \frac{e^{t\gamma} - 1}{\lambda e^{-s\gamma} - 1}, \quad g = \frac{e^{-t\gamma} - 1}{\frac{1}{\lambda} e^{s\gamma} - 1},$$

$$\frac{(1-a)^{s+t}}{a^t} = \lambda^t \frac{(1-\theta)^{s+t}}{\theta^t},$$

with $\theta = -\dfrac{t}{s} \neq 1, a.$

The author also gave all the possible meromorphic functions f and g if the inequality in (1) is replaced by

$$\overline{N}_{1)}\left(r, \frac{1}{f-a}\right) = S(r,f). \tag{2}$$

In 2003, W.-R. Lü and H.-X. Yi considered another conditions different from that in (1) and (2), and proved the following results.

Theorem B ([6]). *Let f and g be two distinct meromorphic functions sharing* 0, 1 *and* ∞ *CM. Suppose that*

$$\limsup_{r\to\infty} \frac{N_{1)}(r,f) + N_{1)}\left(r, \frac{1}{f}\right)}{T(r,f)} < 1.$$

Then

$$f = \frac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, \quad g = \frac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1},$$

where s and k are positive integers $(1 \leq s \leq k)$ such that $(s, k+1) = 1$, and γ is a nonconstant entire function.

Theorem C ([6]). *Let f and g be two meromorphic functions sharing* 0, 1 *and* ∞ *CM, and suppose that $N_{1)}(r,f) = S(r,f)$. If*

$$\limsup_{r\to\infty}\frac{N_{1)}\left(r,\dfrac{1}{f-1}\right)+N_{1)}\left(r,\dfrac{1}{f}\right)}{T(r,f)}<2,\quad (r\in I),$$

then f and g assume one of the following forms:

(i) $f=e^{k\gamma}+e^{(k-1)\gamma}+\cdots+e^{\gamma}+1,\ g=e^{-k\gamma}+e^{-(k-1)\gamma}+\cdots+e^{-\gamma}+1;$

(ii) $f=-e^{k\gamma}-e^{(k-1)\gamma}-\cdots-e^{\gamma},\ g=-e^{-k\gamma}-e^{-(k-1)\gamma}-\cdots-e^{-\gamma},$

where k is a positive integer and γ is a nonconstant entire function.

In this paper, we shall find all the possible forms of two nonconstant entire functions f and g that share two finite values CM. In fact, we shall prove the following result.

THEOREM 1. *Suppose that f and g are two distinct nonconstant entire functions. If f and g share the values* 0 *and* 1 *CM, then they assume one of the following cases*:

(i) $f=c(1-e^{\zeta}),\ g=(1-c)(1-e^{-\zeta});$

(ii) $f=e^{-n\zeta}\sum_{j=0}^{n}e^{j\zeta},\ g=\sum_{j=0}^{n}e^{j\zeta},\ n=1,2,\dots;$

(iii) $f=-e^{-(n+1)\zeta}\sum_{j=0}^{n}e^{j\zeta},\ g=-e^{\zeta}\sum_{j=0}^{n}e^{j\zeta},\ n=0,1,2,\dots,$

where c $(\neq 0,1)$ is a constant, and ζ is a nonconstant entire function.

Theorem 1 can be generalized to the case (see, Section 3) that f and g are meromorphic functions with the conditions $\overline{N}(r,f)=S(r,f)$ and $\overline{N}(r,g)=S(r,g)$. By Theorem 1, we can get the following results easily. The first one gives a positive answer to Osgood-Yang's conjecture, and the others extend some known results which had some restrictions for the order.

COROLLARY 1. *If nonconstant entire functions f and g share two values counting multiplicities, then $T(r,f)=T(r,g)+O(1)$.*

COROLLARY 2. *Suppose that f and g are two distinct nonconstant entire functions sharing the values* 0 *and* 1 *CM. If $\delta(0,f)>0$, and if $\delta(0,f)\neq 1/p$ for any integer p (≥ 2), then $f=e^{\zeta}$ and $g=e^{-\zeta}$, where ζ is a nonconstant entire function.*

COROLLARY 3. *Suppose that f and α are nonconstant entire functions, and a_1, a_2 are two nonzero constant. If f share* 0 *and* 1 *CM with $g=a_1e^{\alpha}+a_2e^{-\alpha}$, then $f=g$.*

2. Lemmas and proof of the main result

The following two lemmas will be used in the proof of the main theorem.

LEMMA 1 ([4]). *Let f_1 and f_2 be two nonconstant meromorphic functions satisfying*

$$\overline{N}(r, f_i) + \overline{N}\left(r, \frac{1}{f_i}\right) = S(r), \quad i = 1, 2.$$

If $f_1^s f_2^t - 1$ *is not identically zero for all integers s and t* ($|s| + |t| > 0$), *then for any positive number* ε, *we have*

$$N_0(r, 1; f_1, f_2) \leq \varepsilon T(r) + S(r),$$

where $N_0(r, 1; f_1, f_2)$ *denotes the reduced counting function of* f_1 *and* f_2 *related to the common* 1*-points, which counts such points only once ignoring multiplicities, and* $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))$ *as* $r \to \infty$, *except for a set of r of finite linear measure.*

Lemma 2 ([1] or [7]). *If f and g are nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM and f is not a Möbius transformation of g, then*

$$T(r, f) + T(r, g) = N(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - 1}\right) + N_0(r) + S(r),$$

where $N_0(r)$ *denotes the reduced counting function of the zeros of* $f - g$ *which are not the* 0*-points,* 1*-points or poles of f and g, and* $S(r) := S(r, f) = S(r, g)$.

For convenience, we introduce the notation $S^*(r, f)$ which is defined to be any quantity such that for any positive number ε there exists a $S(r, f)$ satisfying the following inequality:

$$|S^*(r, f)| \leq \varepsilon T(r, f) + S(r, f).$$

Suppose that $\mathscr{M}(\mathbf{C})$ is the set of all meromorphic functions on $\mathbf{C}$. For $f \in \mathscr{M}(\mathbf{C})$, Let

$$S(f) = \{g \in \mathscr{M}(\mathbf{C}) : T(r, g) = S(r, f)\},$$
$$S^*(f) = \{g \in \mathscr{M}(\mathbf{C}) : T(r, g) = S^*(r, f)\}.$$

It is obvious that both $S(f)$ and $S^*(f)$ are fields of functions, which are closed under products and differentiating, and $S(f) \subset S^*(f)$. It is easily seen that we can not find any set I of infinite linear measure such that $T(r, f) \leq S^*(r, f)$, $r \in I$.

Now we prove the main result. Suppose that f and g are two distinct nonconstant entire functions, and share 0, 1 CM. Then there exists two entire functions α and β such that

$$\frac{f}{g} = e^{\alpha}, \quad \frac{f - 1}{g - 1} = e^{\beta}. \tag{3}$$

Since f and g share 0 and 1 CM, by Nevanlinna's second fundamental theorem, we get

$$T(r, f) \leq 2T(r, g) + S(r, f) \quad \text{and} \quad T(r, g) \leq 2T(r, f) + S(r, g).$$

Therefore, an $S(r,f)$ is also an $S(r,g)$, and vice versa. If $e^{\alpha} = c$ is a constant, then it is easily seen that $c \neq 0, 1$. Note that f and g share 1 CM. We see that both 1 and c are the exceptional values of f. This is impossible. Hence e^{α} is not a constant. Similarly, e^{β} is not a constant either.

If $e^{\alpha-\beta} = c$ is a constant, then $c \neq 1$, otherwise $f = g$. It follows from (3) that

$$f = \frac{c}{c-1}(1-e^{\beta}), \quad g = \frac{1}{c-1}(e^{-\beta}-1).$$

Therefore, f and g assume the first form in Theorem 1.

In the sequel, we suppose that e^{α}, e^{β} and $e^{\alpha-\beta}$ are not constants, and distinguish two cases below.

Case 1. $(e^{-\beta})^s - (e^{\alpha-\beta})^t$ is not identically zero for any integers s and t $(|s|+|t|>0)$.

In this case, by Lemma 1 we see that

$$\overline{N}_0(r,1;e^{-\beta},e^{\alpha-\beta}) \leq \varepsilon(T(r,e^{-\beta}) + T(r,e^{\alpha-\beta})) + S(r,f) \tag{4}$$

holds for any positive number ε. It follows from (3) that

$$f = \frac{e^{\alpha-\beta}-e^{\alpha}}{e^{\alpha-\beta}-1}, \quad g = \frac{e^{-\beta}-1}{e^{\alpha-\beta}-1}. \tag{5}$$

Since f and g are entire, we see that any 1-points of $e^{\alpha-\beta}$ is a common 1-points of $e^{-\beta}$ and $e^{\alpha-\beta}$. Therefore,

$$\overline{N}\left(r,\frac{1}{e^{\alpha-\beta}-1}\right) \leq \overline{N}_0(r,1;e^{-\beta},e^{\alpha-\beta}).$$

This and (4) imply $\overline{N}\left(r,\dfrac{1}{e^{\alpha-\beta}-1}\right) = S^*(r,f)$. And thus

$$T(r,e^{\alpha-\beta}) = S^*(r,f). \tag{6}$$

Since $N_0(r)$ is the reduced counting function which counts the zeros of $f-g$, but does not count 0-points, 1-points or poles of g, we have

$$N_0(r) \leq \overline{N}_0(r,1;e^{-\beta},e^{\alpha-\beta}) \leq S^*(r,f).$$

By Lemma 2, we get

$$\begin{aligned} T(r,f) + T(r,g) &= N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-1}\right) + N(r,f) + N_0(r) + S(r,f) \\ &= N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{g-1}\right) + S^*(r,f), \end{aligned}$$

which implies

$$m\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{g-1}\right)=S^*(r,f). \tag{7}$$

Let

$$\varphi=\frac{f-g}{f(g-1)}. \tag{8}$$

Then we have $N(r,\varphi)=0$. By (7), we have $m(r,\varphi)=S^*(r,f)$. Therefore,

$$T(r,\varphi)=S^*(r,f). \tag{9}$$

It is obvious that $\varphi\neq 0,-1$ and $g=(1+\varphi)f/(1+\varphi f)$. Therefore, $N(r,1/(\varphi f+1))=S^*(r,f)$. Let

$$\psi=\frac{\varphi' f+\varphi f'}{\varphi f+1}. \tag{10}$$

By the lemma of logarithmic derivative, we have $m(r,\psi)=S^*(r,f)$. Thus, $T(r,\psi)=S^*(r,f)$. Let

$$\gamma=\frac{1}{e^{\alpha-\beta}-1}. \tag{11}$$

It follows from (6) that $T(r,\gamma)=S^*(r,f)$. By (5) and (11), we deduce that

$$f=\gamma+1-\gamma e^{\alpha}. \tag{12}$$

Therefore,

$$\begin{aligned} f' &= \gamma'-(\gamma'+\gamma\alpha')e^{\alpha} \\ &= \gamma'-(\gamma'+\gamma\alpha')\left(\frac{\gamma+1-f}{\gamma}\right) \\ &= \gamma'-\left(\frac{\gamma'}{\gamma}+\alpha'\right)(\gamma+1-f). \end{aligned}$$

This and (10) imply

$$\psi\varphi f+\psi=\left(\varphi'+\varphi\left(\frac{\gamma'}{\gamma}+\alpha'\right)\right)f-\varphi\left(\frac{\gamma'}{\gamma}+\alpha'\gamma+\alpha'\right). \tag{13}$$

If $\psi\varphi\neq\varphi'+\varphi\left(\frac{\gamma'}{\gamma}+\alpha'\right)$, then the above equation implies $T(r,f)=S^*(r,f)$, which is impossible. Suppose $\psi\varphi=\varphi'+\varphi\left(\frac{\gamma'}{\gamma}+\alpha'\right)$. We get

$$\psi=\frac{\varphi'}{\varphi}+\frac{\gamma'}{\gamma}+\alpha'.$$

Combining this and (10), we have

$$\frac{\varphi' f + \varphi f'}{\varphi f + 1} = \frac{\varphi'}{\varphi} + \frac{\gamma'}{\gamma} + \alpha'.$$

By integration, we get

$$\varphi f + 1 = c(\varphi\gamma\alpha),$$

where c is a nonzero constant. This also implies $T(r, f) = S^*(r, f)$, a contradiction. Case 1 has been ruled out.

Case 2. There exist two nonzero integers s and t such that $(e^{-\beta})^s = (e^{\alpha-\beta})^t$.

Without loss of generality, we assume $s > 0$. Let d be the maximal common factor of s and t. Thus there exist two integers u and v such that $us + vt = d$. Let $h = (e^{-\beta})^{v/d}(e^{\alpha-\beta})^{u/d}$. Then we have $e^{\alpha-\beta} = h^s$ and $e^{-\beta} = h^t$. If $s > d$, then any $e^{2\pi i/s}$ point of h is an 1-point of $e^{\alpha-\beta}$, but not an 1-point of $e^{-\beta}$. Note that any 1-point of $e^{\alpha-\beta}$ must be an 1-point of $e^{-\beta}$. Hence $s = d$. Therefore, $e^{-\beta} = (e^{\alpha-\beta})^n$, where $n = t/d$. If $n > 0$, then it follows from (5) that

$$f = e^{-(n-1)(\alpha-\beta)} \sum_{j=0}^{n-1} e^{(\alpha-\beta)j}, \quad g = \sum_{j=0}^{n-1} e^{(\alpha-\beta)j}.$$

Hence f and g assume the second case in Theorem 1. If $n < 0$, then from (5) we get

$$f = -e^{n(\beta-\alpha)} \sum_{j=0}^{-n-1} e^{(\beta-\alpha)j}, \quad g = -e^{\beta-\alpha} \sum_{j=0}^{-n-1} e^{(\beta-\alpha)j}.$$

Hence f and g assume the third case in Theorem 1. This completes the proof of Theorem 1.

3. Concluding remark

A meromorphic function a ($\not\equiv \infty$) is called a small function with respect to f provided that $T(r, a) = S(r, f)$. Let f and g be two nonconstant meromorphic functions, and let a be a small function with respect to f and g. Denote by $\overline{N}_E(r, f = a = g)$ the reduced counting function of the common a-points of f and g with the same multiplicities. We say that f and g share a in the sense of CM^*, if

$$\overline{N}\left(r, \frac{1}{f - a}\right) - \overline{N}_E(r, f = a = g) = S(r, f),$$

and

$$\overline{N}\left(r, \frac{1}{g - a}\right) - \overline{N}_E(r, f = a = g) = S(r, g).$$

By the arguments similar to that in the proof of Theorem 1, we can proof the following result.

THEOREM 2. *Suppose that f and g are two distinct nonconstant meromorphic functions satisfying $\overline{N}(r,f) = S(r,f)$ and $\overline{N}(r,g) = S(r,g)$. If f and g share the values* 0 *and* 1 *CM^*, then they assume one of the following cases*:

(i) $f = c(1-h),\ g = (1-c)(1-1/h)$;

(ii) $f = h^{-n}\sum_{j=0}^{n} h^j,\ g = \sum_{j=0}^{n} h^j,\ n = 1, 2, \ldots;$

(iii) $f = -h^{-(n+1)}\sum_{j=0}^{n} h^j,\ g = -h\sum_{j=0}^{n} h^j,\ n = 0, 1, 2, \ldots,$

where h is a nonconstant meromorphic function satisfying

$$\overline{N}(r,h) + \overline{N}\left(r, \frac{1}{h}\right) = S(r,h),$$

and c $(\not\equiv 0, 1)$ is a small function of h.

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