H. M. AL-QASSEM
KODAI MATH. J.
28 (2005), 494–510

MAXIMAL OPERATORS RELATED TO BLOCK SPACES

H. M. Al-Qassem

Abstract

In this paper, we prove appropriate L^p bounds for a class of maximal operators $\mathscr{S}_\Omega$ related to singular integrals with kernels which belong to block spaces and are supported by subvarieties. Also, we show that our condition on the kernel is optimal for the L^2 boundedness of $\mathscr{S}_\Omega$. Our results improve substantially the main result obtained by L. K. Chen and H. Lin in [CL].

1. Introduction and statement of results

Let $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Throughout this paper, p' will denote the dual exponent to p, that is $1/p + 1/p' = 1$. Also, we shall let Ω be a homogeneous function of degree zero which satisfies $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(u)\, d\sigma(u) = 0. \tag{1.1}$$

Let $\mathscr{H}$ = the set of all radial functions h satisfying

$$\left(\int_0^\infty |h(r)|^2 \frac{dr}{r}\right)^{1/2} \leq 1.$$

Also, for $d \neq 0$, we say that a smooth function $\Psi : \mathbf{R}_+ \to \mathbf{R}$ belongs to the class $\Gamma_d(\mathbf{R}_+)$ if for some positive constants C_1, C_2, C_3, and C_4 independent of t, the following growth conditions are satisfied:

$$|\Psi(t)| \leq C_1 t^d, \quad |\Psi''(t)| \leq C_2 t^{d-2}, \tag{1.2}$$

$$C_3 t^{d-1} \leq |\Psi'(t)| \leq C_4 t^{d-1}. \tag{1.3}$$

Now, a smooth function $\Psi : \mathbf{R}_+ \to \mathbf{R}$ belongs to the class $\Gamma(\mathbf{R}_+)$ if $\Psi \in \Gamma_d(\mathbf{R}_+)$ for some $d \neq 0$.

For a function $\Psi \in \Gamma(\mathbf{R}_+)$ we define the maximal operator $\mathscr{S}_{\Omega,\Psi}$ by

---

2000 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B15, 42B25.

Key words and phrases. Maximal operator, Rough kernel, Singular integral, L^p boundedness, Block spaces.

Received June 1, 2004; revised January 31, 2005.

$$\mathscr{S}_{\Omega,\Psi}f(x) = \sup_{h\in\mathscr{H}}\left|\int_{\mathbf{R}^n} f(x-\Psi(|y|)y')h(|y|)\Omega(y')|y|^{-n}\,dy\right|, \tag{1.4}$$

where $y' = y/|y| \in \mathbf{S}^{n-1}$ and $f \in \mathscr{S}(\mathbf{R}^n)$, the space of Schwartz functions.

The maximal operator $\mathscr{S}_{\Omega,\Psi}$ is closely related to the singular integral operator $T_{\Psi,\Omega,h}$ given by

$$T_{\Psi,\Omega,h}f(x) = p.v.\int_{\mathbf{R}^n} f(x-\Psi(|y|)y')\frac{\Omega(y')}{|y|^n}h(|y|)\,dy, \tag{1.5}$$

where h is a measurable function on $\mathbf{R}_+$.

For the sake of simplicity, we denote $\mathscr{S}_{\Omega,\Psi} = \mathscr{S}_\Omega$ and $T_{\Psi,\Omega,h} = T_{\Omega,h}$ if $\Psi(t) = t$.

In [CL], L. K. Chen and H. Lin studied the L^p boundedness of the maximal operator $\mathscr{S}_\Omega$ under a smoothness condition on Ω. In fact, they proved the following:

Theorem A [CL]. *Assume $n \geq 2$ and $\Omega \in C(\mathbf{S}^{n-1})$ satisfying* (1.1). *Then*

$$\|\mathscr{S}_\Omega(f)\|_{L^p(\mathbf{R}^n)} \leq C_p\|f\|_{L^p(\mathbf{R}^n)}$$

for $2n/(2n-1) < p < \infty$ and $f \in L^p$. Moreover, the range of p is the best possible.

On the other hand, the L^p boundedness of the singular integral operator $T_{\Psi,\Omega,h}$ is known to hold under much weaker conditions on Ω (see [CZ], [Fe], [DR], [LTW], [FP], [AA1], [AP]). For example, if $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$, Calderón-Zygmund showed that $T_{\Omega,1}$ is bounded on L^p for all $p \in (1,\infty)$ and the condition $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$ is essentially the weakest possible size condition on Ω for the L^p boundedness of $T_{\Omega,1}$ to hold ([CZ]). Some years later, Connett ([Co]) and Coifman-Weiss ([CW]) obtained an improvement over the result of Calderón and Zygmund by considering Ω in the Hardy space $H^1(\mathbf{S}^{n-1})$. The study of the L^p $(1 < p < \infty)$ boundedness of the singular operator $T_{\Omega,h}$ began in R. Fefferman in [Fe] if $h \in L^\infty(\mathbf{R}^+)$ and Ω satisfies some Lipschitz condition on S^{n-1} and subsequently by many authors under various conditions on Ω and h (see for example, [Na], [Ch], [DR]). In 1997, Fan and Pan introduced the more general class of operators $T_{\Psi,\Omega,h}$ and showed that $T_{\Psi,\Omega,h}$ is bounded on $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ if $H^1(\mathbf{S}^{n-1})$ and $h \in L^\infty(\mathbf{R}_+)$. Another condition on Ω was given by Jiang and Lu who introduced a special class of block spaces $B_q^{(0,v)}(\mathbf{S}^{n-1})$ and proved the following L^2 boundedness result.

Theorem B ([LTW]). *Let $T_{\Omega,h}$ be given as above. Then if $h \in L^\infty(\mathbf{R}_+)$ and $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ with $q > 1$, $T_{\Omega,h}$ is a bounded operator on $L^2(\mathbf{R}^n)$.*

Some years later, the L^p boundedness of the more general operator $T_{\Psi,\Omega,h}$ was proved for all $p \in (1,\infty)$ under the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ (see for example, [AA1]). Also, it was proved in [AAP1] that the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$

is the best possible for the L^p boundedness of $T_{\Omega,1}$ to hold. Namely, the L^p boundedness of $T_{\Omega,1}$ may fail for any p if it is replaced by a weaker condition $\Omega \in B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ for any $-1 < \upsilon < 0$ and $q > 1$. The definition of the block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ will be recalled in Section 2.

The results cited in [AA1] and [AAP1] above on singular integrals give rise to the problem whether similar results hold for the maximal integral operator $\mathscr{S}_{\Omega,\Psi}$. More precisely, we have the following:

Problem. *Determine wether the L^p boundedness of the operator $\mathscr{S}_\Omega$ holds under a condition in the form of $\Omega \in B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$, $-1 < \upsilon$, and, if so, what is the best possible value of υ.*

The main focus of this paper is to obtain a solution to the above problem. Our main result in this paper is the following:

Theorem 1.1. *Let $n \geq 2$ and $\mathscr{S}_{\Omega,\Psi}$ be given as in* (1.4). *Then*

(a) *If $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$, $q > 1$, and satisfies* (1.1), *$\mathscr{S}_{\Omega,\Psi}$ is bounded on $L^p(\mathbf{R}^n)$ for $2 \leq p < \infty$;*

(b) *If $\Omega \in L^q(\mathbf{S}^{n-1})$ (for some $q > 1$) and satisfies* (1.1), *$\mathscr{S}_{\Omega,\Psi}$ is bounded on $L^p(\mathbf{R}^n)$ for $2n\delta/(2n + n\delta - 2) < p < \infty$, where $\delta = \max\{2, q'\}$.*

(c) *There exists an Ω which lies in $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ for all $-1 < \upsilon < -\frac{1}{2}$ and satisfies* (1.1) *such that $\mathscr{S}_\Omega$ is not bounded on $L^2(\mathbf{R}^n)$.*

We remark that on $\mathbf{S}^{n-1}$, for any $q > 1$ and $-1 < \upsilon$, the following inclusions hold and are proper:

$$C^1(\mathbf{S}^{n-1}) \subset \bigcup_{r>1} L^r(\mathbf{S}^{n-1}) \subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}). \tag{1.6}$$

By the relationship in (1.6) remarked above one sees that parts (a) and (b) represent a substantial improvement of the main result of L. K. Chen and H. Lin, while part (c) shows that the condition $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ is nearly optimal.

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one in each occurrence.

Acknowledgment. The author would like to thank very much the referee for his very valuable comments and suggestions.

2. Definitions and lemmas

The block spaces originated in the work of M. H. Taibleson and G. Weiss on the convergence of the Fourier series (see [TW]) in connection with the developments of the real Hardy spaces. Below we shall recall the definition of block spaces on $\mathbf{S}^{n-1}$. For further background information about the theory of spaces generated by blocks and its applications to harmonic analysis, see the book [LTW].

DEFINITION 2.1. *A q-block on* $\mathbf{S}^{n-1}$ *is an* L^q $(1 < q \le \infty)$ *function* $b(x)$ *that satisfies*

$$\text{(i)} \quad \operatorname{supp}(b) \subset I; \quad \text{(ii)} \quad \|b\|_{L^q} \le |I|^{-1/q'},$$

where $|I| = \sigma(I)$, *and* $I = B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$ *is a cap on* $\mathbf{S}^{n-1}$ *for some* $x_0' \in \mathbf{S}^{n-1}$ *and* $\theta_0 \in (0, 1]$.

Jiang and Lu introduced (see [LTW]) the class of block spaces $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ (for $\upsilon > -1$) with respect to the study of the singular integral operators $T_{\Omega,h}$.

DEFINITION 2.2. *The block space* $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ *is defined by*

$$B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) = \left\{ \Omega \in L^1(\mathbf{S}^{n-1}) \colon \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu, \, M_q^{(0,\upsilon)}(\{\lambda_\mu\}) < \infty \right\},$$

where each λ_μ *is a complex number*; *each* b_μ *is a q-block supported on a cap* I_μ *on* $\mathbf{S}^{n-1}$, $\upsilon > -1$ *and*

$$M_q^{(0,\upsilon)}(\{\lambda_\mu\}) = \sum_{\mu=1}^{\infty} |\lambda_\mu| \{1 + \log^{(\upsilon+1)}(|I_\mu|^{-1})\}. \tag{2.7}$$

We remark that the definition of $B_q^{(0,\upsilon)}([a,b])$, $a, b \in \mathbf{R}$ will be the same as that of $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ except for minor modifications.

Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})} = N_q^{(0,\upsilon)}(\Omega) = \inf\{M_q^{(0,\upsilon)}(\{\lambda_\mu\}) \colon \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu$ and each b_μ is a q-block function supported on a cap I_μ on $\mathbf{S}^{n-1}\}$. Then $\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})}$ is a norm on the space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ and $(B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}), \|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})})$ is a Banach space.

In their investigations of block spaces, Keitoku and Sato in [KS] showed that these spaces enjoy the following properties:

$$B_q^{(0,\upsilon_2)}(\mathbf{S}^{n-1}) \subset B_q^{(0,\upsilon_1)}(\mathbf{S}^{n-1}) \quad \text{if } \upsilon_2 > \upsilon_1 > -1;$$

$$B_{q_2}^{(0,\upsilon)}(\mathbf{S}^{n-1}) \subset B_{q_1}^{(0,\upsilon)}(\mathbf{S}^{n-1}) \quad \text{if } 1 < q_1 < q_2 \text{ and for any } \upsilon > -1;$$

$$\bigcup_{q>1} B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) \nsubseteq \bigcup_{q>1} L^q(\mathbf{S}^{n-1}) \quad \text{for any } \upsilon > -1.$$

The proof of Theorem 1.1 (c) will rely heavily on the following lemma from [AAP1].

LEMMA 2.3. *For any* $\upsilon > -1$, $a, b \in \mathbf{R}$,

(i) *If* $f \in B_q^{(0,\upsilon)}([a,b])$ *and* g *is a measurable on* $[a,b]$ *with* $|g| \le |f|$, *then* $g \in B_q^{(0,\upsilon)}([a,b])$ *with*

$$N_q^{(0,\upsilon)}(g) \le N_q^{(0,\upsilon)}(f);$$

(ii) *Let* I_1 *and* I_2 *be two disjoint intervals in* $[a, b]$ *with* $|I_1|, |I_2| < 1$ *and* $\alpha_1, \alpha_2 \in \mathbf{R}^+$. *Then*

$$N_q^{(0,v)}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) \geq N_q^{(0,v)}(\alpha_1\chi_{I_1}) + N_q^{(0,v)}(\alpha_2\chi_{I_2});$$

(iii) *Let* I *be an interval in* $[a, b]$ *with* $|I| < 1$. *Then*

$$N_q^{(0,v)}(\chi_I) \geq |I|(1 + \log^{v+1}(|I|^{-1})).$$

Lemma 2.4. *Let* $q > 1$, $\mu \in \mathbf{N} \cup \{0\}$ *and* $\tilde{b}_\mu$ *be a function on* $\mathbf{S}^{n-1}$ *satisfying* (i) $\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(y)\, d\sigma(y) = 0$; (ii) $\|\tilde{b}_\mu\|_q \leq |I_\mu|^{-1/q'}$ *for some cap* I_μ *on* $\mathbf{S}^{n-1}$ *with* $|I_\mu| < e^{-2}$; *and* (iii) $\|\tilde{b}_\mu\|_1 \leq 1$. *Assume that* Ψ *belongs to the class* $\Gamma(\mathbf{R}_+)$ *for some* $d \neq 0$. *Let* $\omega_\mu = 2^{\log(|I_\mu|^{-1})}$ *and for* $\xi \in \mathbf{R}^n$, *let*

$$Y_{\mu,k}(\xi) = \left(\int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left|\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(x) e^{-i\Psi(t)(\xi\cdot x)}\, d\sigma(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Then there exist positive constants C *and* α *with* $0 < \alpha q' < 1$ *such that*

$$|Y_{\mu,k}(\xi)| \leq C(\log|I_\mu|^{-1})^{1/2}; \tag{2.8}$$

$$|Y_{\mu,k}(\xi)| \leq C(\log|I_\mu|^{-1})^{1/2} |\omega_\mu^{kd}\xi|^{\pm\alpha/\log(|I_\mu|^{-1})}, \tag{2.9}$$

where $t^{\pm\alpha} = \inf\{t^\alpha, t^{-\alpha}\}$ *and* C *is a positive constant independent of* k, ξ *and* μ.

Proof. We shall prove our estimates only for the case $d > 0$, because the proof for the case $d < 0$ is essentially the same. First, by condition (iii), it is easy to verify that (2.8) holds.

Next, since

$$\left|\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(x) e^{-i\Psi(t)(\xi\cdot x)}\, d\sigma(x)\right|^2 = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \tilde{b}_\mu(x)\overline{\tilde{b}_\mu(y)} e^{-i\Psi(t)\xi\cdot(x-y)}\, d\sigma(x)\, d\sigma(y)$$

we get

$$|Y_{\mu,k}(\xi)|^2 = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \tilde{b}_\mu(x)\overline{\tilde{b}_\mu(y)} \left(\int_1^{\omega_\mu} e^{-i\Psi(\omega_\mu^k t)\xi\cdot(x-y)} \frac{dt}{t}\right) d\sigma(x)\, d\sigma(y). \tag{2.10}$$

By integration by parts, it is easy to verify that

$$\left|\int_1^{\omega_\mu} e^{-i\Psi(\omega_\mu^k t)\xi\cdot(x-y)} \frac{dt}{t}\right| \leq C\omega_\mu |\omega_\mu^{kd}\xi|^{-1} |\xi'\cdot(y-x)|^{-1}$$

which when combined with the trivial estimate

$$\left|\int_1^{\omega_\mu} e^{-i\Psi(\omega_\mu^k t)\xi\cdot(x-y)} \frac{dt}{t}\right| \le C(\log|I_\mu|^{-1})$$

yields

$$\left|\int_1^{\omega_\mu} e^{-i\Psi(\omega_\mu^k t)\xi\cdot(x-y)} \frac{dt}{t}\right| \le C(\log|I_\mu|^{-1})\omega_\mu^\alpha|\omega_\mu^{kd}\xi|^{-\alpha}|\xi'\cdot(y-x)|^{-\alpha} \tag{2.11}$$

for any $0 < \alpha < 1$.

By Hölder's inequality, condition (ii) on $\tilde{b}_\mu$ and (2.10)–(2.11) we get

$$|Y_{\mu,k}(\xi)| \le C(\log|I_\mu|^{-1})^{1/2}\omega_\mu^{\alpha/2}|\omega_\mu^{kd}\xi|^{-\alpha/2}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} |\xi'\cdot(x-y)|^{-\alpha q'}\,d\sigma(x)\,d\sigma(y)\right)^{1/2}$$

By choosing $\alpha q' < 1$, we get

$$|Y_{\mu,k}(\xi)| \le C(\log|I_\mu|^{-1})^{1/2}\omega_\mu^{\alpha/2}|\omega_\mu^{kd}\xi|^{-\alpha/2}$$

which, when combined with the trivial estimate (2.8), yields the estimate (2.9) with a minus sign in the exponent. To get the second estimate, we use the cancellation condition (i) on $\tilde{b}_\mu$ to get

$$|Y_{\mu,k}(\xi)|^2 \le \int_1^{\omega_\mu}\left(\int_{\mathbf{S}^{n-1}} |\tilde{b}_\mu(x)|\,|e^{-i\Psi(\omega_\mu^k t)\xi\cdot x} - 1|\,d\sigma(x)\right)^2 \frac{dt}{t}.$$

By (1.2) and condition (iii), we get

$$|Y_{\mu,k}(\xi)| \le C(\log|I_\mu|^{-1})^{1/2}|\omega_\mu^{kd}\xi|. \tag{2.12}$$

By interpolation between this estimate with the trivial estimate (2.8) we get the second estimate in (2.9). This completes the proof of the lemma.

By the proof of Lemma 3.1 in [AAP2], we get the following:

LEMMA 2.5. *Let $\{\nu_k : k \in \mathbf{Z}\}$ be a sequence of non negative Borel measures on $\mathbf{R}^n$. Suppose that for all $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$, for some $a \ge 2$, $\alpha, C > 0$ and for some constant $B > 1$ we have*

(i) $\|\nu_k\| \le B$;

(ii) $|\hat{\nu}_k(\xi)| \le CB|a^{kB}\xi|^{-\alpha/B}$;

(iii) $|\hat{\nu}_k(\xi) - 1| \le CB|a^{kB}\xi|^{\alpha/B}$.

Then the inequality

$$\|\nu^*(f)\|_p \le CB\|f\|_p \tag{2.13}$$

holds for all $1 < p \le \infty$ and f in $L^p(\mathbf{R}^n)$, where $\nu^*(f) = \sup_{k\in\mathbf{Z}}|\nu_k * f|$. The constant C_p is independent of B.

For any $\mu \in \mathbf{N}$ and $\tilde{b}_\mu \in L^1(\mathbf{S}^{n-1})$, we define the maximal operator

$$v_\mu^* f(x) = \sup_{k \in \mathbf{Z}} \left| \int_{\omega_\mu^k \le |y| < \omega_\mu^{k+1}} f(x - \Psi(|y|)y') \frac{\tilde{b}_\mu(y')}{|y|^n} \, dy \right|, \tag{2.14}$$

where $\omega_\mu = 2^{\log(|I_\mu|^{-1})}$ for some cap I_μ on $\mathbf{S}^{n-1}$.

By Lemmas 2.4 and 2.5, we get immediately the following:

Lemma 2.6. *Let $\mu \in \mathbf{N}$ and let $\tilde{b}_\mu$ be a function on $\mathbf{S}^{n-1}$ satisfying* (i) $\|\tilde{b}_\mu\|_q \le |I_\mu|^{-1/q'}$ *for some $q > 1$ and for some cap I_μ on $\mathbf{S}^{n-1}$*; (ii) $\|\tilde{b}_\mu\|_1 \le 1$. *Assume that Ψ belongs to the class $\Gamma(\mathbf{R}_+)$ for some $d \ne 0$. Then*

$$\|v_\mu^*(f)\|_p \le C_p(\log|I_\mu|^{-1})\|f\|_p \tag{2.15}$$

for $1 < p \le \infty$ and $f \in L^p$, where C_p is independent of μ and f.

Let $\mathscr{M}_S$ be the spherical maximal operator defined by

$$\mathscr{M}_S f(x) = \sup_{r>0} \int_{\mathbf{S}^{n-1}} |f(x - r\theta)| \, d\sigma(\theta).$$

By the results of E. M. Stein [St3] and J. Bourgain [Bo] we have

Lemma 2.7. *Suppose that $n \ge 2$ and $p > n/(n-1)$. Then $\mathscr{M}_S(f)$ is bounded on $L^p(\mathbf{R}^n)$.*

3. Proof of the main result

Proof of Theorem 1.1 (a). Assume that $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and satisfies (1.1). Thus Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu$, where $\lambda_\mu \in \mathbf{C}$, b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$ and $M_q^{(0,-1/2)}(\{\lambda_\mu\}) < \infty$. To each block function $b_\mu(\cdot)$, let $\tilde{b}_\mu(\cdot)$ be a function defined by

$$\tilde{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u) \, d\sigma(u). \tag{3.1}$$

Let $\mathbf{J} = \{\mu \in \mathbf{N} : |I_\mu| < e^{-2}\}$. Let $\tilde{b}_0 = \Omega - \sum_{\mu \in \mathbf{J}}^{\infty} \lambda_\mu \tilde{b}_\mu$. Then for some positive constant C, the following holds for all $\mu \in \mathbf{J} \cup \{0\}$:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(u) \, d\sigma(u) = 0, \tag{3.2}$$

$$\|\tilde{b}_\mu\|_q \le C|I_\mu|^{-1/q'}, \tag{3.3}$$

$$\|\tilde{b}_\mu\|_1 \le C, \tag{3.4}$$

$$\Omega = \sum_{\mu \in \mathbf{J} \cup \{0\}} \lambda_\mu \tilde{b}_\mu, \tag{3.5}$$

where I_0 is any cap on $\mathbf{S}^{n-1}$ with $|I_0| = e^{-3}$. By (3.5) we have

$$\|\mathscr{S}_{\Omega,\Psi} f\|_p \leq \sum_{\mu \in \mathbf{J} \cup \{0\}} |\lambda_\mu| \, \|\mathscr{S}_{\tilde{b}_\mu,\Psi} f\|_p. \tag{3.6}$$

Thus, by (3.6), Theorem 1.1 (a) is proved if we can show that

$$\|\mathscr{S}_{\tilde{b}_\mu,\Psi} f\|_p \leq C_p (\log|I_\mu|^{-1})^{1/2} \|f\|_p \tag{3.7}$$

for each $\mu \in \mathbf{J} \cup \{0\}$, $2 \leq p < \infty$ and for some positive constant C_p independent of μ. To this end, we let $\{\varphi_{k,\mu}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the intervals $[\omega_\mu^{-kd-|d|}, \omega_\mu^{-kd+|d|}]$. To be precise, we require the following:

$$\varphi_{k,\mu} \in C^\infty, \quad 0 \leq \varphi_{k,\mu} \leq 1, \quad \sum_k \varphi_{k,\mu}(t) = 1,$$

$$\operatorname{supp} \varphi_{k,\mu} \subseteq [\omega_\mu^{-kd-|d|}, \omega_\mu^{-kd+|d|}], \quad \left| \frac{d^s \varphi_{k,\mu}(t)}{dt^s} \right| \leq \frac{C_s}{t^s},$$

where C_s is independent of the lacunary sequence $\{\omega_\mu^k : k \in \mathbf{Z}\}$. Define the partial sum operators $S_{k,\mu}$ on $\mathbf{R}^n$ by

$$(\widehat{S_{k,\mu} f})(\xi) = \varphi_{k,\mu}(|\xi|) \hat{f}(\xi).$$

Since $f(x) = \sum_{k \in \mathbf{Z}} (S_{k+l,\mu} f)(x)$ for any $f \in \mathscr{S}(\mathbf{R}^n)$ and $l \in \mathbf{Z}$, by duality and applying Minkowski's inequality we get

$$\begin{aligned}
\mathscr{S}_{\tilde{b}_\mu,\Psi} f(x) &\leq \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} f(x - \Psi(t)\xi) \tilde{b}_\mu(\xi) \, d\sigma(\xi) \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= \left(\sum_{k \in \mathbf{Z}} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left| \int_{\mathbf{S}^{n-1}} f(x - \Psi(t)\xi) \tilde{b}_\mu(\xi) \, d\sigma(\xi) \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= \left(\sum_{k \in \mathbf{Z}} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left| \sum_{l \in \mathbf{Z}} E_{k+l,t,\tilde{b}_\mu} f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\
&\leq \sum_{l \in \mathbf{Z}} \left(\sum_{k \in \mathbf{Z}} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} |E_{k+l,t,\tilde{b}_\mu} f(x)|^2 \frac{dt}{t} \right)^{1/2},
\end{aligned}$$

where

$$E_{l,t,\Omega} f(x) = \int_{\mathbf{S}^{n-1}} \Omega(\xi) (S_{l,\mu} f)(x - \Psi(t)\xi) \, d\sigma(\xi).$$

Now if we let

$$T_{l,\mu,\tilde{b}_\mu}f(x)=\left(\sum_{k\in\mathbf{Z}}\int_{\omega_\mu^k}^{\omega_\mu^{k+1}}|E_{k+l,t,\tilde{b}_\mu}f(x)|^2\,\frac{dt}{t}\right)^{1/2},$$

then we have

$$\mathscr{S}_{\tilde{b}_\mu,\Psi}f(x)\le\sum_{l\in\mathbf{Z}}T_{l,\mu,\tilde{b}_\mu}f(x). \tag{3.8}$$

Therefore, to prove (3.7), it suffices to prove

$$\|T_{l,\mu,\tilde{b}_\mu}(f)\|_p\le C_p(\log|I_\mu|^{-1})^{1/2}2^{-\theta_p|l|\,|d|}\|f\|_p \tag{3.9}$$

for some positive constants C_p, θ_p and for all $2\le p<\infty$.

To prove (3.9), let us first compute the L^2 norm of $T_{l,\mu,\tilde{b}_\mu}(f)$. By Plancherel's theorem and using Lemma 2.4 we obtain

$$\begin{aligned}\|T_{l,\mu,\tilde{b}_\mu}(f)\|_2^2&=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\omega_\mu^k}^{\omega_\mu^{k+1}}|E_{k+l,t,\tilde{b}_\mu}f(x)|^2\,\frac{dt}{t}\,dx\\&\le\sum_{k\in\mathbf{Z}}\int_{\Delta_{k+l}}|Y_{\mu,k}(\xi)|^2|\hat{f}(\xi)|^2\,d\xi\\&\le C(\log|I_\mu|^{-1})2^{-2\alpha|l|\,|d|}\sum_{k\in\mathbf{Z}}\int_{\Delta_{k+l}}|\hat{f}(\xi)|^2\,d\xi\\&\le C(\log|I_\mu|^{-1})2^{-2\alpha|l|\,|d|}\|f\|_2^2,\end{aligned}$$

where $Y_{\mu,k}(\xi)$ is defined as in Lemma 2.4 and

$$\Delta_k=\{\xi\in\mathbf{R}^n:\omega_\mu^{-kd-|d|}\le|\xi|\le\omega_\mu^{-kd+|d|}\}.$$

Therefore, we have

$$\|T_{l,\mu,\tilde{b}_\mu}(f)\|_2\le C(\log|I_\mu|^{-1})^{1/2}2^{-\alpha|l|\,|d|}\|f\|_2. \tag{3.10}$$

Now, let us compute the L^p-norm of $T_{l,\mu,\tilde{b}_\mu}(f)$ for $p>2$. By duality, there is a function g in $L^{(p/2)'}(\mathbf{R}^n)$ with $\|g\|_{(p/2)'}\le 1$ such that

$$\begin{aligned}\|T_{l,\mu,\tilde{b}_\mu}(f)\|_p^2&=\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{\omega_\mu^k}^{\omega_\mu^{k+1}}|E_{k+l,t,\tilde{b}_\mu}f(x)|^2\,\frac{dt}{t}|g(x)|\,dx\\&\le C\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{\omega_\mu^k}^{\omega_\mu^{k+1}}\int_{\mathbf{S}^{n-1}}|\tilde{b}_\mu(\xi)|\,|S_{k+l,\mu}f(x)|^2|g(x+\Psi(t)\xi)|\,d\sigma(\xi)\,\frac{dt}{t}\,dx\\&\le C\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|S_{k+l,\mu}f(x)|^2v_\mu^*(\tilde{g})(-x)\,dx,\quad\text{where }\tilde{g}(x)=g(-x)\\&\le C\left\|\sum_{k\in\mathbf{Z}}|S_{k+l,\mu}f|^2\right\|_{(p/2)}\|v_\mu^*(\tilde{g})\|_{(p/2)'}.\end{aligned}$$

Thus, using Lemma 2.6, the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [St1], p. 96, we have

$$\|T_{l,\mu,\tilde{b}_\mu}(f)\|_p \le C_p(\log|I_\mu|^{-1})^{1/2}\|f\|_p \quad \text{for } 2 \le p < \infty. \tag{3.11}$$

Interpolating between (3.10) and (3.11) we get (3.9) which in turn ends the proof of Theorem 1.1 (a).

Proof of Theorem 1.1 (b). Assume that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. By Theorem 1.1 (a) and since for any fixed $r > 1$, $B_r^{(0,-1/2)}(\mathbf{S}^{n-1}) \supset L^q(\mathbf{S}^{n-1})$ for all $q > 1$, we need to prove Theorem 1.1 (b) only for $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ and $2n\delta/(2n + n\delta - 2) < p < 2$. By the same arguments employed in the proof of Theorem 1.1 (a), it suffices to show that

$$\|T_{l,0,\Omega}(f)\|_p \le C_p\|f\|_p \quad \text{for } 2n\delta/(2n + n\delta - 2) < p < 2 \text{ and } q > 1. \tag{3.12}$$

By definition of $T_{l,0,\Omega}$ and by a simple change of variable we have

$$T_{l,0,\Omega}f(x) \le \left(\sum_{k\in\mathbf{Z}}\int_1^2 |F_{k,l,t}f(x)|^2\,\frac{dt}{t}\right)^{1/2},$$

where

$$F_{k,l,t}f(x) = \int_{\mathbf{S}^{n-1}} \Omega(\xi)(S_{k+l,0}f)(x - \Psi(2^k t)\xi)\,d\sigma(\xi).$$

We notice that, to prove $T_{l,0,\Omega}(f) \in L^p(\mathbf{R}^n)$, it suffices to show that $F_{k,l,t}f(x) \in L^p\left(l^2\left[L^2\left([1,2],\frac{dt}{t}\right),k\right],dx\right)$. By duality, there is a function $g = g_k(x,t)$ satisfying $\|g\| \le 1$ and

$$g \in L^{p'}\left(l^2\left[L^2\left([1,2),\frac{dt}{t}\right),k\right],dx\right)$$

such that

$$\begin{aligned}
&\|T_{l,0,\Omega}(f)\|_p \\
&\quad = \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_1^2\int_{\mathbf{S}^{n-1}} \Omega(\xi)g_k(x,t)(S_{k+l,0}f)(x - \Psi(2^k t)\xi)\,d\sigma(\xi)\,\frac{dt}{t}\,dx \\
&\quad = \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_1^2\int_{\mathbf{S}^{n-1}} (S_{k+l,0}f)(x)\Omega(\xi)g_k(x + \Psi(2^k t)\xi, t)\,d\sigma(\xi)\,\frac{dt}{t}\,dx \\
&\quad \le \|(X(g))^{1/2}\|_{p'}\left\|\left(\sum_{k\in\mathbf{Z}}|S_{k+l,0}f|^2\right)^{1/2}\right\|_p,
\end{aligned}$$

where

$$X(g)(x) = \sum_{k\in\mathbf{Z}} \left(\int_1^2 \int_{\mathbf{S}^{n-1}} \Omega(\xi) g_k(x + \Psi(2^k t)\xi, t)\, d\sigma(\xi) \frac{dt}{t} \right)^2.$$

By the Littlewood-Paley theory we have

$$\|T_{l,0,\Omega}(f)\|_p \le C_p \|f\|_p \|(X(g))^{1/2}\|_{p'}.$$

Since $\|(X(g))^{1/2}\|_{p'} = \|X(g)\|_{p'/2}^{1/2}$ and $p' > 2$, there is a function $a \in L^{(p'/2)'}(\mathbf{R}^n)$ such that $\|a\|_{(p'/2)'} \le 1$ and

$$\|X(g)\|_{p'/2} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \left(\int_1^2 \int_{\mathbf{S}^{n-1}} \Omega(\xi) g_k(x + \Psi(2^k t)\xi, t)\, d\sigma(\xi) \frac{dt}{t} \right)^2 |a(x)|\, dx.$$

Now, we need to consider two cases:

Case 1. $2n\delta/(2n + n\delta - 2) < p < 2$ and $q \ge 2$. In this case we have $2n/(2n-1) < p < 2$.

By Hölder's inequality, we have

$$\begin{aligned}(3.13) \qquad & \left(\int_1^2 \int_{\mathbf{S}^{n-1}} \Omega(\xi) g_k(x + \Psi(2^k t)\xi, t)\, d\sigma(\xi) \frac{dt}{t} \right)^2 \\ & \le \|\Omega\|_q^2 \int_1^2 \left(\int_{\mathbf{S}^{n-1}} |g_k(x + \Psi(2^k t)\xi, t)|^{q'}\, d\sigma(\xi) \right)^{2/q'} \frac{dt}{t} \\ & \le \|\Omega\|_q^2 \int_1^2 \int_{\mathbf{S}^{n-1}} |g_k(x + \Psi(2^k t)\xi, t)|^2\, d\sigma(\xi) \frac{dt}{t}.\end{aligned}$$

By (3.13) and a change of variable we get

$$\begin{aligned}& \|X(g)\|_{p'/2} \\ & \quad \le \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^2 |g_k(x,t)|^2 \left(\int_{\mathbf{S}^{n-1}} |a(x + \Psi(2^k t)\xi)|\, d\sigma(\xi) \right) \frac{dt}{t}\, dx \\ & \quad \le \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} \int_1^2 |g_k(x,t)|^2 \frac{dt}{t} \right) \mathscr{M}_S(|a|)(x)\, dx.\end{aligned}$$

By noticing that $(p'/2)' > n/(n-1)$, applying Hölder's inequality, the choice of a and Lemma 2.7 we get

$$\|X(g)\|_{p'/2} \le C$$

which in turn gives (3.12) for $2n\delta/(2n + n\delta - 2) < p < 2$ and $q \ge 2$.

Case 2. $2n\delta/(2n + n\delta - 2) < p < 2$ and $1 < q < 2$.

By Hölder's inequality, Fubini's theorem and a change of variable we have

$$
\begin{aligned}
&\|X(g)\|_{p'/2} \\
&\quad \le \|\Omega\|_q^q \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^2 \int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{2-q} |g_k(x+\Psi(2^k t)\xi, t)|^2 \, d\sigma(\xi) \frac{dt}{t} |a(x)| \, dx \\
&\quad \le C \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^2 |g_k(x,t)|^2 \left(\int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{2-q} |a(x+\Psi(2^k t)\xi)| \, d\sigma(\xi) \right) \frac{dt}{t} \, dx \\
&\quad \le C \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^2 |g_k(x,t)|^2 \left(\int_{\mathbf{S}^{n-1}} |a(x+\Psi(2^k t)\xi)|^{q'/2} \, d\sigma(\xi) \right)^{2/q'} \frac{dt}{t} \, dx \\
&\quad \le C \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} \int_1^2 |g_k(x,t)|^2 \frac{dt}{t} \right) (\mathscr{M}_S(|a|^{q'/2})(x))^{2/q'} \, dx \\
&\quad \le C \left\| \left(\sum_{k\in\mathbf{Z}} \int_1^2 |g_k(x,t)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{p'/2} \|(\mathscr{M}_S(|a|^{q'/2}))^{2/q'}\|_{(p'/2)'}.
\end{aligned}
$$

Since $(2/q')(p'/2)' > n/(n-1)$, by Lemma 2.7 we get

$$\|X(h)\|_{p'/2} \le C$$

which implies (3.12) for $2n\delta/(2n+n\delta-2) < p < 2$ and $1 < q < 2$. This completes the proof of Theorem 1.2 (b).

Proof of Theorem 1.1 (c). By duality, the operator $\mathscr{S}_\Omega$ is simply

$$\mathscr{S}_\Omega f(x) = \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} f(x-ty)\Omega(y) \, d\sigma(y) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

It is obvious that $\mathscr{S}_\Omega$ is bounded on $L^2(\mathbf{R}^n)$ if and only if the multiplier

$$m(\xi) = \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} e^{-2\pi i t \xi\cdot x} \Omega(x) \, d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \tag{3.14}$$

is an L^∞ function. It is easy to see that

$$m(\xi) = \lim_{N\to\infty,\, \varepsilon\to 0} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)} \int_{\varepsilon|\xi|}^{N|\xi|} \left(e^{-2\pi i t \xi'\cdot(x-y)} \frac{dt}{t} \right) d\sigma(x) \, d\sigma(y).$$

Notice that

$$\int_{\varepsilon|\xi|}^{N|\xi|} (e^{-2\pi i t\xi'\cdot(x-y)} - \cos(2\pi t)) \frac{dt}{t} \to \log|\xi'\cdot(x-y)|^{-1} - i\frac{\pi}{2} sgn(\xi'\cdot(x-y))$$

as $N \to \infty$ and $\varepsilon \to 0$, and the integral is bounded, uniformly in ε and N, $C(1+\log|\xi'\cdot(x-y)|)$. Therefore, using (1.1) and the Lebesgue dominated convergence theorem we obtain

$$m(\xi) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)}(\log|\xi'\cdot(y-x)|^{-1} - i\frac{\pi}{2}sgn(\xi'\cdot(y-x)))\,d\sigma(x)\,d\sigma(y).$$

Now, if Ω is a real-valued function, we have

$$m(\xi) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\Omega(y)(\log|\xi'\cdot(x-y)|^{-1})\,d\sigma(x)\,d\sigma(y).$$

Now, we are ready to prove part (c) of Theorem 1.1. For the sake of simplicity we shall present the construction of Ω only in the case $n=2$ and $q=\infty$. Other cases can be obtained by minor modifications. Also, we shall work on $[-1,1]$ instead of $\mathbf{S}^1$. We follow a similar argument as in [AA2]. For $x \in [-1,1]$, let

$$\Omega(x) = \sum_{k=1}^{\infty} \lambda_k b_k(x) \tag{3.15}$$

where $I_k = \left[\frac{1}{k+1}, \frac{1}{k}\right)$ for $k \geq 2$,

$$\lambda_1 = \sum_{k=2}^{\infty} \frac{1}{(k+1)(\log k)^{3/2}}, \quad b_1(x) = -\chi_{[-1,0]}(x),$$

$$\lambda_k = \frac{1}{(k+1)(\log k)^{3/2}}, \quad b_k(x) = |I_k|^{-1}\chi_{I_k}(x).$$

Then Ω has the desired properties. More precisely, Ω satisfies the following:

$$\int_{-1}^{1} \Omega(x)\,dx = 0; \tag{3.16}$$

$$\Omega \in B_{\infty}^{(0,\upsilon)}([-1,1]) \quad \text{for each } \upsilon,\ -1 < \upsilon < -\frac{1}{2}; \tag{3.17}$$

$$\Omega \notin B_{\infty}^{(0,-1/2)}([-1,1]); \tag{3.18}$$

$$S_1 = \int_{[0,1]^2} (\Omega(x)\Omega(y)\log|x-y|^{-1})\,dxdy = \infty; \tag{3.19}$$

$$S_2 = \int_{[-1,1]^2\setminus[0,1]^2} |\Omega(x)\Omega(y)\log|x-y|^{-1}|\,dxdy < \infty. \tag{3.20}$$

The proof of (3.16)–(3.17) is straightforward. Now we turn to the proof of (3.18). We first notice that each b_k is an ∞-block supported on the interval I_k. So to prove (3.18), we only need to show that $N_{\infty}^{(0,-1/2)}(\Omega) = \infty$. To this end, by Lemma 2.3 we have for each l,

$$\begin{aligned} N_\infty^{(0,-1/2)}(\Omega + \lambda_1 \chi_{[-1,0]}) &\geq \sum_{k=2}^{m} |\lambda_k|\,|I_k|^{-1} N_\infty^{(0,-1/2)}(\chi_{I_k}) \\ &\geq \sum_{k=2}^{m} |\lambda_k|(1+\log^{1/2}(|I_k|^{-1})). \end{aligned}$$

Letting $m \to \infty$, we get $N_\infty^{(0,-1/2)}(\Omega + \lambda_1 \chi_{[-1,0]}) = \infty$. Since, $N_\infty^{(0,-1/2)}(\lambda_1 \chi_{[-1,0]}) < \infty$ we get $N_\infty^{(0,-1/2)}(\Omega) = \infty$.

Now, we verify (3.19). Notice that for $(x,y) \in I_k \times I_j$, we have $\log|x-y|^{-1} \geq 0$. Therefore, we have

$$S_1 \geq \sum_{j=2}^{\infty} \sum_{k=2(j+1)}^{\infty} \frac{kj}{(\log k)^{3/2} (\log j)^{3/2}} \int_{I_k \times I_j} \log|x-y|^{-1}\, dxdy.$$

We notice that, for each $(x,y) \in I_k \times I_j$ with $k \geq 2(j+1)$, we have $y \geq 2x$. Thus $|x-y| = y - x \leq \frac{1}{j}$ and so $\log|x-y|^{-1} \geq \log j$ which in turn leads to

$$\begin{aligned} S_1 &\geq C \sum_{j=2}^{\infty} \frac{1}{j(\log j)^{1/2}} \left(\sum_{k=2(j+1)}^{\infty} \frac{1}{k(\log k)^{3/2}} \right) \\ &\geq C \sum_{j=2}^{\infty} \frac{1}{j(\log j)} = \infty. \end{aligned}$$

Finally, we verify (3.20). To this end, we divide the integral domain $[-1,1]^2 \backslash [0,1]^2$ into three parts: $[-1,0] \times [0,1]$, $[0,1] \times [-1,0]$, and $[-1,0] \times [-1,0]$. First, the integral over $[-1,0] \times [0,1]$ is dominated from above by

$$C \sum_{k=2}^{\infty} \frac{k}{(\log k)^{3/2}} \left| \int_{I_k} \int_{-1}^{0} \log|x-y|^{-1}\, dxdy \right|.$$

By straightforward calculations, we have

$$\left| \int_{I_k} \int_{-1}^{0} (\log|x-y|^{-1})\, dxdy \right| \leq C \frac{1}{k^2}$$

for some positive constant independent of k. Thus, we have

$$C \sum_{k=2}^{\infty} \frac{k}{(\log k)^{3/2}} \left| \int_{I_k} \int_{-1}^{0} \log|x-y|^{-1}\, dxdy \right| \leq C \sum_{k=2}^{\infty} \frac{1}{k(\log k)^{3/2}} < \infty.$$

Similarly, the integral over $[0,1] \times [-1,0]$ is finite. Finally, the integral over $[-1,0] \times [-1,0]$ is finite because $\left(\sum_{k=2}^{\infty} \frac{1}{(k+1)(\log k)^{3/2}} \right)^2 \chi_{[-1,0]\times[-1,0]} \in L^\infty$. This finishes the proof of Theorem 1.1 (c).

4. A further result

In this section, we are concerned with a maximal operator related to the Marcinkiewicz integral operator $\mu_{\Omega,h}$ which is defined by

$$\mu_{\Omega,h}f(x) = \left(\int_0^\infty \left|\int_{|y|\le t} f(x-y)\frac{\Omega(y/|y|)}{|y|^{n-1}}h(|y|)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where h is a measurable function on $\mathbf{R}^+$ and $\Omega \in L^1(\mathbf{S}^{n-1})$ is a function satisfying (1.1).

Marcinkiewicz integral operators have been investigated by many authors, dating back to the investigations of such operators by A. Zygmund on the circle and E. Stein on $\mathbf{R}^n$. For a sampling of past studies, see [St2], [BCP], [Wa], [AACP] and [AA2]. In particular, Al-Qassem and Al-Salman in [AA2] showed that $\mu_{\Omega,1}$ is bounded on $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ if $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ and there exists an Ω which lies in $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ for all $-1 < \upsilon < -\frac{1}{2}$ such that $\mu_{\Omega,1}$ is not bounded on $L^2(\mathbf{R}^n)$.

Motivated by the definition of $\mathscr{S}_\Omega$, a maximal operator S_Ω corresponding to $\mu_{\Omega,h}$ can be defined by

$$\mathscr{M}_\Omega f(x) = \sup_{h\in\mathscr{H}}\left(\int_0^\infty \left|\int_{|y|\le t} f(x-y)\frac{\Omega(y/|y|)}{|y|^{n-1}}h(|y|)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{4.1}$$

where $\Omega \in L^1(\mathbf{S}^{n-1})$ is a function satisfies the cancellation condition (1.1).

We have the following result concerning this maximal operator $\mathscr{M}_\Omega$:

Theorem 4.1. *Let $n \ge 2$ and $\mathscr{M}_\Omega$ be given as in* (4.1). *Then*

(a) *If $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ (for some $q > 1$) and satisfies* (1.1), *then $\mathscr{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for $2 \le p < \infty$;*

(b) *If $\Omega \in L^q(\mathbf{S}^{n-1})$ (for some $q > 1$) and satisfies* (1.1), *then $\mathscr{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for $2n\delta/(2n + n\delta - 2) < p < \infty$, where $\delta = \max\{2, q'\}$.*

Proof. By Minkowski's inequality we obtain

$$\begin{aligned}
&\left(\int_0^\infty \left|\int_{|y|\le t} f(x-y)\frac{\Omega(y')}{|y|^{n-1}}h(|y|)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad\le \left(\int_0^\infty \left(\int_0^\infty \left|\int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)\,d\sigma(y)\right| |h(s)|\chi_{[0,t]}(s)\,ds\right)^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad\le \int_0^\infty \left(\int_0^\infty \left|\int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)\,d\sigma(y)\right|^2 |h(s)|^2\chi_{[0,t]}(s)\frac{dt}{t^3}\right)^{1/2} ds
\end{aligned}$$

$$= \int_0^\infty \left| \int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)\, d\sigma(y) \right| |h(s)| \left(\int_s^\infty \frac{dt}{t^3} \right)^{1/2} ds$$
$$= \frac{1}{\sqrt{2}} \int_0^\infty \left| \int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)\, d\sigma(y) \right| |h(s)| \frac{ds}{s}.$$

Thus, by Hölder's inequality we have

$$\mathscr{M}_\Omega f(x) \leq \frac{1}{\sqrt{2}} \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)\, d\sigma(y) \right|^2 \frac{ds}{s} \right)^{1/2} = \frac{1}{\sqrt{2}} \mathscr{S}_\Omega f(x). \tag{4.2}$$

Therefore, Theorem 4.1 follows immediately from by Theorem 1.1.

References

[AA1] H. Al-Qassem and A. Al-Salman, L^p boundedness of a class of singular integral operators with rough kernels, Turkish J. Math. **25** (2001), 519–533.

[AA2] H. Al-Qassem and A. Al-Salman, A note on Marcinkiewicz integral operators, J. Math. Anal. Appl. **282** (2003), 698–710.

[AACP] A. Al-Salman, H. Al-Qassem, L. Cheng and Y. Pan, L^p bounds for the function of Marcinkiewicz, Math. Res. Lett. **9** (2002), 697–700.

[AAP1] H. Al-Qassem, A. Al-Salman and Y. Pan, Singular integrals associated to homogeneous mappings with rough kernels, Hokkaido Math. J. **33** (2004), 551–569.

[AAP2] H. Al-Qassem, A. Al-Salman and Y. Pan, Rough singular integrals with kernels supported by submanifolds of finite type, Asian J. Math. **9** (2005), 19–30.

[AP] H. Al-Qassem and Y. Pan, L^p estimates for singular integrals with kernels belonging to certain block spaces, Revista Matemática Iberoamericana **18** (2002), 701–730.

[BCP] A. Benedek, A. Calderón and R. Panzone, Convolution operators on Banach space valued functions, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365.

[Bo] J. Bourgain, Average in the plane over convex curves and maximal operators, J. Analyse Math. **47** (1986), 69–85.

[CZ] A. P. Calderón and A. Zygmund, On singular integrals, Amer. J. Math. **78** (1956), 289–309.

[CL] L. K. Chen and H. Lin, A maximal operator related to a class of singular integrals, Illi. Jour. Math. **34** (1990), 120–126.

[CW] R. Coifman and G. Weiss, Extension of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. **83** (1977), 569–645.

[Co] W. C. Connett, Singular integrals near L^1, Proc. Sympos. Pure. Math. (S. Wainger and G. Weiss, eds.) **35**, Amer. Math. Soc., Providence, RI, 1979, 163–165.

[DR] J. Duoandikoetxea and J. L. Rubio de Francia, Maximal functions and singular integral operators via Fourier transform estimates, Invent. Math. **84** (1986), 541–561.

[Fe] R. Fefferman, A note on singular integrals, Proc. Amer. Math. Soc. **74** (1979), 266–270.

[KS] M. Keitoku and E. Sato, Block spaces on the unit sphere in $\mathbf{R}^n$, Proc. Amer. Math. Soc. **119** (1993), 453–455.

[LTW] S. Lu, M. Taibleson and G. Weiss, Spaces generated by blocks, Beijing Normal University Press, Beijing, 1989.

[St1] E. M. Stein, On the functions of Littlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[St2] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, NJ, 1970.

[St3] E. M. Stein, Maximal functions: spherical means, Proc. Nat. Acad. Sci. USA **73** (1976), 2174–2175.

[St4] E. M. Stein, Harmonic analysis real-variable methods, orthogonality and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

[TW] M. H. Taibleson and G. Weiss, Certain function spaces associated with a.e. convergence of Fourier series, Univ. of Chicago Conf. in honor of Zygmund, Woodsworth, 1983.

[Wa] T. Walsh, On the function of Marcinkiewicz, Studia Math. **44** (1972), 203–217.

Department of Mathematics
Yarmouk University
Irbid
Jordan
E-mail: husseink@yu.edu.jo