S. JIANG
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# ANTI-MAGIC SQUARES OF EVEN ORDER

Sheng Jiang

**Abstract**

A systematic method for constructing anti-magic squares of every even order ($>2$) was found. This partially answered an open question asked by Gakuho Abe.

## 1. Introduction

An anti-magic square of order $n$ is an $n$ by $n$ matrix with entries $1, 2, \dots, n^2$ such that the set of all the sums of $n$ numbers in each row, each column and each main diagonal consists of consecutive numbers.

Gakuho Abe [1] collected several unsolved problems on magic squares, in which the last one is the following

Problem 2.23. Find a method of constructing an anti-magic square of every order.

The aim of this paper is to solve above problem for the case of even orders.

In section 2, we shall prove a classification theorem. It says that all anti-magic squares can be divided into two classes, called type $+$ and type $-$, for each class the consecutive sums are completely determined by order $n$.

Then in section 3, we consider the case $n = 4k$ $(k = 1, 2, 3, \dots)$, and give a method of constructing anti-magic squares for these $n$.

Finally, section 4 deals with the same topic for $n = 4k + 2$ $(k = 1, 2, 3, \dots)$, and obtain similar results. And hence Abe's problem 2.23 is solved for all even orders.

## 2. Classification

Let $A = (a_{ij})$ be an anti-magic square of order $n$. Denote by $r_i$ the sum of $n$ numbers in $i$-th row of $A$, $c_j$ the sum of $j$-th column, $d_1$ and $d_2$ the sums for each of main diagonals $(i, j = 1, 2, \dots, n)$, respectively. Let $S_0$ be the average of all $r_i$ $(i = 1, 2, \dots, n)$, then it is also the average of all $c_j$ $(j = 1, 2, \dots, n)$, and

$$S_0 = \frac{1}{2}n(n^2 + 1).$$

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Now we can prove the following

THEOREM 1. *Let* $A = (a_{ij})$ *be an anti-magic square of order* $n$. *Then in the* $(2n+2)$ *sums* $d_1, d_2, r_i$ *and* $c_j$ $(i, j = 1, 2, \ldots, n)$, *there is always a set of* $(2n+1)$ *sums consists of* $\{S_0, S_0 \pm 1, \ldots, S_0 \pm n\}$, *and the rest one is equal to either* $S_0 - (n+1)$ *or* $S_0 + (n+1)$.

*Proof.* Denote by

$$s = \frac{1}{2n+2}\left(d_1 + d_2 + \sum_{i=1}^{n} r_i + \sum_{j=1}^{n} c_j\right),$$

then

$$d_1 + d_2 = 2(n+1)s - n^2(n^2+1).$$

Let $\alpha$ and $\omega$ be the smallest one and largest one of $(2n+2)$ sums $d_1, d_2, r_i$ and $c_j$ $(i, j = 1, 2, \ldots, n)$, respectively, then

$$2s = \alpha + \omega = 2\alpha + 2n + 1.$$

Since

$$2\alpha + 1 \le d_1 + d_2 \le 2\omega - 1,$$

we have

$$\frac{1}{2}n(n^2-1) - \frac{3}{2} \le \alpha \le \frac{1}{2}n(n^2-1) + \frac{1}{2}.$$

But $\alpha$ is an integer, hence only two values are possible:

$$\alpha_1 = \frac{1}{2}n(n^2-1) - 1, \quad \alpha_2 = \frac{1}{2}n(n^2-1).$$

If $\alpha = \alpha_1$, then the $(2n+2)$ sums are $S_0, S_0 \pm 1, \ldots, S_0 \pm n$, and $S_0 - (n+1)$. If $\alpha = \alpha_2$, then the $(2n+2)$ sums are $S_0, S_0 \pm 1, \ldots, S_0 \pm n$, and $S_0 + (n+1)$.

*Remark*. According to Theorem 1, all anti-magic squares can be divided into two classes. For one class, each square has a sum $S_0 - (n+1)$, and will be called of type $-$. And the other class will be called of type $+$, in which every square has a sum $S_0 + (n+1)$.

The Theorem 1 is true for all possible orders. But in the following sections we shall see that even orders maybe are more interesting, because one can get a simple method to construct them.

## 3. Order $n = 4k$

Obviously, any square of order 2 cannot be anti-magic. Hence, for an anti-magic square of even order $n$, the smallest possible value of $n$ is 4.

In this section we consider the case $n = 4k$ $(k = 1, 2, 3, \ldots)$.

LEMMA 1. *There exist anti-magic squares of order* 4 *both for type* $-$ *and type* $+$.

*Proof.* Let $P$ be a $4 \times 4$ matrix with entries $1, 2, 3, \ldots, 16$, such that

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = (a_{\alpha\beta})$, $B = (b_{\alpha\beta})$, $C = (c_{\alpha\beta})$ and $D = (d_{\alpha\beta})$ are $2 \times 2$ submatrices satisfying

$$a_{\alpha 1} + a_{\alpha 2} = d_{1\beta} + d_{2\beta} = b_{\alpha\beta} + c_{\beta\alpha} = 17 \quad (\alpha, \beta = 1, 2).$$

By computer search, we get many examples of anti-magic squares in this special form, one is showed in Table 1. It is of type $-$ and denoted by $P^-$. The numbers out of frame show the sums of rows, columns and main diagonals of this square.

Table 1. $P^-$

| | | | | | |
|---|---|---|---|---|---|
| | 4 | 13 | 12 | 1 | 30 |
| | 11 | 6 | 2 | 14 | 33 |
| | 5 | 15 | 10 | 8 | 38 |
| | 16 | 3 | 7 | 9 | 35 |
| 34 | 36 | 37 | 31 | 32 | 29 |

Further, in the matrix $P^-$, changing rows of block $A$ as well as columns of block $D$, we get an anti-magic square of type $+$ as Table 2. Denote the new square by $P^+$.

Table 2. $P^+$

| | | | | | |
|---|---|---|---|---|---|
| | 11 | 6 | 12 | 1 | 30 |
| | 4 | 13 | 2 | 14 | 33 |
| | 5 | 15 | 8 | 10 | 38 |
| | 16 | 3 | 9 | 7 | 35 |
| 34 | 36 | 37 | 31 | 32 | 39 |

*Remark.* Squares $P^-$ and $P^+$ have the same corresponding row sums $r_i$ and column sums $c_j$ (for each $i$ and each $j$), as well as one diagonal sum $d_1$, but only different in the other's. Such a closely related pair will be called *twin* anti-magic squares.

THEOREM 2. *For $n = 4k$ $(k = 1, 2, 3, \ldots)$, there exist anti-magic squares of order $n$ both for type $-$ and type $+$.*

*Proof.* When $k = 1$, i.e. $n = 4$, we have squares $P^-$ and $P^+$ by Lemma 1.

When $k > 1$, we shall construct by bordered block matrix. For convenience, let us consider the case $n = 8$ in detail. The procedure contains three steps.

(1) Substituting $(1/2)((n-4)^2+1) = 17/2$ from every entries of $P^-$, we obtain a new square, denoted by $P^-_*$, as Table 3.

Table 3. $P^-_*$

| | | | | | |
|---|---|---|---|---|---|
| | −4.5 | 4.5 | 3.5 | −7.5 | −4 |
| | 2.5 | −2.5 | −6.5 | 5.5 | −1 |
| | −3.5 | 6.5 | 1.5 | −0.5 | 4 |
| | 7.5 | −5.5 | −1.5 | 0.5 | 1 |
| 0 | 2 | 3 | −3 | −2 | −5 |

(2) Now construct an $8 \times 8$ block matrix $M_8$ in the form

$$M_8 = \begin{pmatrix} A^* & E & B^* \\ F & P^-_* & G \\ C^* & H & D^* \end{pmatrix},$$

where $A^* = (a^*_{\alpha\beta})$, $B^* = (b^*_{\alpha\beta})$, $C^* = (c^*_{\alpha\beta})$ and $D^* = (d^*_{\alpha\beta})$ are $2 \times 2$ submatrices satisfying

$$a^*_{\alpha 1} = -a^*_{\alpha 2}, \quad d^*_{1\beta} = -d^*_{2\beta}, \quad b^*_{\alpha\beta} = -c^*_{\beta\alpha} \quad (\alpha, \beta = 1, 2);$$

$E = (e_{\alpha x})$, $H = (h_{\alpha x})$, ${}^tF = (f_{\alpha x})$ and ${}^tG = (g_{\alpha x})$ (here "$t$" denotes the transpose of matrix) are $2 \times 4$ submatrices satisfying

$$e_{1x} = -e_{2x}, \quad f_{1x} = -f_{2x}, \quad g_{1x} = -g_{2x}, \quad h_{1x} = -h_{2x}$$
$$(x = 1, 2, 3, 4).$$

Note that the numbers in blocks $E$, $F$, $G$ and $H$ are in the range $-31.5$ to $-8.5$ and $8.5$ to $31.5$.

Now let us set

$$\begin{aligned} \frac{n^2-1}{2} &= 31.5 = c^*_{21} = c^*_{12} + 1 = a^*_{12} + 2 \\ &= a^*_{21} + 3 = d^*_{11} + 4 = d^*_{22} + 5, \end{aligned}$$

and

$$\begin{aligned} b^*_{11} &= c^*_{21} - n = 23.5, \\ b^*_{22} &= b^*_{11} + 2 = 25.5. \end{aligned}$$

For the arragement of numbers in $E$, $H$ and ${}^tF$, the only requirement is that every sum of 4 numbers in any row equals zero, but the sums for ${}^tG$ are $\pm 1$. Thus $M_8$ is determined.

(3) Adding to each entries of $M_8$ by $(n^2+1)/2 = 32.5$, we get an anti-magic square of order 8 as Table 4, which is of type $-$ and denoted by $M^-_8$.

Table 4. $M_8^-$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 3 | 62 | 49 | 15 | 14 | 52 | 56 | 1 | 252 |
| | 61 | 4 | 16 | 50 | 51 | 13 | 2 | 58 | 255 |
| | 21 | 44 | 28 | 37 | 36 | 25 | 57 | 8 | 256 |
| | 43 | 22 | 35 | 30 | 26 | 38 | 10 | 55 | 259 |
| | 42 | 23 | 29 | 39 | 34 | 32 | 11 | 54 | 264 |
| | 24 | 41 | 40 | 27 | 31 | 33 | 53 | 12 | 261 |
| | 9 | 63 | 45 | 19 | 18 | 48 | 60 | 6 | 268 |
| | 64 | 7 | 20 | 46 | 47 | 17 | 5 | 59 | 265 |
| 260 | 267 | 266 | 262 | 263 | 257 | 258 | 254 | 253 | 251 |

Table 5. $M_8^+$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 61 | 4 | 49 | 15 | 14 | 52 | 56 | 1 | 252 |
| | 3 | 62 | 16 | 50 | 51 | 13 | 2 | 58 | 255 |
| | 21 | 44 | 35 | 30 | 36 | 25 | 57 | 8 | 256 |
| | 43 | 22 | 28 | 37 | 26 | 38 | 10 | 55 | 259 |
| | 42 | 23 | 29 | 39 | 32 | 34 | 11 | 54 | 264 |
| | 24 | 41 | 40 | 27 | 33 | 31 | 53 | 12 | 261 |
| | 9 | 63 | 45 | 19 | 18 | 48 | 6 | 60 | 268 |
| | 64 | 7 | 20 | 46 | 47 | 17 | 59 | 5 | 265 |
| 260 | 267 | 266 | 262 | 263 | 257 | 258 | 254 | 253 | 269 |

Changing rows or columns in suitable blocks of $M_8^-$, we obtain an anti-magic square of type $+$, denoted by $M_8^+$, as Table 5.

In a similar manner, for any $n = 4k$ $(k = 2, 3, \ldots)$, we can generate a pair of twin anti-magic squares $M_n^-$ and $M_n^+$ from $M_{n-4}^-$ or $M_{n-4}^+$.

## 4. Order $n = 4k + 2$

The rest even orders we need consider are of form $n = 4k + 2$ $(k = 1, 2, 3, \ldots)$.

LEMMA 2. *There exist anti-magic squares of order* 6 *both for type* $-$ *and type* $+$.

*Proof.* Using block matrix and by means of computer search, we get a pair of twin anti-magic squares of order 6. They are denoted by $Q^-$ and $Q^+$, and showed in Tables 6 and 7, respectively.

Table 6. $Q^{-}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 5 | 32 | 23 | 14 | 30 | 1 | 105 |
| | 31 | 6 | 16 | 21 | 2 | 33 | 109 |
| | 11 | 26 | 17 | 18 | 34 | 9 | 115 |
| | 25 | 12 | 19 | 20 | 3 | 28 | 107 |
| | 7 | 36 | 22 | 13 | 27 | 8 | 113 |
| | 35 | 4 | 15 | 24 | 10 | 29 | 117 |
| 111 | 114 | 116 | 112 | 110 | 106 | 108 | 104 |

Table 7. $Q^{+}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 31 | 6 | 23 | 14 | 30 | 1 | 105 |
| | 5 | 32 | 16 | 21 | 2 | 33 | 109 |
| | 11 | 26 | 17 | 18 | 34 | 9 | 115 |
| | 25 | 12 | 19 | 20 | 3 | 28 | 107 |
| | 7 | 36 | 22 | 13 | 8 | 27 | 113 |
| | 35 | 4 | 15 | 24 | 29 | 10 | 117 |
| 111 | 114 | 116 | 112 | 110 | 106 | 108 | 118 |

Theorem 3. *For $n = 4k+2$ $(k = 1, 2, 3, \ldots)$, there exist anti-magic squares of order $n$ both for type $-$ and type $+$.*

*Proof.* Similar to the construction in previous section (with some small modification), for every integer $k > 1$, one can generate a pair of twin anti-magic squares $M^{-}_{4k+2}$ and $M^{+}_{4k+2}$ from $M^{-}_{4k-2}$ or $M^{+}_{4k-2}$. Especially, $M^{-}_{10}$ and $M^{+}_{10}$ can be constructed from $Q^{-}$ or $Q^{+}$.

As an example, here we write out $M^{-}_{10}$ in detail, see Table 8.

Table 8. $M^{-}_{10}$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | 98 | 87 | 13 | 12 | 91 | 8 | 94 | 90 | 1 | 497 |
| | 97 | 4 | 14 | 88 | 89 | 10 | 93 | 7 | 2 | 92 | 496 |
| | 86 | 15 | 37 | 64 | 55 | 46 | 62 | 33 | 69 | 32 | 499 |
| | 16 | 85 | 63 | 38 | 48 | 53 | 34 | 65 | 31 | 70 | 503 |
| | 17 | 84 | 43 | 58 | 49 | 50 | 66 | 41 | 30 | 71 | 509 |
| | 83 | 18 | 57 | 44 | 51 | 52 | 35 | 60 | 72 | 29 | 501 |
| | 82 | 19 | 39 | 68 | 54 | 45 | 59 | 40 | 73 | 28 | 507 |
| | 20 | 81 | 67 | 36 | 47 | 56 | 42 | 61 | 27 | 74 | 511 |
| | 11 | 99 | 21 | 79 | 78 | 24 | 25 | 75 | 96 | 6 | 514 |
| | 100 | 9 | 80 | 22 | 23 | 77 | 76 | 26 | 5 | 95 | 513 |
| 505 | 515 | 512 | 508 | 510 | 506 | 504 | 500 | 502 | 495 | 498 | 494 |

Combining Theorems 2 and 3, we obtain the answer for the question 2.23 of Abe [1] in the case of all even orders.

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## References

[1] G. Abe, Unsolved problems on magic squares, Discrete Math., **127** (1994), 3–13.

Department of Mathematics
Yangzhou University
Yangzhou, Jiangsu 225002
P. R. China