

A NOTE ON BOUNDARY REGULARITY OF SUBELLIPTIC HARMONIC MAPS*

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Abstract

Let $M^m = \Omega$ be a bounded domain of \mathbf{R}^m . Jost and Xu in [7] introduced the subelliptic harmonic map from it and proved the regularity up to the boundary of subelliptic harmonic maps into small balls supposing that the boundary of the domain is noncharacteristic and smooth. In this note, we investigate the boundary regularity of this kind of maps from Ω with non-smooth and partial noncharacteristic boundary.

1. Introduction, main result

Let M and N be two Riemannian manifolds. A critical point in Sobolev space $H^1(M, N)$ of the energy functional is called a (weakly) harmonic map. Denote a geodesic ball with radius R and center $p \in N$ by $B_R(p)$. $B_R(p)$ is said to be small, if $R < \min\left\{\frac{\pi}{2\sqrt{\kappa}}, i(p)\right\}$ where κ is a positive upper bound of the sectional curvature of N and $i(p)$ is the injective radius of p . For instance, any ball of the Euclidean space is small, and so is any geodesic ball with radius less than $\pi/2$ in unit spheres. Hildebrandt et al. proved that harmonic maps into a small ball are regular up to the boundary (see [6]).

Let $M^m = \Omega$ be a bounded domain of \mathbf{R}^m . Jost and Xu in [7] introduced the subelliptic harmonic map (see the next section), which is a variant of the classical harmonic map, and proved the regularity up to the boundary of subelliptic harmonic maps into small balls supposing that the boundary of the domain is noncharacteristic and smooth. In this note, we investigate the boundary regularity of this kind of maps from Ω with non-smooth and partial noncharacteristic boundary.

We call a vector ν a outward normal to $\partial\Omega$ at $x \in \partial\Omega$, if there is an Euclidean ball B with center x_0 , such that $\bar{B} \cap \bar{\Omega} = \{x\}$ and $\nu = k(x_0 - x)$ for some positive number k . Apparently, if x is a smooth point of the boundary, ν is the usual outward normal vector.

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Let $X = \{X_1, \dots, X_{k_0}\}$ be a family of Hörmander’s vectors, and Δ_X be the subLaplacian defined by X (see next section). Write the terms of the second order of $-\Delta_X$ by $\sum_{i,j=1}^m a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$. Let $x \in \partial\Omega$. If there exists a outward normal vector $v = (v^1, \dots, v_m)$ at x to $\partial\Omega$ such that $\sum_{i,j=1}^m a_{ij} v^i v^j > 0$, x is called a non-characteristic point of $\partial\Omega$ with respect to X . The set of all noncharacteristic points is denoted by S_1 .

Our main theorem is

THEOREM. *Suppose that $u : \Omega \rightarrow B_R(p) \subset N$ is subelliptic harmonic, where $B_R(p)$ is a small ball. If $u = \phi$ on $\partial\Omega$ where $\phi \in C^2(\Omega \cup S_1, B_R(p))$, then u is continuous to S_1 .*

The plan of this paper is: in Section 2, we collect some useful definitions and facts; in Section 3, we prove the main theorem. In the last two sections, we proof two lemmas which are used in the proof of the main theorem.

2. Preliminaries

Let Ω be an open domain in \mathbf{R}^m , $X = (X_1, X_2, \dots, X_{k_0})$ be a family of C^∞ vector fields on Ω . We call X_α to be Hömander’s vector fields if they satisfy the following Hörmander’s condition:

X_α, \dots, X_{k_0} together with their commutators up to a fixed length r span the tangent space at each point of Ω .

For any collection of vector fields $X = (X_\alpha, \alpha = 1, \dots, k_0)$, we define an associated operator of second order as follows:

$$(1) \quad \Delta_X := \sum X_\alpha^* X_\alpha$$

where, X_α^* is the adjoint operator of X_α with respect to the Lebesgue’s measure. If X satisfies Hörmander’s condition, the operator Δ_X is hypoelliptic, i.e. if $\Delta_X u$ is smooth, then so is u . In this case, the operator is called Hömander’s operator, or called a subelliptic Laplacian or a sub-Laplacian. In general, this is a de-generated elliptic operator.

Let N be a Riemannian manifold with Riemannian metric g . Without loss of generality, we assume that N is a submanifold of \mathbf{R}^v by Nash’s embedding theorem. Define

$$(2) \quad M^{1,p}(\Omega, \mathbf{R}^v) := \{u : \Omega \rightarrow \mathbf{R}^v \mid u \in L^p; X_\alpha u \in L^p, \alpha = 1, \dots, k_0\}$$

and define $M_0^{1,p}(\Omega, \mathbf{R}^v)$ to be the closure of $C_0^\infty(\Omega, \mathbf{R}^v)$ in $M^{1,p}(\Omega, \mathbf{R}^v)$ with respect to the norm

$$(3) \quad \|u\|_{M^{1,p}}^p := \int_\Omega |u|^p \, dx + \int_\Omega \sum |X_\alpha u|^p \, dx.$$

$$(4) \quad M^1(\Omega, N) := \{u \in M^{1,2}(\Omega, \mathbf{R}^v) : u(\Omega) \subset N \text{ a.e.}\}.$$

$$(5) \quad M_0^1(\Omega, N) := \{u \in M_0^{1,2}(\Omega, \mathbf{R}^v) : u(\Omega) \subset N \text{ a.e.}\}.$$

Let u be in $M^1(\Omega, N)$ whose coordinate representation is also denoted by u . Define the energy density of u by

$$(6) \quad e(u)(x) = \frac{1}{2} \sum g_{ij}(u(x)) X_\alpha u^i(x) X_\alpha u^j(x)$$

and the energy by $E(u) = \int_\Omega e(u) \, dx$.

A critical point of the energy functional is called a subelliptic harmonic map. The Euler–Lagrange equations of the energy functional reads (see [11])

$$(7) \quad \tau(u) =: \sum X_\alpha^* X_\alpha u^l - \sum \Gamma_{ik}^l X_\alpha u^i X_\alpha u^k = 0.$$

When N is the Euclidean line, the above system is linear, and its solutions are called subelliptic functions. We have

LEMMA 1. *Suppose that $\Omega \subset \mathbf{R}^n$ is a bounded domain. If $g \in M^1 \cap L^\infty(\Omega, \mathbf{R})$ satisfies*

$$(8) \quad \begin{aligned} \Delta_X g &\leq 0 \quad \text{in } \Omega \\ g &\leq 0 \quad \text{on } \partial\Omega \end{aligned}$$

Then, $g \leq 0$ in $\bar{\Omega}$.

Proof. Let $g^+ = \max(g, 0)$. Then by hypothesis, $g^+ \in M_0^1(\Omega, \mathbf{R})$. So we can use it as a test function and obtain

$$(9) \quad \begin{aligned} \int_\Omega \Delta_X g \cdot g^+ \, dx &= \int_\Omega X_\alpha g^+ X_\alpha g^+ \, dx \\ &\leq 0. \end{aligned}$$

Hence we have $X_\alpha g^+ = 0$ on Ω for any $\alpha = 1, 2, \dots, k_0$. Hence $[X_\alpha, X_\beta]g^+ = (X_\alpha X_\beta - X_\beta X_\alpha)g^+ = 0$, $[[X_\alpha, X_\beta], X_\gamma]g^+ = ([X_\alpha, X_\beta]X_\gamma - X_\gamma[X_\alpha, X_\beta])g^+ = 0, \dots$. By Hörmander’s condition we have $Xg^+ = 0$ for any tangent vector field X on Ω . Therefore $g^+ = \text{const.}$ whence $g^+ = 0$ because $g^+|_{\partial\Omega} = 0$. Q.E.D.

3. Proof of the main theorem

Let $x_0 \in \partial\Omega$. Denote f be a convex function over $\overline{B_R(p)} \subset N$ with unique minimum 0 at $u(x_0) \in B_R(p)$. Such a function exists if $B_R(p)$ is a small ball. For example, we can take

$$f(x) = \frac{1 - \cos(\sqrt{\kappa} \, \text{dist}(u(x_0), x))}{\kappa \cos(\sqrt{\kappa} \, \text{dist}(p, x))}.$$

The proof of the convexity of f is arranged in the last section (see Lemma 3). Note that $\text{dist}^2(p, x)$ is convex in regular ball $\overline{B_R(p)}$, but $u(x_0)$ is not its minimum, and that $\text{dist}(u(x_0), x)$ has unique minimum 0 at $u(x_0)$, but it may not be convex in small ball $\overline{B_R(p)}$ because $\text{dist}(u(x_0), x)$ may be bigger than $\frac{\pi}{2\sqrt{\kappa}}$. If we let κ tend to zero, then $f(x)$ tend to $\frac{1}{2}(\text{dist}(u(x_0), x))^2$. Hence we call this function the modified distance. Let g be a bounded solution of the following problem:

$$(10) \quad \begin{aligned} \Delta_X g &= 0 && \text{in } \Omega \\ g &= f \circ u && \text{on } \partial\Omega. \end{aligned}$$

The solvability of this problem is proved in Section 4 (see Lemma 2). The chain rule is (see [7] or [11])

$$(11) \quad -\Delta_X(f \circ u) = \sum \text{Hess } f(X_\alpha u, X_\alpha u) - \langle \text{grad } f, \tau(u) \rangle.$$

Because u is subelliptic harmonic and f is convex, we have $-\Delta_X(f \circ u) \geq 0$ by the chain rule. So

$$(12) \quad -\Delta_X(f \circ u - g) \geq 0.$$

But (recall ϕ is the boundary value of u)

$$(13) \quad (f \circ u - g)|_{\partial\Omega} = (f \circ \phi - f \circ \phi)|_{\partial\Omega} = 0$$

which combined with Eq. (12) implies

$$(14) \quad f \circ u - g \leq 0$$

in Ω by Lemma 1. Hence we have

$$(15) \quad \begin{aligned} 0 &\geq f(u(x)) - g(x) \\ &= f(u(x)) - f(\phi(x_0)) + f(\phi(x_0)) - g(x) \\ &= f(u(x)) - f(\phi(x_0)) + g(x_0) - g(x) \\ &= f(u(x)) + g(x_0) - g(x) \end{aligned}$$

because $f(\phi(x_0)) = f(u(x_0)) = 0$ by the definition of f . Hence we get

$$(16) \quad f(u(x)) \leq g(x) - g(x_0)$$

On the other hand, by Taylor's expansion and the convexity of f (Lemma 3), we have

$$(17) \quad \begin{aligned} f(u(x)) &= \frac{1}{2} \text{Hess } f(\xi)(h, h) \quad \text{where } h = \exp_{u(x_0)}^{-1}(u(x)) \\ &\geq \frac{1}{2} \lambda |h|^2 \\ &= \frac{1}{2} \lambda \text{dist}(u(x), u(x_0))^2 \end{aligned}$$

where $\text{Hess } f$ is the Hessian of f , λ is the infimum of the smallest eigenvalue of $\text{Hess } f$ on $B_R(p)$, and $\xi \in B_{|\eta|}(u(x_0))$. So

$$(18) \quad \text{dist}(u(x), u(x_0)) < \sqrt{\frac{2}{\lambda}(g(x) - w(g_0))}.$$

If $x_0 \in S_1$, then g is continuous to x_0 by Lemma 2. Letting x tend to x_0 , we have $\text{dist}(u(x), u(x_0)) \rightarrow 0$, since $g(x) \rightarrow g(x_0) = f(u(x_0)) = 0$. Q.E.D.

4. Linear boundary valued problem

In this section, we solve the linear boundary valued problem which is used in Section 3. We have

LEMMA 2. *Let S_1 be the noncharacteristic part of the boundary of Ω which satisfies the exterior ball condition. Suppose that $\phi \in C^2(\Omega \cup S_1, \mathbf{R})$. Then there exists a bounded solution u to the following linear boundary valued problem:*

$$(19) \quad \begin{cases} \Delta_X u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

The solution u is smooth in Ω and continuous to S_1 .

Proof. Denote

$$(20) \quad \mathcal{B}_R(\phi) = \{u \in M^1(\Omega; N) \mid u - \phi \in M_0^1(\Omega; N) \text{ and } \text{dist}(u(x), p) \leq R\}.$$

Let $R < R' < \min\{i(p), \pi/2\sqrt{\kappa}\}$. Then, we have ([7]):

Let $u \in \mathcal{B}_{R'}(\phi)$ satisfy $E(u) = \inf_{v \in \mathcal{B}_{R'}(\phi)} E(v)$. If $\phi(\Omega) \subset B_R(p)$, then, $u \in \mathcal{B}_R(\phi)$, and u is weakly subelliptic harmonic.

If $N = \mathbf{R}$, the one dimensional Euclidean space, the above fact implies the existence of bounded weak solutions of (19).

Because Δ_X is hypoelliptic, u is smooth in Ω . In the following, we discuss the boundary regularity of u . Let $v = u - \phi$. Then v solves the boundary valued problem

$$(21) \quad \begin{cases} \Delta_X v = -\Delta_X \phi & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Write Δ_X as $-\sum_{i,j=1}^m a_{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^m b_j \frac{\partial}{\partial x^j}$. Let $x_1 \in S_1$ and let $\nu = (\nu^i)$ be a outward normal vector to $\partial\Omega$ at x_1 such that $\sum a_{ij} \nu^i \nu^j > 0$. Let $B_r(x_0)$ be the Euclidean ball centered at x_0 such that $\overline{B_r(x_0)} \cap \overline{\Omega} = x_1$, and that $\nu = k(x_0 - x_1)$ for some positive number k . Let $h(x) = e^{-k|x-x_0|^2} - e^{-k|x_0-x_1|^2}$, where $|x|$ denotes the Euclidean norm of x . It is easily to check $h(x) < 0$ for all $x \in \overline{\Omega} \setminus \{x_1\}$ and $h(x_1) = 0$. Then we have

$$(22) \quad -\Delta_X h(x_1) = \left(4k^2 \sum_{i,j=1}^m a_{ij}(x_1^j - x_0^j)(x_1^i - x_0^i) + \text{lower terms of } k \right) e^{-k|x_1 - x_0|^2} \geq 2c$$

where c is a positive constant, if we let k large enough. Hence there is a neighborhood U of x_1 such that $-\Delta_X h(x) \geq c > 0$ in U . Let $V = U \cap \Omega$ and $\partial V = B_1 \cup B_2$, where $B_1 = \partial\Omega \cap \partial V$ and $B_2 = \partial V \setminus B_1$. Then $v|_{B_1} = 0$, $h|_{B_1} \leq 0$ and $h|_{B_2} < -\varepsilon$ for some small positive number ε . Choose a positive number M large enough, we have (note that $\Delta\phi$ is bounded on $\Omega \cup S_1$, because $\phi \in C^2(\Omega \cup S_1, \mathbf{R})$.)

$$(23) \quad \begin{aligned} -\Delta_X(Mh \pm v) &\geq Mc \mp \Delta_X\phi \geq 0 && \text{in } V \\ (Mh \pm v) &\leq 0 && \text{on } \partial V. \end{aligned}$$

By Lemma 1 and also noting $h \leq 0$, we reach $\pm v \leq -Mh = M|h|$. Thus

$$(24) \quad |v| \leq M|h| \quad \text{in } V.$$

Because $h(x) \rightarrow h(x_1) = 0$ as $x \rightarrow x_1$, we have $v(x) \rightarrow 0 = v(x_1)$. Therefore, v is continuous at x_1 and hence u . Q.E.D.

5. Convexity of the modified distance

Let N^n be a Riemannian manifold, the sectional curvature of which is bounded above by a positive number κ . Let $p \in N$ and $B_R(p)$ be a regular ball. For a fixed point $q \in B_R(p)$, define

$$f(x) = \frac{1 - \cos(\sqrt{\kappa} \text{dist}(q, x))}{\kappa \cos(\sqrt{\kappa} \text{dist}(p, x))}.$$

In this section, we prove the following lemma:

LEMMA 3. Hess $f(v, v) \geq \lambda|v|^2$ on $B_R(p)$, where λ is a positive number, v is any tangent vector at the point under consideration.

Proof. The following calculation is at $x \in B_R(p)$. Let Φ be a smooth function on $B_R(p)$, ∇ the Riemannian connection, grad the gradient operator. Then

$$(25) \quad \text{grad } e^\Phi = e^\Phi \text{ grad } \Phi,$$

$$(26) \quad \nabla_v \text{ grad } e^\Phi = e^\Phi \langle \text{grad } \Phi, v \rangle + e^\Phi \nabla_v \text{ grad } \Phi.$$

Hence

$$(27) \quad \text{Hess } e^\Phi(v, v) = e^\Phi \langle \text{grad } \Phi, v \rangle^2 + e^\Phi \text{ Hess } \Phi(v, v).$$

Let w be a smooth function on $B_R(p)$. By a straightforward calculation, we have

$$(28) \quad \begin{aligned} \text{Hess}(e^\Phi w) &= w \text{Hess } e^\Phi(v, v) + e^\Phi \text{Hess } w(v, v) \\ &\quad + 2\langle \text{grad } e^\Phi, v \rangle \langle \text{grad } w, v \rangle. \end{aligned}$$

Inserting (25) and (27) into (28) yields

$$(29) \quad \begin{aligned} e^{-\Phi} \text{Hess}(e^\Phi w)(v, v) &= \text{Hess } w(v, v) + w \text{Hess } \Phi(v, v) + w \langle \text{grad } \Phi, v \rangle^2 \\ &\quad + 2\langle \text{grad } \Phi, v \rangle \langle \text{grad } w, v \rangle. \end{aligned}$$

Let $\Phi = \omega \circ \psi$, where $\psi : B_R(p) \rightarrow \mathbf{R}$ and $\omega : \mathbf{R} \rightarrow \mathbf{R}$. Then

$$(30) \quad \text{grad } \Phi = \dot{\omega} \circ \psi \text{ grad } \psi,$$

$$(31) \quad \text{Hess } \Phi(v, v) = \ddot{\omega} \circ \psi \langle \text{grad } \psi, v \rangle^2 + \dot{\omega} \circ \psi \text{Hess } \psi(v, v).$$

Substituting (30) and (31) into (29), we have

$$(32) \quad \begin{aligned} e^{-\Phi} \text{Hess}(e^\Phi w)(v, v) &= \text{Hess } w(v, v) + w(\ddot{\omega} \circ \psi \langle \text{grad } \psi, v \rangle^2 + \dot{\omega} \circ \psi \text{Hess } \psi(v, v)) \\ &\quad + w(\dot{\omega} \circ \psi)^2 \langle \text{grad } \psi, v \rangle^2 + 2\dot{\omega} \circ \psi \langle \text{grad } \psi, v \rangle \langle \text{grad } w, v \rangle. \end{aligned}$$

If $\ddot{\omega} = \dot{\omega}^2$, then

$$(33) \quad \begin{aligned} e^{-\Phi} \text{Hess}(e^\Phi w)(v, v) &= \text{Hess } w(v, v) + 2w(\dot{\omega} \circ \psi)^2 \langle \text{grad } \psi, v \rangle^2 + w\dot{\omega} \circ \psi \text{Hess } \psi(v, v) \\ &\quad + 2\dot{\omega} \circ \psi \langle \text{grad } \psi, v \rangle \langle \text{grad } w, v \rangle. \end{aligned}$$

Let $w > 0$. By Young's inequality: $ab \leq \frac{1}{2}ta^2 + \frac{1}{2t}b^2$ with $a = -2\dot{\omega} \circ \psi \langle \text{grad } \psi, v \rangle$, $b = \langle \text{grad } w, v \rangle$, and $t = w$, we have

$$(34) \quad -2\dot{\omega} \circ \psi \langle \text{grad } \psi, v \rangle \langle \text{grad } w, v \rangle \leq 2w(\dot{\omega} \circ \psi)^2 \langle \text{grad } \psi, v \rangle^2 + \frac{1}{2w} \langle \text{grad } w, v \rangle^2.$$

Applying (34) to (33) we have

$$(35) \quad \begin{aligned} e^{-\Phi} \text{Hess}(e^\Phi w)(v, v) &\geq \text{Hess } w(v, v) + w\dot{\omega} \circ \psi \text{Hess } \psi(v, v) \\ &\quad - \frac{1}{2w} \langle \text{grad } w, v \rangle^2. \end{aligned}$$

Set $c_\kappa(t) = \cos(\sqrt{\kappa}t)$, $s_\kappa(t) = \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t)$. It is easily to get $\dot{c}_\kappa = -\kappa s_\kappa$, $\dot{s}_\kappa = c_\kappa$, and $\kappa s_\kappa^2 + c_\kappa^2 = 1$. Let $r_q(x) = \text{dist}(q, x)$, $q_\kappa(t) = \frac{1}{\kappa}(1 - c_\kappa(t))$, and $w = q_\kappa \circ r_q$. Then

$$(36) \quad \text{grad } w = s_\kappa \circ r_q \text{ grad } r_q.$$

Furthermore, if $x \neq q$ we have $w > 0$. By a straightforward calculation, one has

$$(37) \quad \text{Hess } w(v, v) = \dot{s}_\kappa \circ r_q \langle \text{grad } r_q, v \rangle^2 + s_\kappa \circ r_q \text{ Hess } r_q(v, v).$$

Let γ be the minimizing geodesic from $\gamma(0) = q$ to $\gamma(\rho) = x$, where $\rho = r_q(x)$. Denote $v^\top = \langle v, \dot{\gamma}(\rho) \rangle \dot{\gamma}(\rho)$, $v^\perp = v - v^\top$. By [6] (p. 59), we have (it is sufficient to take $v_1 = 0$ and $v_2 = v$)

$$(38) \quad \text{Hess } r_q(v, v) \geq \frac{\dot{s}_\kappa \circ r_q}{s_\kappa \circ r_q} |v^\perp|^2.$$

Apply (38) to (37). We have

$$(39) \quad \begin{aligned} \text{Hess } w(v, v) &\geq \dot{s}_\kappa \circ r_q \langle \text{grad } r_q, v \rangle^2 + \dot{s}_\kappa \circ r_q |v^\perp|^2 \\ &= \dot{s}_\kappa \circ r_q |v^\top|^2 + \dot{s}_\kappa \circ r_q |v^\perp|^2 \\ &= \dot{s}_\kappa \circ r_q |v|^2 \\ &= (1 - \kappa w) |v|^2. \end{aligned}$$

Let $\psi = q_\kappa \circ r_p$. Then by (39) we have

$$(40) \quad \text{Hess } \psi(v, v) \geq (1 - \kappa \psi) |v|^2.$$

Take $\omega(t) = -\log(1 - \kappa t)$ and notice that $0 < \kappa \psi < 1$ since $x \neq q$ and $r_p(x) < \frac{1}{2\sqrt{\kappa}}$. Then $\dot{\omega}(t) = \kappa/(1 - \kappa t)$, $\ddot{\omega} = \dot{\omega}^2$. Taking use of (39) and (40) in (35), we get

$$(41) \quad \begin{aligned} e^{-\Phi} \text{Hess}(e^\Phi w)(v, v) &\geq (1 - \kappa w) |v|^2 + w \dot{\omega} \circ \psi (1 - \kappa \psi) |v|^2 - \frac{1}{2w} \langle \text{grad } w, v \rangle^2 \\ &= (1 - \kappa w) |v|^2 + \kappa w |v|^2 - \frac{1}{2w} \langle \text{grad } w, v \rangle^2 \\ &= |v|^2 - \frac{1}{2w} (s_\kappa \circ r_q)^2 \langle \text{grad } r_q, v \rangle^2 \\ &= |v|^2 - \frac{1}{2} (1 + c_\kappa \circ r_q) \langle \text{grad } r_q, v \rangle^2 \\ &> 0 \quad \text{if } x \neq q \text{ and } v \neq 0. \end{aligned}$$

At $x = q$, $w = 0$ and $\text{grad } w = s_\kappa \circ r_q = 0$. Then by (32) and (36) we have

$$(42) \quad e^{-\Phi} \text{Hess}(e^\Phi w)(v, v) = \text{Hess } w(v, v) \geq |v|^2 > 0$$

when $v \neq 0$. Hence $e^\Phi w$ is a convex function on $B_R(p)$ for any $R < \frac{1}{2\sqrt{\kappa}}$, hence for R' where $R < R' < \frac{1}{2\sqrt{\kappa}}$. Therefore, $e^\Phi w$ is convex on $\overline{B_R(p)}$. Let

λ be the infimum of the smallest eigenvalues of $\text{Hess}(e^{\Phi w})$ on $\overline{B_R(p)}$. Then $\text{Hess}(e^{\Phi w})(v, v) \geq \lambda|v|^2$. Apparently, $\lambda > 0$ because of the compactness of $\overline{B_R(p)}$. On the other hand, $e^{\Phi w} = \frac{1 - c_{\kappa} \circ r_q}{\kappa c_{\kappa} \circ r_p} = f$. The lemma follows. Q.E.D.

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