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ZETA FUNCTIONS AND NORMALIZED MULTIPLE SINE FUNCTIONS

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Abstract

By using normalized multiple sine functions we show expressions for special values of zeta functions and L-functions containing $\zeta(3)$, $\zeta(5)$, etc. Our result reveals the importance of division values of normalized multiple sine functions. Properties of multiple Hurwitz zeta functions are crucial for the proof.

1. Introduction

The normalized sine function

$$S_1(x) = 2\sin(\pi x)$$

has the basic importance in number theory. This is expressed as

$$S_1(x) = \Gamma_1(x)^{-1}\Gamma_1(1-x)^{-1}$$

with the normalized gamma function

$$\Gamma_1(x) = \exp\left(\frac{\partial}{\partial s}\zeta(s,x)\bigg|_{s=0}\right),$$

where

$$\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$$

is the Hurwitz zeta function. In fact, Lerch's formula says that

$$\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}}$$

for the usual gamma function $\Gamma(x)$.

We know that the special value of $S_1(x)$ at a rational number $x \in \mathbf{Q}$ with $0 < x < 1$ is an algebraic integer

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$$\begin{aligned} S_1(x) &= (e^{-\pi i x} - e^{\pi i x})i \\ &= |1 - e^{2\pi i x}|. \end{aligned}$$

(Since $S_1(1/3) = \sqrt{3}$, the factor "2" is needed in $S_1(x)$ to assure the integrality.) This algebraic integer is intimately related to the cyclotomic unit, and at the same time it appears in the socalled the class number formula of Dirichlet

$$L(1,\chi) = -\frac{\tau(\chi)}{N} \log\left(\prod_{k=1}^{N-1} S_1\left(\frac{k}{N}\right)^{\bar{\chi}(k)}\right),$$

where

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

is the Dirichlet L-function for a non-trivial primitive even Dirichlet character χ modulo N and

$$\tau(\chi) = \sum_{k=1}^{N-1} \chi(k) e^{2\pi i k/N}$$

is the Gauss sum. We notice that the Dirichlet's formula is written also as

$$L'(0,\chi) = -\frac{1}{2} \log\left(\prod_{k=1}^{N-1} S_1\left(\frac{k}{N}\right)^{\chi(k)}\right)$$

via the functional equation.

The purpose of this paper is to generalize such a formula to the case of $L(r,\chi)$ for $r \geq 2$ containing the Riemann zeta case $\chi = \mathbf{1}$ by using the normalized multiple sine function $S_r(x)$, which was constructed and studied in the previous paper [KK] (see §2 for a survey). We recall the construction. For $\omega_1, \ldots, \omega_r > 0$ and $x > 0$, the multiple Hurwitz zeta function is defined by Barnes [B] as

$$\zeta_r(s, x; (\omega_1, \ldots, \omega_r)) = \sum_{n_1, \ldots, n_r = 0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}$$

in $\mathrm{Re}(s) > r$. This has the analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function, and it is holomorphic at $s = 0$. Then the normalized multiple gamma function is defined as

$$\Gamma_r(x, (\omega_1, \ldots, \omega_r)) = \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x; (\omega_1, \ldots, \omega_r))\bigg|_{s=0}\right).$$

This is a constant multiple of multiple gamma function $\Gamma_r^B(x; (\omega_1, \ldots, \omega_r))$ of Barnes [B]:

$$\Gamma_r(x;(\omega_1,\dots,\omega_r)) = \Gamma_r^B(x;(\omega_1,\dots,\omega_r))/\rho_r(\omega_1,\dots,\omega_r).$$

Now, the normalized multiple sine function is

$$S_r(x;(\omega_1,\dots,\omega_r)) = \Gamma_r(x;(\omega_1,\dots,\omega_r))^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-x;(\omega_1,\dots,\omega_r))^{(-1)^r}.$$

Hence, by the zeta regularized product (see Manin [M]), we can write

$$\begin{aligned} &S_r(x;(\omega_1,\dots,\omega_r)) \\ &\quad = \prod_{n_1,\dots,n_r=0}^{\infty}(n_1\omega_1+\cdots+n_r\omega_r+x)\left(\prod_{n_1,\dots,n_r=1}^{\infty}(n_1\omega_1+\cdots+n_r\omega_r-x)\right)^{(-1)^{r-1}}. \end{aligned}$$

For example

$$\begin{aligned} S_1(x,\omega) &= \Gamma_1(x,\omega)^{-1}\Gamma_1(\omega-x,\omega)^{-1} \\ &= \prod_{n=0}^{\infty}(n\omega+x)\prod_{n=1}^{\infty}(n\omega-x) \\ &= 2\sin(\pi x/\omega) \end{aligned}$$

since we have

$$\Gamma_1(x,\omega) = (2\pi)^{-1/2}\Gamma(x/\omega)\omega^{x/\omega-1/2}$$

from

$$\zeta_1(s,x,\omega) = \omega^{-s}\zeta(s,x/\omega).$$

To simplify the notation we put $S_r(x) = S_r(x;(1,\dots,1))$, $\Gamma_r(x) = \Gamma_r(x;(1,\dots,1))$ and $\zeta_r(s,x) = \zeta_r(s,x;(1,\dots,1))$. Hence

$$S_r(x) = \Gamma_r(x)^{-1}\Gamma_r(r-x)^{(-1)^r}$$

and

$$\Gamma_r(x) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s,x)\Big|_{s=0}\right).$$

This normalized multiple sine function $S_r(x)$ has good properties similar to the usual sine function $S_1(x) = 2\sin(\pi x)$. We refer to §2 for details. For example, it has the periodicity and the duplication formula:

$$S_r(x+1) = S_r(x)S_{r-1}(x)^{-1}$$

and

$$S_r(2x) = \prod_{k=0}^{r} S_r\left(x+\frac{k}{2}\right)^{\binom{r}{k}}.$$

Moreover $S_r(x)$ satisfies the following differential equation:

$$\frac{S_r'(x)}{S_r(x)} = Q_r(x)\cot \pi x$$

with $Q_r(x) = (-1)^{r-1}\pi\binom{x-1}{r-1}$. So, $S_r(x)$ is a solution of the algebraic differential equation

$$S_r''(x) + (\pi Q_r(x)^{-1} - 1)S_r'(x)^2 S_r(x)^{-1} - Q_r'(x)Q_r(x)^{-1}S_r'(x) + \pi Q_r(x)S_r(x) = 0.$$

We also note that $S_r(x)$ has the Weierstrass product expression similar to

$$\begin{aligned} S_1(x) &= 2\pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) \\ &= 2\pi x \prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_1H_n}\left(1-\frac{x}{n}\right)^{{}_1H_{-n}}\right). \end{aligned}$$

For example

$$\begin{aligned} S_2(x) &= 2\pi x e^{-x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{n+1}\left(1-\frac{x}{n}\right)^{-n+1}e^{-2x}\right) \\ &= 2\pi x e^{-x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_2H_n}\left(1-\frac{x}{n}\right)^{{}_2H_{-n}}e^{-2x}\right) \end{aligned}$$

and

$$\begin{aligned} S_3(x) &= 2\pi e^{-\zeta'(-2)} x e^{x^2/4-(3/2)x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{n^2/2+3n/2+1}\left(1-\frac{x}{n}\right)^{n^2/2-3n/2+1}e^{x^2/2-3x}\right) \\ &= 2\pi e^{-\zeta'(-2)} x e^{x^2/4-(3/2)x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_3H_n}\left(1-\frac{x}{n}\right)^{{}_3H_{-n}}e^{x^2/2-3x}\right) \end{aligned}$$

(see §2).

Our main results are as follows. The first result expresses the values of the Riemann zeta function at positive odd integers.

Theorem 1.1. *Let* $n = 1, 2, 3, \ldots,$ *and for* $k = 1, 2, \ldots, n$ *put*

$$a(2n+1, k) = \sum_{l=1}^{k}(-1)^{k-l} l^{2n}\binom{2n+1}{k-l},$$

which is a positive integer. Then we have:

(1)

$$\zeta'(-2n) = -\log\left(\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}\right).$$

(2)
$$\zeta(2n+1) = \frac{(-1)^{n-1}2^{2n+1}\pi^{2n}}{(2n)!} \log\left(\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}\right).$$

Example 1.2. We have

$$\zeta(3) = 4\pi^2 \log S_3(1), \tag{1.1}$$

$$\zeta(5) = -\frac{4\pi^4}{3} \log(S_5(1)S_5(2)^{11}), \tag{1.2}$$

$$\zeta(7) = \frac{8\pi^6}{45} \log(S_7(1)S_7(2)^{57}S_7(3)^{302}). \tag{1.3}$$

The above formula (1.1) was proved in [KK] previously.

Remark 1.3. By the formula

$$S_r(k) = \prod_{l=0}^{k-1} S_{r-l}(1)^{\binom{k-1}{l}(-1)^l} \tag{1.4}$$

for $1 \le k < r$ (cf. §2), we can also express $\zeta(2n+1)$ in terms of $S_l(1)$ $(2 \le l \le 2n+1)$:

$$\zeta(2n+1) = \frac{(-1)^{n-1}2^{2n+1}\pi^{2n}}{(2n)!} \log\left(\prod_{l=2}^{2n+1} S_l(1)^{b(2n+1,l)}\right)$$

with $b(2n+1,l) \in \mathbf{Z}$.

Example 1.4. Since $S_5(2) = S_5(1)S_4(1)^{-1}$ (see §2),

$$\zeta(5) = -\frac{4\pi^4}{3} \log(S_5(1)^{12}S_4(1)^{-11}).$$

Next, let χ be a non-trivial primitive Dirichlet character modulo N, and

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} \tag{1.5}$$

the Dirichlet L-function. Then the values $L(r,\chi)$ for $r = 1,2,3,\dots$ are classified as

$$L(r,\chi) = \begin{cases} \pi^r \cdot (\chi\text{-Bernoulli number}) & \cdots \quad \chi(-1) = (-1)^r \\ \text{“difficult”} & \cdots \quad \chi(-1) = (-1)^{r+1}. \end{cases}$$

Here “difficult” means that these values have not been calculated explicitly yet except for the $r = 1$ case appearing in the Dirichlet’s class number formula.

We generalize Dirichlet's result to some difficult case.

THEOREM 1.5. *Let χ be a primitive odd character modulo N. Then*:
(1)

$$L'(-1,\chi) = -\frac{1}{2}\log\prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k\right)^{\chi(k)}.$$

(2)

$$L(2,\chi) = \frac{2\pi i\tau(\chi)}{N^2}\log\prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k\right)^{\bar{\chi}(k)}.$$

Example 1.6. We have

$$\begin{aligned} L\left(2,\left(\frac{-4}{*}\right)\right) &= -\frac{\pi}{4}\log\left(S_2\left(\frac{1}{4}\right)^4 S_1\left(\frac{1}{4}\right)S_2\left(\frac{3}{4}\right)^{-4}S_1\left(\frac{3}{4}\right)^{-3}\right) \\ &= \frac{\pi}{4}\log\left(2^3 S_2\left(\frac{1}{4}\right)^{-8}\right), \\ L\left(2,\left(\frac{-3}{*}\right)\right) &= -\frac{2\sqrt{3}\pi}{9}\log\left(S_2\left(\frac{1}{3}\right)^3 S_1\left(\frac{1}{3}\right)S_2\left(\frac{2}{3}\right)^{-3}S_1\left(\frac{2}{3}\right)^{-2}\right) \\ &= \frac{4\sqrt{3}\pi}{9}\log\left(3S_2\left(\frac{1}{3}\right)^{-3}\right), \end{aligned}$$

where we used the following relations (see §2):

$$\begin{aligned} S_2(1-x) &= S_2(1+x)^{-1} \\ &= (S_2(x)S_1(x)^{-1})^{-1} \\ &= S_2(x)^{-1}S_1(x). \end{aligned}$$

THEOREM 1.7. *Let χ be a non-trivial primitive even character modulo N. Then*:
(1)

$$L'(-2,\chi) = -\frac{1}{2}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\chi(k)}.$$

(2)

$$L(3,\chi) = \frac{2\pi^2\tau(\chi)}{N^3}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\bar{\chi}(k)}.$$

Example 1.8.

$$L\left(3, \left(\frac{12}{*}\right)\right) = \frac{\sqrt{3}\pi^2}{432} \log\left(S_3\left(\frac{1}{12}\right)^{288} S_2\left(\frac{1}{12}\right)^{-408} S_1\left(\frac{1}{12}\right) S_3\left(\frac{5}{12}\right)^{-288}\right.$$
$$\times S_2\left(\frac{5}{12}\right)^{312} S_1\left(\frac{5}{12}\right)^{-25} S_3\left(\frac{7}{12}\right)^{-288} S_2\left(\frac{7}{12}\right)^{264} S_1\left(\frac{7}{12}\right)^{-49}$$
$$\left.\times S_3\left(\frac{11}{12}\right)^{288} S_2\left(\frac{11}{12}\right)^{-164} S_1\left(\frac{11}{12}\right)^{121}\right).$$

Thus the values $S_r(a)$ for $a \in \mathbf{Q}$ satisfying $0 < a < r$ are quite interesting in relation to zeta values. We formulate our expectation as

Expectation 1.9. *$S_r(a) \in \bar{\mathbf{Q}}$ for $a \in \mathbf{Q}$ satisfying $0 < a < r$.*

The situation would become transparent when we generalize it as below:

Expectation 1.10. *$S_r\left(\frac{k_1\omega_1 + \cdots + k_r\omega_r}{N}; \underline{\omega}\right) \in \bar{\mathbf{Q}}$ for $N = 1, 2, 3, \ldots$ and $k_i = 0, 1, \ldots, N-1$.*

It is easy to see that Expectation 1.9 is contained in Expectation 1.10 for $\underline{\omega} = (1, \ldots, 1)$, and Expectation 1.10 clearly indicates that we are studying division values of multiple sine functions.

We note that Shintani [Sh] deeply studied $S_2(x, (1, \varepsilon))$ for a fundamental unit ε of a real quadratic field. In particular, he showed its appearance in the expression for a special value of a suitable L-function, and he obtained certain algebraicity such as

$$S_2\left(\frac{1}{3}, (1, \varepsilon)\right) S_2\left(1 + \frac{\varepsilon}{3}, (1, \varepsilon)\right) S_2\left(\frac{2 + 2\varepsilon}{3}, (1, \varepsilon)\right) = \sqrt{\frac{\frac{1+\sqrt{21}}{2} - \sqrt{\frac{3+\sqrt{21}}{2}}}{2}}$$

for $\varepsilon = \dfrac{5 + \sqrt{21}}{2}$, which is the fundamental unit of $\mathbf{Q}(\sqrt{21})$. Moreover, Shintani studied Kronecker's Jugendtraum for a real quadratic field by using $S_2(x, (1, \varepsilon))$ (he denoted it by $F(x; (1, \varepsilon))^{-1}$). It might be valuable to report the following general product formula

$$\prod_{\substack{k_1, \ldots, k_r = 0 \\ (k_1, \ldots, k_r) \neq (0, \ldots, 0)}}^{N-1} S_r\left(\frac{k_1\omega_1 + \cdots + k_r\omega_r}{N}; (\omega_1, \ldots, \omega_r)\right) = N$$

for an integer $N \geq 2$. (See §2.)

THEOREM 1.11. (1) *Expectations* 1.9 *and* 1.10 *are valid for* $r = 1$.
(2) *Expectations* 1.9 *and* 1.10 *are valid for* $r = 2$ *with* $N = 2$. *Actually*

$$S_2\left(\frac{\omega_1}{2};\underline{\omega}\right) = S_2\left(\frac{\omega_2}{2};\underline{\omega}\right) = \sqrt{2}$$

and

$$S_2\left(\frac{\omega_1+\omega_2}{2};\underline{\omega}\right) = 1.$$

Remark 1.12. This paper was referred to in [KK] as a preprint in 2001.

2. Multiple sine functions

The basic properties of multiple sine functions were proved in [K] and [KK]. Here we recall some of them.

THEOREM 2.1 [KK, Theorem 2.1]. *The multiple sine function* $S_r(x,\underline{\omega})$ *satisfies the following identities*:

(a) *For* $\underline{\omega} = (\omega_1,\dots,\omega_r) \in \mathbf{R}_+^r$ *put* $\underline{\omega}(i) = (\omega_1,\dots,\omega_{i-1},\omega_{i+1},\dots,\omega_r) \in \mathbf{R}_+^{r-1}$, *then we have*

$$S_r(x+\omega_i,\underline{\omega}) = S_r(x,\underline{\omega})S_{r-1}(x,\underline{\omega}(i))^{-1}, \tag{2.1}$$

where we put $S_0(x,\cdot) \equiv -1$.

(b) *For a positive integer* N, *we have*

$$S_r(Nx,\underline{\omega}) = \prod_{0\le k_i\le N-1} S_r\left(x+\frac{\mathbf{k}\cdot\omega}{N},\underline{\omega}\right), \tag{2.2}$$

where the product is taken over the vectors $\mathbf{k} = (k_1,\dots,k_r)$.

(c)

$$\prod_{\substack{0\le k_i\le N-1\\ \mathbf{k}\ne\mathbf{0}}} S_r\left(\frac{\mathbf{k}\cdot\omega}{N},\underline{\omega}\right) = N.$$

(d)

$$S_r(0,\underline{\omega}) = 0.$$

(e) *We have for any* $c > 0$ *the homogeneity*

$$S_r(cx,c\underline{\omega}) = S_r(x,\underline{\omega}).$$

THEOREM 2.2. (a) *For* $r \ge 2$ *we have*

$$S_r(x+1) = S_r(x)S_{r-1}(x)^{-1}.$$

(b)

$$S_r(2x) = \prod_{k=0}^{r} S_r\left(x + \frac{k}{2}\right)^{\binom{r}{k}}.$$

(c) *Put* $Q_r(x) = (-1)^{r-1}\pi\binom{x-1}{r-1}$, *then*

$$\frac{S_r'(x)}{S_r(x)} = Q_r(x)\cot(\pi x).$$

(d)

$$S_r''(x) + (\pi Q_r(x)^{-1} - 1)S_r'(x)^2 S_r(x)^{-1} - Q_r'(x)Q_r(x)^{-1}S_r'(x) + \pi Q_r(x)S_r(x) = 0.$$

(e)

$$\begin{aligned} S_2(x) &= 2\pi x e^{-x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{n+1}\left(1-\frac{x}{n}\right)^{-n+1}e^{-2x}\right) \\ &= 2\pi x e^{-x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_2H_n}\left(1-\frac{x}{n}\right)^{{}_2H_{-n}}e^{-2x}\right). \end{aligned}$$

(f)

$$\begin{aligned} S_3(x) &= 2\pi e^{-\zeta'(-2)} x e^{x^2/4-(3/2)x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{n^2/2+3n/2+1}\left(1-\frac{x}{n}\right)^{n^2/2-3n/2+1}e^{x^2/2-3x}\right) \\ &= 2\pi e^{-\zeta'(-2)} x e^{x^2/4-(3/2)x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_3H_n}\left(1-\frac{x}{n}\right)^{{}_3H_{-n}}e^{x^2/2-3x}\right). \end{aligned}$$

Proof. The assertions (a) and (b) are immediate consequences from [KK, Theorem 2.1]. The differential equation (c) is proved in [KK, Theorem 2.15]. We compute from (c) that

$$\begin{aligned} \left(Q_r(x)^{-1}\frac{S_r'}{S_r}(x)\right)' &= (\cot \pi x)' \\ &= -\frac{\pi}{\sin^2 \pi x} \\ &= -\pi(\cot^2(\pi x) + 1) \\ &= -\pi\left(\left(Q_r(x)^{-1}\frac{S_r'}{S_r}(x)\right)^2 + 1\right), \end{aligned}$$

which gives the proof of (d). Finally (e) and (f) are deduced from [KK, Examples 3.6], where we express the normalized multiple sine functions $S_r(x)$ in terms of primitive multiple sine functions which are defined by the Hadamard product. ■

3. The Riemann zeta function

LEMMA 3.1. *There exist uniquely determined integers* $a(r,k)$ *such that*

$$x^{r-1} = \sum_{k=1}^{r-1} a(r,k)_r H_{x-k} \tag{3.1}$$

with ${}_rH_{x-k} = \dfrac{(x-k+r-1)\cdots(x-k+1)}{(r-1)!}$ *for an indeterminate* x. *Indeed* $a(r,k)$ *are given as follows*:

$$a(r,k) = \sum_{l=1}^{k} (-1)^{k-l} l^{r-1} \binom{r}{k-l}. \tag{3.2}$$

Moreover,

$$a(r,r-k) = a(r,k). \tag{3.3}$$

Proof. The existence of $a(r,k)$ follows from the fact that the $(r-1)$ polynomials ${}_rH_{x-k}$ $(k = 1,\dots,r-1)$ are linearly independent over $\mathbf{Q}$. By putting $x = k$ in (3.1), we have

$$k^{r-1} = a(r,1)\binom{k+r-2}{r-1} + a(r,2)\binom{k+r-3}{r-1} + \cdots + a(r,k)\cdot 1.$$

This leads to

$$a(r,k) = k^{r-1} - \sum_{j=1}^{k-1} a(r,j)\binom{k+r-1-j}{r-1}.$$

Thus (3.2) is proved by induction on k. Next, from (3.1)

$$(-x)^{r-1} = \sum_{k=1}^{r-1} a(r,k)_r H_{-x-k}$$

and

$${}_rH_{-x-k} = \frac{(-x-k+r-1)\cdots(-x-k+1)}{(r-1)!} = (-1)^{r-1}{}_rH_{x-(r-k)},$$

so

$$x^{r-1} = \sum_{k=1}^{r-1} a(r,k)_r H_{x-(r-k)} = \sum_{k=1}^{r-1} a(r,r-k)_r H_{x-k}.$$

Hence, by the uniquenes of $a(r,k)$ we have $a(r,r-k) = a(r,k)$. ■

Example 3.2. For $x = n \in \mathbf{Z}$ and $r = 2,3,4,5$ we have

$$\begin{aligned}
n &= {}_2H_{n-1}, \\
n^2 &= {}_3H_{n-1} + {}_3H_{n-2}, \\
n^3 &= {}_4H_{n-1} + 4{}_4H_{n-2} + {}_4H_{n-3}, \\
n^4 &= {}_5H_{n-1} + 11{}_5H_{n-2} + 11{}_5H_{n-3} + {}_5H_{n-4}.
\end{aligned}$$

Proof of Theorem 1.1. For $r \geq 2$ we have by Lemma 3.1

$$\begin{aligned}
\zeta(s+1-r) &= \sum_{n=1}^{\infty} \frac{n^{r-1}}{n^s} \\
&= \sum_{k=1}^{r-1} a(r,k) \sum_{n=1}^{\infty} \frac{{}_rH_{n-k}}{n^s} \\
&= \sum_{k=1}^{r-1} a(r,k) \zeta_r(s,k),
\end{aligned}$$

where $\zeta_r(s,k)$ is the multiple Hurwitz zeta function

$$\zeta_r(s,k) = \sum_{n=0}^{\infty} \frac{{}_rH_n}{(n+k)^s}.$$

Thus we have

$$\zeta'(1-r) = \sum_{k=1}^{r-1} a(r,k) \log \Gamma_r(k).$$

In case $r = 2n+1$, it follows that

$$\begin{aligned}
\zeta'(-2n) &= \sum_{k=1}^{2n} a(2n+1,k) \log \Gamma_{2n+1}(k) \\
&= -\sum_{k=1}^{n} a(2n+1,k) \log S_{2n+1}(k) \\
&= -\log\left(\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}\right),
\end{aligned}$$

where we used $S_{2n+1}(k) = \Gamma_{2n+1}(k)^{-1}\Gamma_{2n+1}(2n+1-k)^{-1}$ and $a(2n+1, 2n+1-k) = a(2n+1, k)$. ■

Example 3.3. We saw in [KK, Theorem 3.8(c)] that

$$\zeta(3) = 4\pi^2 \log S_3(1).$$

Combining this with the fact that

$$S_3(1) = \sqrt{2} S_3\left(\frac{1}{2}\right)^{-4/3},$$

which can be obtained by the facts

$$\begin{aligned} S_3(1) &= S_3\left(2 \cdot \frac{1}{2}\right) \\ &= S_3\left(\frac{1}{2}\right) S_3(1)^3 S_3\left(\frac{3}{2}\right)^3 S_3(2) \\ &= S_3(1)^4 S_3\left(\frac{1}{2}\right)^4 S_2\left(\frac{1}{2}\right)^{-3} \end{aligned}$$

and that $S_2\left(\frac{1}{2}\right) = \sqrt{2}$, we have

$$\zeta(3) = \frac{16\pi^2}{3} \log\left(S_3\left(\frac{1}{2}\right)^{-1} 2^{3/8}\right)$$

which was proved in [KK, Theorem 3.8(b)] by another method (using a "primitive multiple sine function").

4. Dirichlet L-functions for odd characters

We prove the formula for $L(2, \chi)$ for odd characters. Since our method follows a proof for Dirichlet's result on $L(1, \chi)$ for even characters, we first recall it. We show the formula for $L'(0, \chi)$. Then the result on $L(1, \chi)$ follows via the functional equation.

Let χ be a non-trivial primitive Dirichlet character modulo N. We have

$$\begin{aligned} L(s, \chi) &= \sum_{k=1}^{N-1} \chi(k) \sum_{m=0}^{\infty} \frac{1}{(mN+k)^s} \\ &= N^{-s} \sum_{k=1}^{N-1} \chi(k) \zeta\left(s, \frac{k}{N}\right), \end{aligned}$$

where

$$\zeta(s,x) = \sum_{m=0}^{\infty} \frac{1}{(m+x)^s}$$

is the Hurwitz zeta function. Hence

$$L(0,\chi) = \sum_{k=1}^{N-1} \chi(k)\zeta\left(0,\frac{k}{N}\right)$$

and

$$\begin{aligned} L'(0,\chi) &= \sum_{k=1}^{N-1} \chi(k)\zeta'\left(0,\frac{k}{N}\right) - (\log N)\sum_{k=1}^{N-1} \chi(k)\zeta\left(0,\frac{k}{N}\right) \\ &= \sum_{k=1}^{N-1} \chi(k)\zeta'\left(0,\frac{k}{N}\right) - (\log N)L(0,\chi). \end{aligned}$$

When χ is even, it holds that $L(0,\chi) = 0$ (this is the reason of "difficult"), so we have

$$\begin{aligned} L'(0,\chi) &= \sum_{k=1}^{N-1} \chi(k)\zeta'\left(0,\frac{k}{N}\right) \\ &= \sum_{k=1}^{N-1} \chi(k) \log \Gamma_1\left(\frac{k}{N}\right) \\ &= \frac{1}{2}\sum_{k=1}^{N-1} \chi(k)\left(\log \Gamma_1\left(\frac{k}{N}\right) + \log \Gamma_1\left(\frac{N-k}{N}\right)\right) \\ &= -\frac{1}{2}\sum_{k=1}^{N-1} \chi(k) \log S_1\left(\frac{k}{N}\right). \end{aligned}$$

This gives the Dirichlet's result.

Proof of Theorem 1.5. We prove (1), then (2) is obtained via the functional equation. Since

$$\begin{aligned} \zeta(s-1,x) &= \sum_{n=0}^{\infty} \frac{n+x}{(n+x)^s} = \sum_{n=0}^{\infty} \frac{n+1}{(n+x)^s} + (x-1)\sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \\ &= \zeta_2(s,x) + (x-1)\zeta_1(s,x), \end{aligned}$$

we have

$$\zeta'(-1,x) = \zeta_2'(0,x) + (x-1)\zeta_1'(0,x),$$

as $\zeta_1(s,x) = \zeta(s,x)$. Now that χ is odd and that $L(-1,\chi) = 0$, we compute

$$\begin{aligned}
L'(-1,\chi) &= N \sum_{k=1}^{N-1} \chi(k)\zeta'\left(-1, \frac{k}{N}\right) \\
&= N \sum_{k=1}^{N-1} \chi(k)\zeta_2'\left(0, \frac{k}{N}\right) + N \sum_{k=1}^{N-1} \chi(k)\left(\frac{k}{N} - 1\right)\zeta_1'\left(0, \frac{k}{N}\right) \\
&= N \sum_{k=1}^{N-1} \chi(k) \log \Gamma_2\left(\frac{k}{N}\right) + N \sum_{k=1}^{N-1} \chi(k)\left(\frac{k}{N} - 1\right) \log \Gamma_1'\left(\frac{k}{N}\right) \\
&= N \sum_{k=1}^{N-1} \chi(k) \log\left(\Gamma_2\left(\frac{k}{N}\right)\Gamma_1\left(\frac{k}{N}\right)^{k/N-1}\right) \\
&= \frac{N}{2} \sum_{k=1}^{N-1} \chi(k) \log\left(\frac{\Gamma_2\left(\frac{k}{N}\right)}{\Gamma_2\left(1 - \frac{k}{N}\right)} \frac{\Gamma_1\left(\frac{k}{N}\right)^{k/N-1}}{\Gamma_1\left(1 - \frac{k}{N}\right)^{-k/N}}\right) \\
&= \frac{N}{2} \sum_{k=1}^{N-1} \chi(k) \log\left(\frac{\Gamma_2\left(\frac{k}{N}\right)}{\Gamma_2\left(2 - \frac{k}{N}\right)} \left(\Gamma_1\left(\frac{k}{N}\right)\Gamma_1\left(1 - \frac{k}{N}\right)\right)^{k/N-1}\right) \\
&= -\frac{N}{2} \sum_{k=1}^{N-1} \chi(k) \log\left(S_2\left(\frac{k}{N}\right)S_1\left(\frac{k}{N}\right)^{k/N-1}\right) \\
&= -\frac{N}{2} \sum_{k=1}^{N-1} \chi(k) \log\left(S_2\left(\frac{k}{N}\right)S_1\left(\frac{k}{N}\right)^{k/N}\right),
\end{aligned}$$

where we used the fact $S_1\left(\frac{k}{N}\right) = S_1\left(\frac{N-k}{N}\right)$ with $\chi(N-k) = -\chi(k)$. ∎

5. Dirichlet L-functions for even characters

Proof of Theorem 1.7. We again show (1), then (2) is obtained via the functional equation. Since

$$(n+x)^2 = 2{}_3H_n + (2x-3){}_2H_n + (x-1)^2{}_1H_n,$$

we have

$$\zeta(s-2,x) = \sum_{n=0}^{\infty} \frac{(n+x)^2}{(n+x)^s} = 2\zeta_3(s,x) + (2x-3)\zeta_2(s,x) + (x-1)^2\zeta_1(s,x).$$

Therefore we have

$$\zeta'(-2,x) = 2\zeta_3'(0,x) + (2x-3)\zeta_2'(0,x) + (x-1)^2\zeta_1'(0,x).$$

Now that χ is even and that $L(-2,\chi) = 0$, we compute

$$\begin{aligned}
L'(-2,\chi) &= N^2 \sum_{k=1}^{N-1} \chi(k)\zeta'\left(-2, \frac{k}{N}\right) \\
&= N^2 \sum_{k=1}^{N-1} \chi(k)\left(2\zeta_3'\left(0,\frac{k}{N}\right) + \left(2\frac{k}{N}-3\right)\zeta_2'\left(0,\frac{k}{N}\right) + \left(\frac{k}{N}-1\right)^2\zeta_1'\left(0,\frac{k}{N}\right)\right) \\
&= N^2 \sum_{k=1}^{N-1} \chi(k)\log\left(\Gamma_3\left(\frac{k}{N}\right)^2 \Gamma_2\left(\frac{k}{N}\right)^{2k/N-3} \Gamma_1\left(\frac{k}{N}\right)^{(k/N-1)^2}\right) \\
&= -\frac{1}{2}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\chi(k)}. \qquad \blacksquare
\end{aligned}$$

6. Division values of normalized multiple sines

Proof of Theorem 1.11. Since $S_1(x,\omega) = 2\sin\left(\frac{\pi x}{\omega}\right)$ by [KK, §2], we have

$$S_1\left(\frac{k\omega}{N},\omega\right) = 2\sin\left(\frac{k\pi}{N}\right) = -i(e^{i\pi k/N} - e^{-i\pi k/N}) \in \bar{\mathbf{Q}},$$

which leads to (1).

Recall that

$$S_2(x,(\omega_1,\omega_2)) = \frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))}.$$

First

$$S_2\left(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2)\right) = \frac{\Gamma_2\left(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2)\right)}{\Gamma_2\left(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2)\right)} = 1.$$

Secondly

$$S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right) = \frac{\Gamma_2\left(\frac{\omega_1}{2}+\omega_2,(\omega_1,\omega_2)\right)}{\Gamma_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right)}.$$

Here we use ([KK, §2])

$$\Gamma_2(x+\omega_2,(\omega_1,\omega_2)) = \Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_1)^{-1}.$$

Then

$$\Gamma_2\left(\frac{\omega_1}{2}+\omega_2,(\omega_1,\omega_2)\right) = \Gamma_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right)\Gamma_1\left(\frac{\omega_1}{2},\omega_1\right)^{-1}.$$

Hence

$$S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right) = \Gamma_1\left(\frac{\omega_1}{2},\omega_1\right)^{-1}.$$

Now ([KK, §2])

$$\Gamma_1(x,\omega) = \frac{\Gamma\left(\dfrac{x}{\omega}\right)}{\sqrt{2\pi}}\omega^{x/\omega-1/2},$$

so

$$\Gamma_1\left(\frac{\omega_1}{2},\omega_1\right) = \frac{\Gamma\left(\dfrac{1}{2}\right)}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}}.$$

Thus

$$S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right) = \sqrt{2}. \qquad \blacksquare$$

Remark 6.1. A suitable restriction on the form of division points such as made in Expectation 1.10 will be needed as the following example shows:

$$S_2(2,(1,\sqrt{2})) \notin \bar{\mathbf{Q}}. \tag{6.1}$$

By this example, we must seriously look at $S_r(a_1\omega_1+\cdots+a_r\omega_r;(\omega_1,\dots,\omega_r))$ for general $a_i \in \mathbf{Q}$. The proof of (6.1) is given by

$$\frac{S_2(2,(1,\sqrt{2}))}{S_2(1,(1,\sqrt{2}))} = \frac{S_2(1+1,(1,\sqrt{2}))}{S_2(1,(1,\sqrt{2}))} = S_1(1,\sqrt{2})^{-1} = \left(2\sin\frac{\pi}{\sqrt{2}}\right)^{-1} \notin \bar{\mathbf{Q}},$$

where

$$\begin{aligned}2\sin\frac{\pi}{\sqrt{2}} &= -i(e^{i\pi/\sqrt{2}} - e^{-i\pi/\sqrt{2}})\\ &= -i((-1)^{1/\sqrt{2}} - ((-1)^{1/\sqrt{2}})^{-1})\end{aligned}$$

and we used the transcendency result of Gelfond-Schneider $(-1)^{1/\sqrt{2}} \notin \bar{\mathbf{Q}}$. Moreover we appeal to the following facts:

$$S_2(\omega_1, (\omega_1, \omega_2)) = \sqrt{\frac{\omega_2}{\omega_1}},$$

$$S_2(\omega_2, (\omega_1, \omega_2)) = \sqrt{\frac{\omega_1}{\omega_2}}.$$

In particular

$$S_2(1, (1, \sqrt{2})) = 2^{1/4} \in \bar{\mathbf{Q}}.$$

Thus we obtain (6.1).

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