J. Math. Soc. Japan
Vol. 72, No. 3 (2020) pp. 819–845
doi: 10.2969/jmsj/82128212

Finite-to-one zero-dimensional covers of dynamical systems

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(Received Feb. 5, 2019)

Abstract. In this paper, we study the existence of finite-to-one zero-dimensional covers of dynamical systems. Kulesza showed that any homeomorphism $f : X \to X$ on an n-dimensional compactum X with zero-dimensional set $P(f)$ of periodic points can be covered by a homeomorphism on a zero-dimensional compactum via an at most $(n+1)^n$-to-one map. Moreover, Ikegami, Kato and Ueda showed that in the theorem of Kulesza, the condition of at most $(n+1)^n$-to-one map can be strengthened to the condition of at most 2^n-to-one map. In this paper, we will show that the theorem is also true for more general maps except for homeomorphisms. In fact we prove that the theorem is true for a class of maps containing two-sided zero-dimensional maps. For the special case, we give a theorem of symbolic extensions of positively expansive maps. Finally, we study some dynamical zero-dimensional decomposition theorems of spaces related to such maps.

1. Introduction.

A pair (X, f) is called a *dynamical system* if X is a compact metric space (= compactum) and $f : X \to X$ is a map on X. A dynamical system $(Z, \tilde{f})$ *covers* (X, f) *via* a map $p : Z \to X$ provided that p is an onto map and the following diagram is commutative, i.e., $p\tilde{f} = fp$.

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{f}} & Z \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{f} & X \end{array}$$

Note that (X, f) is also called a *factor* of $(Z, \tilde{f})$ and conversely $(Z, \tilde{f})$ is called a *cover* (or an *extension*) of (X, f). We call the map $p : Z \to X$ a *factor mapping.* If Z is zero-dimensional, then we say that the dynamical system $(Z, \tilde{f})$ is a *zero-dimensional cover* of (X, f). Moreover, if the factor mapping is a finite-to-one map, then we say that the dynamical system $(Z, \tilde{f})$ is a *finite-to-one zero-dimensional cover* of (X, f).

The (symbolic) dynamical systems on Cantor sets have been studied by many mathematicians and also the strong relations between Markov partitions and symbolic dynamics have been studied (e.g., see [**1**], [**3**], [**4**], [**5**], [**11**], [**16**, Proposition 3.19] and [**18**]). In [**1**], Anderson proved that for any dynamical system (X, f), there exists a zero-dimensional cover $(Z, \tilde{f})$ of (X, f), and moreover in [**4**, Theorem A.1] Boyle, Fiebig and Fiebig proved that any dynamical system (X, f) has a zero-dimensional cover $(Z, \tilde{f})$ such that the

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2010 *Mathematics Subject Classification.* Primary 37B10; Secondary 37B45, 37C45, 54H20, 54F45.
Key Words and Phrases. dynamical systems, zero-dimensional covers (extensions) of dynamical systems, finite-to-one map, dimension, periodic point, general position, positively expansive map.

topological entropy $h(f)$ of f is equal to $h(\tilde{f})$, where the factor mappings are not necessarily finite-to-one (see Remark 1 below). In topology, there is a classical theorem by Hurewicz [**8**] that any compactum X is at most n-dimensional if and only if there is a zero-dimensional compactum Z with an onto map $p : Z \to X$ whose fibers have cardinality at most $n+1$. In the theory of dynamical systems, we have the related general problem (e.g., see [**3**], [**4**], [**10**] and [**15**]):

Problem 1.1. What kinds of dynamical systems can be covered by zero-dimensional dynamical systems via finite-to-one maps?

The motivation for this problem comes from (symbolic) dynamics on Cantor sets. To study dynamical properties of the original dynamics (X, f), the finiteness of the fibers of the factor mapping may be very important and so, in this paper we focus on the finiteness of fibers of factor mappings. Related to Problem 1.1, first Kulesza [**15**] proved the following significant theorem:

Theorem 1.2 (Kulesza [**15**]). *For each homeomorphism f on an n-dimensional compactum X with zero-dimensional set $P(f)$ of periodic points, there is a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most $(n+1)^n$-to-one map such that $\tilde{f} : Z \to Z$ is a homeomorphism.*

He also showed that Problem 1.1 needs the assumption $\dim P(f) \leq 0$. In fact, for the disk $X = [0,1]^2$ or some 1-dimensional continuum X, there is a dynamical system (X, f) such that $f : X \to X$ is a homeomorphism on X with $\dim P(f) = 1$ and (X, f) has no zero-dimensional cover via a finite-to-one map (see the proof of Example 2.2 and Remark 2.3 of [**15**]). In [**10**] Ikegami, Kato and Ueda improved the theorem of Kulesza as follows: The condition of at most $(n+1)^n$-to-one map can be strengthened to the condition of at most 2^n-to-one map.

The aim of this paper is to give a partial answer to Problem 1.1. In fact, we show that the above theorem is also true for a class of maps containing two-sided zero-dimensional maps (see Main Theorem 3.18). For the special case that (X, f) is a positively expansive dynamical system with $\dim X = n$, (X, f) can be covered by a subshift (Σ, σ) of the shift map $\sigma : \{1, 2, \dots, k\}^\infty \to \{1, 2, \dots, k\}^\infty$ via an at most 2^n-to-one map. Also, we study some dynamical zero-dimensional decomposition theorems of spaces related to such maps. For the proofs, we need more general and careful arguments than the arguments of [**7**], [**9**] and [**10**]. In this paper, for completeness we give the precise proofs.

2. Preliminaries.

In this paper, all spaces are separable metric spaces and maps are continuous functions. Let $\mathbb{N}$ be the set of all natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z}$ the set of all integers and $\mathbb{Z}_+$ the set of all nonnegative integers, i.e., $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ $(= \{0, 1, 2, \dots\})$. Also, let $\mathbb{R}$ be the real line. If K is a subset of a space X, then $\mathrm{cl}(K)$, $\mathrm{bd}(K)$ and $\mathrm{int}(K)$ denote the closure, the boundary and the interior of K in X, respectively. A subset A of a space X is an F_σ-set of X if A is a countable union of closed subsets of X. Also, a subset B of X is a G_δ-set of X if B is an intersection of countable open subsets of X. A subset B of a space X is *residual* in X if B contains a dense G_δ-set of X. For a space X, $\dim X$ means the topological (covering) dimension of X (e.g., see [**6**]). For a collection $\mathcal{C}$ of subsets of X, we put

$$\mathrm{ord}(\mathcal{C}) = \sup\{\mathrm{ord}_x\mathcal{C} \mid x \in X\},$$

where $\mathrm{ord}_x\mathcal{C}$ is the number of members of $\mathcal{C}$ which contains x. A closed set K in X is *regular closed* in X if $\mathrm{cl}(\mathrm{int}(K)) = K$. A collection $\mathcal{C}$ of regular closed sets in X is called a *regular closed partition* of X provided that $\mathcal{C}$ is a finite family,

$$\bigcup\mathcal{C}\left(=\bigcup\{C \mid C \in \mathcal{C}\}\right) = X$$

and $C \cap C' = \mathrm{bd}(C) \cap \mathrm{bd}(C')$ for each $C, C' \in \mathcal{C}$ with $C \neq C'$. For regular closed partitions $\mathcal{A}$ and $\mathcal{B}$ of X, we consider the following family of closed sets of X;

$$\mathcal{A}@\mathcal{B} = \{\mathrm{cl}[\mathrm{int}(A) \cap \mathrm{int}(B)] \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$

Then we have the following proposition.

Proposition 2.1. *For regular closed partitions $\mathcal{A}$ and $\mathcal{B}$ of X, $\mathcal{A}@\mathcal{B}$ is a regular closed partition of X.*

Proof. First, note that if U is an open set of X, then $\mathrm{cl}(U)$ is regular closed, because that $\mathrm{cl}(U) \subset \mathrm{cl}[\mathrm{int}(\mathrm{cl}(U))] \subset \mathrm{cl}(U)$. Also note that the collection

$$\{\mathrm{int}(A) \cap \mathrm{int}(B) \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is a finite family of mutually disjoint open sets of X. We will prove

$$X = \bigcup\{\mathrm{cl}[\mathrm{int}(A) \cap \mathrm{int}(B)] \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$

Let $x \in X$. Then there is $A \in \mathcal{A}$ with $x \in A = \mathrm{cl}(\mathrm{int}(A))$. We can find a sequence $\{x_n\}_{n=1}^{\infty}$ of points of $\mathrm{int}(A)$ such that $x = \lim_{n\to\infty} x_n$. Since $\mathcal{B}$ is a finite family, we may assume that there is $B \in \mathcal{B}$ with

$$[B(x_n, 1/n) \cap \mathrm{int}(A)] \cap \mathrm{int}(B) \neq \emptyset$$

for each $n \in \mathbb{N}$, where $B(x_n, 1/n)$ is the $1/n$-neighborhood of x_n in X. Then $x \in \mathrm{cl}[\mathrm{int}(A) \cap \mathrm{int}(B)]$ and we see that

$$X = \bigcup\{\mathrm{cl}[\mathrm{int}(A) \cap \mathrm{int}(B)] \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$

Hence $\mathcal{A}@\mathcal{B}$ is a regular closed partition of X. □

Note that $\mathrm{ord}(\mathcal{A}@\mathcal{B}) \leq \mathrm{ord}(\mathcal{A}) \cdot \mathrm{ord}(\mathcal{B})$. A collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of X is called a *swelling* of a collection $\{B_\lambda\}_{\lambda\in\Lambda}$ of subsets of X provided that $B_\lambda \subset A_\lambda$ for each $\lambda \in \Lambda$, and if for any $m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in \Lambda$, we have

$$\bigcap_{i=1}^{m} A_{\lambda_i} \neq \emptyset \text{ if and only if } \bigcap_{i=1}^{m} B_{\lambda_i} \neq \emptyset.$$

Conversely, a family $\{B_\lambda\}_{\lambda\in\Lambda}$ of subsets of X is called a *shrinking* of a cover $\{A_\lambda\}_{\lambda\in\Lambda}$ of X if $\{B_\lambda\}_{\lambda\in\Lambda}$ is a cover of X and $B_\lambda \subset A_\lambda$ for each $\lambda \in \Lambda$.

Let X and Y be compacta. A map $f : X \to Y$ is *zero-dimensional* if $\dim f^{-1}(y) \leq 0$ for each $y \in Y$. A map $f : X \to Y$ is a *zero-dimension preserving map* if for any zero-dimensional closed subset D of X, $\dim f(D) \leq 0$. Also a map $f : X \to X$ is *two-sided zero-dimensional* if f is zero-dimensional and zero-dimension preserving, i.e., for any zero-dimensional closed subset D of X, $\dim f^{-1}(D) \leq 0$ and $\dim f(D) \leq 0$. In this case, note that if Z is a zero-dimensional F_σ-subset of X, then $\dim f(Z) = 0$ (see Proposition 3.1). A map $f : X \to Y$ is *semi-open* (or *quasi-open*) if for any nonempty open set U of X, $f(U)$ contains a nonempty open set of Y, i.e., $\mathrm{int} f(U) \neq \emptyset$. An onto map $p : X \to Y$ is *at most k-to-one* ($k \in \mathbb{N}$) if for any $y \in Y$, $|p^{-1}(y)| \leq k$.

For a map $f : X \to X$, a subset A of X is *f-invariant* if $f(A) \subset A$. We define the set

$$O(x) = \{f^p(x) \mid p \in \mathbb{Z}_+\}$$

which denotes the (positive) orbit of x. Similarly we define the *eventual orbit* of $x \in X$:

$$\begin{aligned} EO(x) &= \{z \in X \mid \text{there exist } \ i, j \in \mathbb{Z}_+ \text{ such that } f^i(x) = f^j(z)\} \\ &= \{z \in X \mid \text{there exists } \ j \in \mathbb{Z}_+ \text{ such that } f^j(z) \in O(x)\}. \end{aligned}$$

Note that

$$EO(x) = \bigcup_{i,j\in\mathbb{Z}_+} f^{-j}(f^i(x)),$$

the family $\{EO(x) \,|\, x \in X\}$ is a decomposition of X and $EO(x)$ is f-invariant, i.e., $f(EO(x)) \subset EO(x)$. Let $P(f)$ be the set of all periodic points of f;

$$P(f) = \{x \in X \mid f^j(x) = x \text{ for some } j \in \mathbb{N}\}.$$

A point $x \in X$ is *eventually periodic* if there is some $p \in \mathbb{Z}_+$ such that $f^p(x) \in P(f)$. Let $EP(f)$ be the set of all eventually periodic points of f;

$$EP(f) = \bigcup_{p=0}^{\infty} f^{-p}(P(f)).$$

Note that $P(f)$ and $EP(f)$ are F_σ-sets of X. In [**14**], Krupski, Omiljanowski and Ungeheuer showed that the set of maps $f : X \to X$ with zero-dimensional sets $CR(f)$ of all chain recurrent points is a dense G_δ-set of the mapping space $C(X, X)$ if X is a (compact) polyhedron. Note that a point $x \in X$ is a *chain recurrent point* of f if for any $\epsilon > 0$ there is a finite sequence $x = x_0, x_1, \ldots, x_m = x$ of points of X such that $d(f(x_i), x_{i+1}) < \epsilon$ for each $i = 0, 1, \ldots, m-1$. Since $P(f) \subset CR(f)$, we see that the set of maps $f : X \to X$ with zero-dimensional sets $P(f)$ of all periodic points is residual in the mapping space $C(X, X)$ if X is a compact polyhedron. Hence almost all maps on compact polyhedra have zero-dimensional sets of periodic points.

Let X be a compactum and $\mathcal{U}, \mathcal{V}$ be two covers of X. Put

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The quantity $N(\mathcal{U})$ denotes minimal cardinality of subcovers of $\mathcal{U}$. Let $f : X \to X$ be a map and let $\mathcal{U}$ be an open cover of X. Put

$$h(f, \mathcal{U}) = \lim_{n\to\infty} \frac{\log N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n+1}(\mathcal{U}))}{n}.$$

The *topological entropy of* f, denoted by $h(f)$, is the supremum of $h(f, \mathcal{U})$ for all open covers $\mathcal{U}$ of X. Positive topological entropy of maps is one of generally accepted definitions of chaos.

3. Zero-dimensional covers.

In this section, we study zero-dimensional covers of some dynamical systems. First, we need the following well-known results of dimension theory. For dimension theory, e.g., see [**6**], [**19**], [**20**].

Proposition 3.1 ([**6**, Theorems 1.5.3 and 1.5.11]).

(1) *If* $\{F_i \mid i \in \mathbb{N}\}$ *is a sequence of closed subsets of a separable metric space* X *with* $\dim F_i \le n$*, then*

$$\dim \left(\bigcup_{i=1}^{\infty} F_i \right) \le n.$$

(2) *If* M *is a subset of a separable metric space* X *with* $\dim M \le n$*, then there is a* G_δ*-set* M^* *of* X *such that* $M \subset M^*$ *and* $\dim M^* \le n$*.*

Proposition 3.2. *If* X *is a separable metric space with* $\dim X \le n$ $(1 \le n < \infty)$ *and* E *is an* F_σ*-set of* X *with* $\dim E \le n-1$*, then there exists a zero-dimensional* F_σ*-set* Z *of* X *such that*

$$Z \cap E = \emptyset,\ \dim(X - Z) \le n - 1.$$

Proof. Let $\mathcal{B}$ be a countable open base of X such that if $B \in \mathcal{B}$, then $\dim \mathrm{bd}(B) \le n-1$. Let $Y = \bigcup\{\mathrm{bd}(B) \mid B \in \mathcal{B}\} \cup E$. By (1) of Proposition 3.1, $\dim Y \le n-1$. By (2) of Proposition 3.1, we have a G_δ-set Y^* such that $Y \subset Y^*$ and $\dim Y^* \le n-1$. Put $Z = X - Y^*$. Then Z satisfies the desired property. □

Proposition 3.3 ([**6**, Theorem 1.5.13]). *Let* M *be a subset of a separable metric space* X *and* $\dim M \le n$*. For any disjoint closed subsets* A, B *of* X*, there exists a partition* L *between* A *and* B *such that* $\dim(M \cap L) \le n-1$*. In particular, if* $\dim M \le 0$*, there exists a partition* L *between* A *and* B *such that* $M \cap L = \emptyset$*.*

Theorem 3.4 (Hurewicz's theorem [**6**, Theorem 4.3.4]). *If $f : X \to Y$ is a closed map between separable metric spaces and there is $k \geq 0$ such that $\dim f^{-1}(y) \leq k$ for each $y \in Y$, then $\dim X \leq \dim Y + k$.*

We say a collection $\mathcal{G}$ of subsets of a compactum X with $\dim X = n < \infty$ is *in general position* provided that if $\mathcal{S} \subset \mathcal{G}$ and $1 \leq |\mathcal{S}| \leq n+1$, then $\dim(\bigcap \mathcal{S}) \leq n - |\mathcal{S}|$, where $\bigcap \mathcal{S} = \bigcap\{S \mid S \in \mathcal{S}\}$ and $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. We find the following two results in [**10**]. For completeness, we give the proofs again.

Lemma 3.5 ([**10**, Lemma 3.2]). *Let X be a compactum with $\dim X = n < \infty$. Suppose that for any $j \in \mathbb{N}$, $\mathcal{G}(j)$ is a finite collection of F_σ-sets of X and $\mathcal{G}(j)$ is in general position. Then there is a zero-dimensional F_σ-set Z of X such that if A is a subset of X with $A \cap Z = \emptyset$, then $\mathcal{G}(j) \cup \{A\}$ is in general position for each $j \in \mathbb{N}$.*

Proof. Let $j \in \mathbb{N}$. Note that $\{\bigcap \mathcal{S} \mid \mathcal{S} \subset \mathcal{G}(j) \text{ and } \bigcap \mathcal{S} \neq \emptyset\}$ is a finite collection such that each element is an F_σ-set in X. By Proposition 3.2, we choose a zero-dimensional F_σ-set Z' of X such that $\dim(X - Z') \leq n - 1$. Also, for each $\mathcal{S} \subset \mathcal{G}(j)$ with $\bigcap \mathcal{S} \neq \emptyset$, we can choose a zero-dimensional F_σ-set $Z_{\mathcal{S}}$ of $\bigcap \mathcal{S}$ such that

$$\dim\left(\bigcap \mathcal{S} - Z_{\mathcal{S}}\right) \leq \dim\left(\bigcap \mathcal{S}\right) - 1.$$

Note that $\{Z_{\mathcal{S}} \mid \mathcal{S} \subset \mathcal{G}(j) \text{ and } \bigcap \mathcal{S} \neq \emptyset\}$ is a finite collection of zero-dimensional F_σ-sets of X. Then

$$Z = \bigcup \left\{ Z_{\mathcal{S}} \,\middle|\, j \in \mathbb{N},\ \mathcal{S} \subset \mathcal{G}(j) \text{ and } \bigcap \mathcal{S} \neq \emptyset \right\} \cup Z'$$

is also a zero-dimensional F_σ-set of X. We will show that Z is a desired set. Now suppose $A \subset X$ with $A \cap Z = \emptyset$. Let $\mathcal{S} \subset \mathcal{G}(j) \cup \{A\}$ such that $1 \leq |\mathcal{S}| \leq n+1$ and $\bigcap \mathcal{S} \neq \emptyset$. We may assume that $A \in \mathcal{S}$. If $|\mathcal{S}| = 1$, then

$$\dim\left(\bigcap \mathcal{S}\right) = \dim A \leq \dim(X - Z') \leq n - 1.$$

On the other hand, suppose $2 \leq |\mathcal{S}| \leq n+1$. Since $\mathcal{S} - \{A\} \subset \mathcal{G}(j)$ and $1 \leq |\mathcal{S} - \{A\}| \leq n$ and $\bigcap(\mathcal{S} - \{A\}) \neq \emptyset$, we see that

$$\begin{aligned}\dim\left(\bigcap \mathcal{S}\right) &= \dim\left[\bigcap(\mathcal{S} - \{A\}) \cap A\right] \leq \dim\left[\bigcap(\mathcal{S} - \{A\}) - Z_{(\mathcal{S} - \{A\})}\right] \\ &\leq \dim\left[\bigcap(\mathcal{S} - \{A\})\right] - 1 \leq (n - |\mathcal{S} - \{A\}|) - 1 \\ &= n - (|\mathcal{S}| - 1) - 1 = n - |\mathcal{S}|.\end{aligned}$$

Therefore $\mathcal{G}(j) \cup \{A\}$ is in general position for any $j \in \mathbb{N}$. □

Lemma 3.6 ([**10**, Lemma 3.3]). *Let $\mathcal{C} = \{C_i \mid 0 \leq i \leq m\}$ be a finite open cover of a compactum X with $\dim X = n < \infty$, and let $\mathcal{B} = \{B_i \mid 0 \leq i \leq m\}$ be a closed shrinking of $\mathcal{C}$. Suppose that O is an open set of X, Z is an at most zero-dimensional F_σ-set of O, and for each $j \in \mathbb{N}$, $\mathcal{G}(j)$ is a finite collection of F_σ-subsets of O such that* *each $\mathcal{G}(j)$ is in general position. Then there is an open shrinking $\mathcal{C}' = \{C'_i \mid 0 \le i \le m\}$ of $\mathcal{C}$ such that for each $0 \le i \le m$,*

(1) $B_i \subset C'_i \subset C_i$,
(2) $C'_i = C_i$ *if* $\mathrm{bd}(C_i) \cap O = \emptyset$,
(3) $C'_i - O = C_i - O$,
(4) $\mathrm{bd}(C'_i) - O \subset \mathrm{bd}(C_i) - O$,
(5) $\mathrm{bd}(C'_i) \cap Z = \emptyset$,
(6) $\mathcal{G}(j) \cup \{\mathrm{bd}(C'_i) \cap O \mid 0 \le i \le m\}$ *is in general position for any* $j \in \mathbb{N}$.

Proof. Without loss of generality, we may assume that $B_0 = C_0 = \emptyset$ $(i = 0)$. We will construct C'_k by induction on $k = 0, 1, \ldots, m$. For the case $k = 0$, we put $C'_0 = \emptyset$. Next we assume that there is $\{C'_i \mid i < k\}$ $(k \le m)$ satisfying the conditions (1)–(5) and

$$\mathcal{G}(j) \cup \{\mathrm{bd}(C'_i) \cap O \mid i < k\}$$

is in general position for each $j \in \mathbb{N}$. We will construct C'_k as follows. By Lemma 3.5, there is a zero-dimensional F_σ-set Z_k of O such that if $A \subset O$ and $A \cap Z_k = \emptyset$, then $\mathcal{G}(j) \cup \{\mathrm{bd}(C'_i) \cap O \mid i < k\} \cup \{A\}$ is in general position for each $j \in \mathbb{N}$. Consider the following open subspace of X:

$$Y_k = X - [\mathrm{bd}(C_k) - O].$$

Also consider the following closed set of Y_k:

$$B'_k = [B_k \cup (\mathrm{cl}(C_k) - O)] \cap Y_k = B_k \cup (C_k - O).$$

Since $\dim(Z_k \cup Z) \le 0$, by Proposition 3.3 we can choose an open set C'_k of Y_k such that $B'_k \subset C'_k \subset \mathrm{cl}_{Y_k}(C'_k) \subset C_k$ and $\mathrm{bd}_{Y_k}(C'_k) \cap (Z_k \cup Z) = \emptyset$. Then C'_k is an open set of X. Note that

$$\mathrm{bd}_{Y_k}(C'_k) \subset O,\ \mathrm{bd}(C'_k) \subset \mathrm{bd}_{Y_k}(C'_k) \cup (\mathrm{bd}(C_k) - O).$$

Also, note that

$$C'_k - O = B'_k - O = C_k - O,$$
$$\mathrm{bd}(C'_k) - O \subset \mathrm{bd}(C_k) - O,$$
$$\mathrm{bd}(C'_k) \cap O \subset \mathrm{bd}_{Y_k}(C'_k) \cap O.$$

Hence $(\mathrm{bd}(C'_k) \cap O) \cap (Z_k \cup Z) = \emptyset$. By the construction, we see that C'_k satisfies the conditions (1)–(5), and $\mathcal{G}(j) \cup \{\mathrm{bd}(C'_i) \cap O \mid i \le k\}$ is in general position for each $j \in \mathbb{N}$.

By the induction on k, we obtain the desired open shrinking

$$\mathcal{C}' = \{C'_i \mid 0 \le i \le m\}$$

of $\mathcal{C}$. □

Lemma 3.7. *Suppose that $f : X \to X$ is a map of a compactum X. If $x \notin P(f)$,* *then* $f^{-i}(x) \cap f^{-j}(x) = \emptyset = f^{-i}(x) \cap f^{j}(\{x\})$ *for any* $i, j \geq 0$ *with* $i \neq j$*. Moreover if* $x \notin EP(f)$*,* $f^{i}(\{x\}) \cap f^{j}(\{x\}) = \emptyset$ *for any* $i, j \in \mathbb{Z}$ *with* $i \neq j$*.*

Proof. Suppose that for $i, j \geq 0$ with $i \neq j$, $f^{-i}(x) \cap f^{-j}(x) \neq \emptyset$. We may assume $i > j$. Take $y \in f^{-i}(x) \cap f^{-j}(x)$. Then

$$x = f^{i}(y) = f^{i-j}(f^{j}(y)) = f^{i-j}(x).$$

Hence $x \in P(f)$. Similarly, if $f^{-i}(x) \cap f^{j}(\{x\}) \neq \emptyset$, then $f^{i+j}(x) = x$ and hence $x \in P(f)$. Similarly, if $f^{i}(x) = f^{j}(x)$, $f^{i-j}(f^{j}(x)) = f^{i}(x) = f^{j}(x)$ and hence $f^{j}(x) \in P(f)$. Consequently $x \in EP(f)$. $\square$

Lemma 3.8. *Suppose that* $f : X \to X$ *is a zero-dimensional map of a compactum* X*. Then* $EP(f)$ *is an* F_σ*-set of* X *with* $\dim EP(f) = \dim P(f)$*.*

Proof. Note that $P(f)$ is an F_σ-set of X and hence $EP(f)$ is so. Since f is a zero-dimensional map, we see that for each $n \in \mathbb{N}$, $\dim f^{-n}(P(f)) \leq \dim P(f)$. Note that $P(f) \subset EP(f)$. By Proposition 3.1, we see that $\dim EP(f) = \dim P(f)$. $\square$

Proposition 3.9. *If* $f : X \to X$ *is a two-sided zero-dimensional onto map of a compactum* X*, then for any closed subset* A *of* X*,* $\dim A = \dim f(A) = \dim f^{-1}(A)$*. Moreover, if* A *is an* F_σ*-set of* X*, then* $\dim A = \dim f(A) = \dim f^{-1}(A)$*.*

Proof. Note that $\dim f^{-1}(y) \leq 0$ for each $y \in X$. By Hurewicz's theorem, we see that $\dim f^{-1}(A) \leq \dim A \leq \dim f(A)$. By induction on $\dim A \leq k$, we will prove $\dim f(A) \leq k$. For $\dim A \leq k = 0$, by the definition we see that $\dim f(A) \leq 0$. For $k - 1$ ($k \geq 1$), we assume that the claim is true. We will prove that the claim for k is true. Let A be a closed set of X with $\dim A \leq k$. By Proposition 3.2, we choose an F_σ-set Z of A such that $\dim Z = 0$ and $\dim(A - Z) \leq k - 1$. Since $\dim f(Z) = 0$, we can choose a zero-dimensional G_δ-set $\tilde{Z}$ of $f(A)$ with $f(Z) \subset \tilde{Z}$ (see (2) of Proposition 3.1). Note that $[f^{-1}(f(A) - \tilde{Z}) \cap A]$ ($\subset A - Z$) is an F_σ-set of A and $\dim[f^{-1}(f(A) - \tilde{Z}) \cap A] \leq k - 1$. By the assumption, we see that

$$\dim(f(A) - \tilde{Z}) = \dim f(f^{-1}(f(A) - \tilde{Z}) \cap A) \leq k - 1.$$

Since $f(A) = (f(A) - \tilde{Z}) \cup \tilde{Z}$, by [**6**, (1.5.7)] we see $\dim f(A) \leq k$. Consequently, we see that $\dim A = \dim f(A)$. Also note that $\dim f^{-1}(A) = \dim f(f^{-1}(A)) = \dim A$. By Proposition 3.1, we see that this result is true for the case of F_σ-sets of X. $\square$

Proposition 3.10. *Let* $f : X \to X$ *be a two-sided zero-dimensional map of a compactum* X *and* $i_j \in \mathbb{Z}_+$ ($j = 0, 1, \ldots, k$)*. Suppose that* M_{i_j} ($j = 0, 1, \ldots, k$) *are* F_σ*-sets of* X *and* A, B *are disjoint closed subsets of* X*. Then there exists a partition* L *between* A *and* B *in* X *such that*

$$\dim(M_{i_j} \cap f^{-i_j}(L)) \leq \dim M_{i_j} - 1$$

for each j*.*

PROOF. By Proposition 3.2, we can find zero-dimensional F_σ-sets Z_{i_j} ($j = 0, 1, 2, \ldots, k$) of M_{i_j} such that $\dim(M_{i_j} - Z_{i_j}) \leq \dim M_{i_j} - 1$. Note that Z_{i_j} are F_σ-sets of X. Since f is two-sided zero-dimensional, $\dim f^{i_j}(Z_{i_j}) \leq 0$. Since $\bigcup_{j=0}^{k} f^{i_j}(Z_{i_j})$ is zero-dimensional, we can find a partition L between A and B such that

$$L \cap \left[\bigcup_{j=0}^{k} f^{i_j}(Z_{i_j}) \right] = \emptyset.$$

Note that $f^{-i_j}(L) \cap Z_{i_j} = \emptyset$. Hence we see that

$$\dim(M_{i_j} \cap f^{-i_j}(L)) \leq \dim M_{i_j} - 1$$

for each j. □

LEMMA 3.11 (cf. [**10**, Lemma 3.4]). *Let $f : X \to X$ be a two-sided zero-dimensional map of a compactum X such that $\dim X = n < \infty$ and $\dim P(f) \leq 0$. Let F be an F_σ-set of X with $\dim F \leq 0$. Suppose that $\mathcal{C} = \{C_i \mid 1 \leq i \leq M\}$ is a finite open cover of X and let $\mathcal{B} = \{B_i \mid 1 \leq i \leq M\}$ be a closed shrinking of $\mathcal{C}$. Then for each $k = 0, 1, 2, \ldots,$ there is an open shrinking $\mathcal{C}'(k) = \{C'_i \mid 1 \leq i \leq M\}$ of $\mathcal{C}$ such that for each $1 \leq i \leq M$,*

(1) *$B_i \subset C'_i \subset C_i$,*

(2) *$\{f^{-p}(\mathrm{bd}(C'_i)) \mid 1 \leq i \leq M, p = 0, 1, \ldots, k\}$ is in general position,*

(3) *$\mathrm{bd}(C'_i) \cap (EP(f) \cup F) = \emptyset$ for each i.*

PROOF. We will construct an open shrinking $\mathcal{C}'(k)$ of $\mathcal{C}$ by induction on k. Note that $\dim EP(f) = \dim P(f) \leq 0$, because that f is zero-dimensional. In the case $k = 0$, we put $O = X$ and $Z = EP(f) \cup F$. Note that $\dim Z \leq 0$. By Lemma 3.6 there is an open shrinking $\mathcal{C}'(0) = \{C'_i \mid 1 \leq i \leq M\}$ of $\mathcal{C}$ such that the conditions (1)–(3) hold. Next we suppose the result holds for $k - 1$. Then there is an open shrinking $\mathcal{D}(= \mathcal{C}'(k-1)) = \{D_i \mid 1 \leq i \leq M\}$ of $\mathcal{C}$ such that for each i,

(1') $B_i \subset D_i \subset C_i$,

(2') $\{f^{-p}(\mathrm{bd}(D_i)) \mid 1 \leq i \leq M$ and $p = 0, 1, \ldots, k-1\}$ is in general position,

(3') $\mathrm{bd}(D_i) \cap (EP(f) \cup F) = \emptyset$.

Put

$$K = \bigcup_{i=1}^{M} \mathrm{bd}(D_i).$$

Since K contains no eventually periodic points, we see that for any point $z \in K$, $f^t(z) \cap f^{t'}(z) = \emptyset$ for any integers with $t, t' \in \mathbb{Z}$ with $t \neq t'$ (see Lemma 3.7). Take open neighborhoods $U(f^t(z))$ $(-2k \leq t \leq 2k)$ of $f^t(z)$ such that the sets $U(f^t(z))$ $(-2k \leq t \leq 2k)$ are mutually disjoint. Since K is compact, we can choose finite points $z_j (1 \leq j \leq m_k)$ and a finite family

$$\mathcal{O} = \{O_j \mid 1 \leq j \leq m_k\}$$

of open sets of X such that $K \subset \bigcup \mathcal{O}$, $z_j \in O_j \subset U(z_j)$ and $f^t(O_j) \subset U(f^t(z_j))$ for each

$-2k \leq t \leq 2k$.

For convenience, we put $O_0 = \emptyset$. By induction on $j = 0, 1, \ldots, m_k$, we will construct a family $\{\mathcal{D}(j) \mid 0 \leq j \leq m_k\}$ of open shrinkings of $\mathcal{C}$ such that for $0 \leq j \leq m_k$, $\mathcal{D}(j)$ satisfies the following conditions;

(a) $\mathcal{D}(0) = \mathcal{D}$,

(b) $\mathcal{D}(j) = \{D(j)_i \mid 1 \leq i \leq M\}$,

(c) $B_i \subset D(j)_i \subset D(j-1)_i$,

(d) for each $1 \leq i \leq M$,

$$D(j)_i \cap (X - O_j) = D(j-1)_i \cap (X - O_j),$$
$$\mathrm{bd}(D(j)_i) \cap (X - O_j) \subset \mathrm{bd}(D(j-1)_i) \cap (X - O_j),$$

and if $\mathrm{bd}(D(j-1)_i) \cap O_j = \emptyset$, then $D(j)_i = D(j-1)_i$,

(e) the family

$$\mathcal{G}(j) = \{f^{-p}(\mathrm{bd}(D(j)_i)) \mid 0 \leq p \leq k-1, 1 \leq i \leq M\}$$
$$\cup \left\{ f^{-k}\left[\mathrm{bd}(D(j)_i) \cap \left(\bigcup_{s=0}^{j} O_s\right)\right] \,\middle|\, 1 \leq i \leq M \right\}$$

is in general position,

(f) $\mathrm{bd}(D(j)_i) \cap (EP(f) \cup F) = \emptyset$ for each $1 \leq i \leq M$.

We construct $\mathcal{D}(j)$ by the induction on j. For $j = 0$, we have $\mathcal{D}(0) = \mathcal{D}$. Suppose that we have $\mathcal{D}(j)$ satisfying the desired conditions. We will construct $\mathcal{D}(j+1)$. For each t with $-k \leq t \leq k$, we assume that the collection

$$\mathcal{S}_t = \{f^{-p}(\mathrm{bd}(D(j)_i)) \cap U(f^t(z_{j+1})) \mid 0 \leq p \leq k-1, 1 \leq i \leq M\}$$
$$\cup \left\{ f^{-k}\left[\mathrm{bd}(D(j)_i)) \cap \left(\bigcup_{s=0}^{j} O_s\right)\right] \cap U(f^t(z_{j+1})) \,\middle|\, 1 \leq i \leq M \right\}$$

is in general position in $U(f^t(z_{j+1}))$. Note that

$$f^{-t}(\mathcal{S}_t)|_{O_{j+1}} = \{f^{-t}(S) \cap O_{j+1} \mid S \in \mathcal{S}_t\}$$

is also in general position (see Proposition 3.9). By Lemma 3.6, we obtain an open shrinking $\mathcal{D}(j+1)$ of $\mathcal{D}(j)$ such that for each $1 \leq i \leq M$,

(1) $B_i \subset D(j+1)_i \subset D(j)_i$,

(2) $D(j+1)_i = D(j)_i$ if $\mathrm{bd}(D(j)_i) \cap O_{j+1} = \emptyset$,

(3) $D(j+1)_i - O_{j+1} = D(j)_i - O_{j+1}$,

(4) $\mathrm{bd}(D(j+1)_i) - O_{j+1} \subset \mathrm{bd}(D(j)_i) - O_{j+1}$,

(5) $\mathrm{bd}(D(j+1)_i) \cap (EP(f) \cup F) = \emptyset$, and

(6) $f^{-t}(\mathcal{S}_t)|_{O_{j+1}} \cup \{\mathrm{bd}(D(j+1)_i) \cap O_{j+1} \mid 1 \leq i \leq M\}$ is in general position for any $-k \leq t \leq k$.

By the similar arguments of the proofs of [**15**, Lemma 3.5] and [**10**, Lemma 3.4], we can check the condition (e): the family

$$\mathcal{G}(j+1) = \{f^{-p}(\mathrm{bd}(D(j+1)_i)) \mid 0 \leq p \leq k-1, 1 \leq i \leq M\} \\ \cup \left\{ f^{-k} \left[\mathrm{bd}(D(j+1)_i) \cap \left(\bigcup_{s=0}^{j+1} O_s \right) \right] \middle| \ 1 \leq i \leq M \right\}$$

is in general position.

If we continue this procedure, we obtain $\mathcal{D}(m_k)$ $(= \mathcal{C}'(k))$ which is the desired open cover of X. □

LEMMA 3.12 (cf. [**10**, Lemma 3.5]). *Suppose that $f : X \to X$ is a two-sided zero-dimensional map of a compactum X such that $\dim X = n < \infty$ and $\dim P(f) \leq 0$. Let F be an F_σ-set of X with $\dim F \leq 0$. Then, for each $j \in \mathbb{N}$, there is a finite open cover $\mathcal{C}(j) = \{C(j)_i \mid 1 \leq i \leq m_j\}$ of X such that*

(1) $\mathrm{mesh}(\mathcal{C}(j)) < 1/j$,
(2) $\mathrm{ord}(\mathcal{G}) \leq n$, *where* $\mathcal{G} = \{f^{-p}(\mathrm{bd}(C(j)_i)) \mid 1 \leq i \leq m_j,\ j \in \mathbb{N}$ *and* $p \in \mathbb{Z}_+\}$, *and*
(3) $F \cap L = \emptyset$, *where* $L = \bigcup\{\mathrm{bd}(C(j)_i) \mid 1 \leq i \leq m_j, j \in \mathbb{N}\}$.

PROOF. We put $F = \bigcup_{j=1}^{\infty} F_j$, where F_j is a zero-dimensional closed set of X. For each $j \in \mathbb{N}$, we take a finite open cover $\mathcal{D}(j)$ of X such that $\mathrm{mesh}(\mathcal{D}(j)) < 1/j$. We put

$$\mathcal{D}(j) = \{D(j)_i \mid 1 \leq i \leq m_j\}.$$

Also we take an open shrinking $\mathcal{B}(j) = \{B(j)_i \mid 1 \leq i \leq m_j\}$ of $\mathcal{D}(j)$ such that $\overline{\mathcal{B}}(j) = \{\mathrm{cl}(B(j)_i) \mid 1 \leq i \leq m_j\}$ is a closed shrinking of $\mathcal{D}(j)$. For each $j \in \mathbb{N}$ and each $k \in \mathbb{N}$ with $k \geq j$, we will find an open shrinking $\mathcal{D}(j,k) = \{D(j,k)_i \mid 1 \leq i \leq m_j\}$ $(k \geq j)$ of $\mathcal{D}(j)$ and a closed shrinking $\mathcal{B}(j,k) = \{B(j,k)_i \mid 1 \leq i \leq m_j\}$ $(k \geq j)$ of $\mathcal{D}(j,k)$ such that

(a) $\mathcal{D}(j,j) = \mathcal{D}(j)$, $\mathcal{B}(j,j) = \overline{\mathcal{B}}(j)$,
(b) $\mathrm{cl}(B(j)_i) = B(j,j)_i \subset B(j,j+1)_i \subset B(j,j+2)_i \subset \cdots \subset D(j,j+2)_i \subset D(j,j+1)_i \subset D(j,j)_i = D(j)_i$, i.e., $\{B(j,k)_i\}_{k=j}^{\infty}$ is an increasing sequence of sets and $\{D(j,k)_i\}_{k=j}^{\infty}$ is a decreasing sequence of sets such that $B(j,k)_i \subset D(j,k)_i$ for $k \geq j$,
(c) $\mathrm{ord}\{\mathrm{cl}(f^{-p}(D(j,k+1)_i - B(j,k+1)_i)) \mid 1 \leq i \leq m_j, 1 \leq j \leq k, 0 \leq p \leq k\} \leq n$, and
(d) $[D(j,k+1)_i - B(j,k+1)_i] \cap \bigcup_{j=1}^{k+1} F_j = \emptyset$.

We proceed by induction on k. Suppose that we have

$$\mathcal{D}(1,k), \ldots, \mathcal{D}(k-1,k), \mathcal{D}(k,k) = \mathcal{D}(k)$$

and

$$\mathcal{B}(1,k), \ldots, \mathcal{B}(k-1,k), \mathcal{B}(k,k) = \overline{\mathcal{B}}(k)$$

satisfying the desired conditions. We will construct $\mathcal{D}(1,k+1), \ldots, \mathcal{D}(k,k+1)$ and $\mathcal{B}(1,k+1), \ldots, \mathcal{B}(k,k+1)$. Note that $\{D(j,k)_i \mid 1 \leq j \leq k$ and $1 \leq i \leq m_j\}$ is a finite open cover of X and $\{B(j,k)_i \mid 1 \leq j \leq k$ and $1 \leq i \leq m_j\}$ is a closed shrinking of $\{D(j,k)_i \mid 1 \leq j \leq k$ and $1 \leq i \leq m_j\}$. By Lemma 3.11, there is an open shrinking

$\{D(j,k+1)_i \mid 1 \le j \le k \text{ and } 1 \le i \le m_j\}$ of $\{D(j,k)_i \mid 1 \le j \le k \text{ and } 1 \le i \le m_j\}$ such that

(1') $B(j,k)_i \subset D(j,k+1)_i \subset D(j,k)_i$,

(2') $\{f^{-p}(\mathrm{bd}(D(j,k+1)_i)) \mid 1 \le i \le m_j,\ 1 \le j \le k \text{ and } 0 \le p \le k\}$ is in general position, and

(3') $\mathrm{bd}(D(j,k+1)_i) \cap \bigcup_{j=1}^{k+1} F_j = \emptyset$.

Since

$$\{f^{-p}(\mathrm{bd}(D(j,k+1)_i)) \mid 1 \le i \le m_j,\ 1 \le j \le k \text{ and } 0 \le p \le k\}$$

is a finite collection of closed subsets of X with ord $\le n$, there is an open swelling

$$\mathcal{O}(j,k+1) = \{O(j,k+1)_i \mid 1 \le i \le m_j\}$$

of $\{\mathrm{bd}(D(j,k+1)_i) \mid 1 \le i \le m_j\}$ such that

(4') $O(j,k+1)_i \cap B(j,k)_i = \emptyset$,

(5') $\mathrm{ord}\{f^{-p}(\mathrm{cl}(O(j,k+1)_i)) \mid 1 \le i \le m_j,\ 1 \le j \le k \text{ and } 0 \le p \le k\} \le n$, and

(6') $O(j,k+1)_i \cap \bigcup_{j=1}^{k+1} F_j = \emptyset$.

For each $1 \le i \le m_j$ and $j \in \mathbb{N}$, we put

$$B(j,k+1)_i = \mathrm{cl}(D(j,k+1)_i) - O(j,k+1)_i.$$

Then $\mathcal{D}(j,k+1)$ and $\mathcal{B}(j,k+1)$ satisfy the conditions (a)–(c).

Finally we put

$$C(j)_i = \mathrm{int}\left(\bigcap_{k=j}^{\infty} D(j,k)_i\right) \quad \text{and} \quad \mathcal{C}(j) = \{C(j)_i \mid 1 \le i \le m_j\}.$$

Note that $B(j)_i \subset C(j)_i \subset D(j)_i$. Since (1) is obvious, we must check (2). It suffices to show that for each $k \in \mathbb{N}$,

$$\mathcal{G}(k) = \{f^{-p}(\mathrm{bd}(C(j)_i) \mid 1 \le i \le m_j,\ 1 \le j \le k \text{ and } 0 \le p \le k\}$$

is ord $\le n$. However, since

$$\mathrm{bd}(C(j)_i) \subset \mathrm{cl}(D(j,k+1)_i - B(j,k+1)_i) \quad \text{and}$$
$$\mathrm{ord}\{\mathrm{cl}(f^{-p}(D(j,k+1)_i - B(j,k+1)_i)) \mid 1 \le i \le m_j,\ 1 \le j \le k \quad \text{and} \quad 0 \le p \le k\} \le n,$$

we see that (2) holds. Also by (d), we see that $F \cap L = \emptyset$. Consequently, we obtain the desired open cover $\mathcal{C}(j) = \{C(j)_i \mid 1 \le i \le m_j\}$ of X for each $j \in \mathbb{N}$. □

LEMMA 3.13. *Let $f : X \to X$ be a map of a compactum X and let H be a subset of X. Suppose that for $j \in \mathbb{N}$, $\mathcal{C}(j) = \{C(j)_i \mid 1 \le i \le m_j\}$ is a finite open cover of X such that* $\mathrm{mesh}(\mathcal{C}(j)) < 1/j$, $H \cap \bigcup \mathcal{G} = \emptyset$ *and* $\mathrm{ord}(\mathcal{G}) \le n$, *where*

$$\mathcal{G} = \{f^{-p}(\mathrm{bd}(C(j)_i)) \mid 1 \le i \le m_j,\ j \in \mathbb{N} \text{ and } p \in \mathbb{Z}_+\}.$$

Then, for $j \in \mathbb{N}$ there is a finite regular closed partition $\mathcal{D}(j)$ of X such that the following properties hold;

(1) $\mathrm{mesh}(\mathcal{D}(j)) \leq 1/j$,
(2) $\mathcal{D}(j+1)$ *is a refinement of* $\mathcal{D}(j)$,
(3) $\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)} \mathcal{D}(j) \leq 2^n$ *for each* $x \in X$, *and*
(4) *if* $x \in H$, *then* $\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)} \mathcal{D}(j) = 1$.

Proof. Let $j \in \mathbb{N}$. Put

$$c(j)_1 = \mathrm{cl}(C(j)_1),$$
$$c(j)_i = \mathrm{cl}\left(\mathrm{int}\left[(C(j)_i) - \left(\bigcup_{i'<i} C(j)_{i'}\right)\right]\right)$$

for $2 \leq i \leq m_j$. Then $\mathcal{C}'(j) = \{c(j)_i \mid 1 \leq i \leq m_j\}$ is a finite regular closed partition of X. Let

$$\mathcal{D}(j) = @_{1 \leq i \leq j}(\mathcal{C}'(i)).$$

We will show that $\mathcal{D}(j)$ is a desired partition. Since (1), (2) and (4) are obvious, we only need to check (3).

Let $x \in X$. For each $j \in \mathbb{N}$ and $p \in \mathbb{Z}_+$, put

$$m_{j,p} = \mathrm{ord}_{f^p(x)}\{\mathrm{bd}(C(j)_i) \mid 1 \leq i \leq m_j\}.$$

Since $\mathrm{ord}_x \mathcal{G} \leq n$, we have

$$\sum_{\substack{j \in \mathbb{N} \\ p \in \mathbb{Z}_+}} m_{j,p} \leq n.$$

We will show that $\mathrm{ord}_{f^p(x)} \mathcal{C}'(j) \leq m_{j,p} + 1$.

Put $i_0 = \min\{i \leq m_j \mid f^p(x) \in C(j)_i\}$. By [**2**, Lemma 13], we can see the following;

$$\begin{aligned}
\mathrm{ord}_{f^p(x)} \mathcal{C}'(j) &= \mathrm{ord}_{f^p(x)}(\{c(j)_i \mid 1 \leq i < i_0\} \cup \{c(j)_{i_0}\} \cup \{c(j)_i \mid i > i_0\}) \\
&\leq \mathrm{ord}_{f^p(x)}(\{\mathrm{cl}(C(j)_i) - C(j)_i \mid i < i_0\} \cup \{c(j)_{i_0}\}) \\
&\leq m_{j,p} + 1.
\end{aligned}$$

Note that $\mathrm{ord}(\mathcal{A}@\mathcal{B}) \leq \mathrm{ord}(\mathcal{A}) \cdot \mathrm{ord}(\mathcal{B})$ for each regular closed partitions $\mathcal{A}$ and $\mathcal{B}$ of X. Since $m + 1 \leq 2^m$ for each $m = 0, 1, 2, \ldots$,

$$\begin{aligned}
\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)} \mathcal{D}(j) &= \prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)}[@_{1 \leq i \leq j} \mathcal{C}'(i)] \leq \prod_{p=0}^{\infty} \prod_{1 \leq i \leq j} \mathrm{ord}_{f^p(x)} \mathcal{C}'(i) \\
&\leq \prod_{p=0}^{\infty} \prod_{1 \leq i \leq j} (m_{i,p} + 1) \leq \prod_{p=0}^{\infty} \prod_{1 \leq i \leq j} 2^{m_{i,p}} = 2^{\sum m_{i,p}} \leq 2^n.
\end{aligned}$$

Therefore, $\mathcal{D}(j)$ is the desired partition. $\square$

Let $Y_k = \{1, 2, \ldots, k\}$ $(k \in \mathbb{N})$ be the discrete space having k-elements and let $Y_k^{\mathbb{Z}_+} = \prod_0^\infty Y_k$ be the product space. Then the shift map $\sigma : Y_k^{\mathbb{Z}_+} \to Y_k^{\mathbb{Z}_+}$ is defined by $\sigma(x)_i = x_{i+1}$ for $x = (x_0, x_1, x_2, \ldots) \in Y_k^{\mathbb{Z}_+}$.

Lemma 3.14. *Let $f : X \to X$ be a map of a compactum X and let H be a subset of X. Suppose that there is $m \in \mathbb{N}$ and a sequence of finite regular closed partitions $\mathcal{D}(j)$ $(j \in \mathbb{N})$ of X such that*

(1) $\mathrm{mesh}(\mathcal{D}(j)) \leq 1/j$,
(2) *$\mathcal{D}(j+1)$ is a refinement of $\mathcal{D}(j)$,*
(3) *$\prod_{p=0}^\infty \mathrm{ord}_{f^p(x)} \mathcal{D}(j) \leq m$ for each $x \in X$, and*
(4) *$H \cap D = \emptyset$, where $D = \bigcup\{f^{-p}(\mathrm{bd}(d)) \mid d \in \mathcal{D}(j), j \in \mathbb{N}, p \in \mathbb{Z}_+\}$, i.e., if $x \in H$,*

$$\prod_{p=0}^\infty \mathrm{ord}_{f^p(x)} \mathcal{D}(j) = 1.$$

Then there is a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most m-to-one map $p : Z \to X$ such that $|p^{-1}(x)| = 1$ for $x \in H$. Moreover, if X is perfect, then Z can be taken as a Cantor set C.

Proof. We put

$$\mathcal{D}(j) = \{d(j)_1, d(j)_2, \ldots, d(j)_{k_j}\}.$$

Let $Y_{k_j} = \{1, 2, \ldots, k_j\}$ $(j \in \mathbb{N})$ be the discrete space having k_j-elements and let $Y_{k_j}^{\mathbb{Z}_+} = \prod_0^\infty Y_{k_j}$ $(= Y_{k_j} \times Y_{k_j} \times \cdots)$ be the product space. For each $j \in \mathbb{N}$, we consider the shift map $\sigma_j : Y_{k_j}^{\mathbb{Z}_+} \to Y_{k_j}^{\mathbb{Z}_+}$ and the sets

$$\Sigma_j = \left\{ a = (a_p)_{p=0}^\infty \in Y_{k_j}^{\mathbb{Z}_+} \;\middle|\; \bigcap_{p=0}^\infty f^{-p}(d(j)_{a_p}) \neq \emptyset \right\}.$$

Note that Σ_j is a zero-dimensional compactum. Since $\mathcal{D}(j+1)$ is a refinement of $\mathcal{D}(j)$, there is the unique map $h : Y_{k_{j+1}} = \{1, 2, \ldots, k_{j+1}\} \to Y_{k_j} = \{1, 2, \ldots, k_j\}$ defined by $d(j)_{h(k)} \supset d(j+1)_k$ for $1 \leq k \leq k_{j+1}$. Let $h_{j,j+1} : \Sigma_{j+1} \to \Sigma_j$ be the map defined by $h_{j,j+1}(a)_p = h(a_p)$ for $a = (a_p)_{p=0}^\infty \in \Sigma_{j+1}$.

Consider the inverse limit of the inverse sequence $\{\Sigma_j, h_{j,j+1}\}_{j=1}^\infty$:

$$Z = \varprojlim\{\Sigma_j, h_{j,j+1}\} = \left\{ (z^j)_{j=1}^\infty \in \prod_{j=1}^\infty \Sigma_j \;\middle|\; z^j = h_{j,j+1}(z^{j+1}) \text{ for } j \in \mathbb{N} \right\} \subset \prod_{j=1}^\infty \Sigma_j$$

which has the topology inherited as a subspace of the product space $\prod_{j=1}^\infty \Sigma_j$. Let $q_j : Z = \varprojlim\{\Sigma_j, h_{j,j+1}\} \to \Sigma_j$ be the natural projection. Then for each $j \in \mathbb{N}$, we know that the following diagram (a) is commutative:

$$\begin{array}{ccc} \Sigma_j & \xleftarrow{h_{j,j+1}} & \Sigma_{j+1} \\ \downarrow \sigma_j & & \downarrow \sigma_{j+1} \\ \Sigma_j & \xleftarrow{h_{j,j+1}} & \Sigma_{j+1} \end{array} \quad . \tag{a}$$

Note that Z is a zero-dimensional compactum and the sequence $\{\sigma_j\}_{j\in\mathbb{N}}$ induces the map $\tilde{f} : Z = \varprojlim\{\Sigma_j, h_{j,j+1}\} \to Z$, i.e.,

$$\tilde{f}(z) = \tilde{f}(z^1, z^2, \ldots) = (\sigma_1(z^1), \sigma_2(z^2), \ldots)$$

for $z = (z^1, z^2, \ldots) \in Z$ $(= \varprojlim\{\Sigma_j, h_{j,j+1}\})$. Also, we define the natural projection $p : Z \to X$ by $p(z) = x \in X$, where $z = (z^1, z^2, \ldots) \in Z(= \varprojlim\{\Sigma_j, h_{j,j+1}\}), z^j = (z^j_0, z^j_1, \ldots) \in \Sigma_j$ and

$$x = \bigcap\{d(j)_{z^j_0} \mid j \in \mathbb{N}\}.$$

We easily see that p is onto. We will show that p is an at most m-to-one map. Let $x \in X$. Note that

$$q_j(p^{-1}(x)) = \{a = (a_p)_{p=0}^{\infty} \in \Sigma_j \mid f^p(x) \in d(j)_{a_p}\}.$$

By (3), $|q_j(p^{-1}(x))| \leq m$ for each $j \in \mathbb{N}$. This implies that $|p^{-1}(x)| \leq m$. By (4), we see that $|p^{-1}(x)| = 1$ for $x \in H$. Also, by (a) we see that the following diagram is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{f}} & Z \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{f} & X \end{array} \quad .$$

Also, we see that if X is perfect, then Z is a Cantor set (= zero-dimensional perfect compactum). This completes the proof. □

Now, we need the definition of topological entropy by Bowen [**5**]. Let $f : X \to X$ be any map of a compactum X. A subset E of X is (n, ϵ)-*separated* if for any $x, y \in E$ with $x \neq y$, there is an integer j such that $0 \leq j < n$ and $d(f^j(x), f^j(y)) > \epsilon$. If K is any nonempty closed subset of X, $s_n(\epsilon; K)$ denotes the largest cardinality of any set $E \subset K$ which is (n, ϵ)-separated. Also we define

$$s(\epsilon; K) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\epsilon; K),$$
$$h(f; K) = \lim_{\epsilon\to 0} s(\epsilon; K).$$

It is well known that the topological entropy $h(f)$ of f is equal to $h(f; X)$ (see [**5**]).

By use of the above results, we will prove the following theorem.

Theorem 3.15. *Suppose that $f : X \to X$ is a two-sided zero-dimensional map of a compactum X with* $\dim X = n < \infty$. *If* $\dim P(f) \leq 0$, *then there exist a dense G_δ-set*

H of X and a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most 2^n-to-one onto map p such that $P(f) \subset H$ and $|p^{-1}(x)| = 1$ for $x \in H$. Moreover, if X is perfect, then Z can be chosen as a Cantor set. In particular, $h(f) = h(\tilde{f})$.

Proof. By Lemma 3.12 and Lemma 3.13, for $j \in \mathbb{N}$ there is a finite regular closed partition $\mathcal{D}(j)$ of X such that
(1) $\mathrm{mesh}(\mathcal{D}(j)) \leq 1/j$,
(2) $\mathcal{D}(j+1)$ is a refinement of $\mathcal{D}(j)$,
(3) $\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)} \mathcal{D}(j) \leq 2^n$ for each $x \in X$, and
(4) $P(f) \cap D = \emptyset$, where $D = \bigcup \{f^{-p}(\mathrm{bd}(d)) \mid d \in \mathcal{D}(j), j \in \mathbb{N}, p \in \mathbb{Z}_+\}$.

Put $H = X - D$. By Lemma 3.14, we can conclude that there exists a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most 2^n-to-one onto map p such that $P(f) \subset H$ and $|p^{-1}(x)| = 1$ for $x \in H$. By Bowen's theorem (e.g., see [**18**, Theorem 7.1]), we have

$$h(f) \leq h(\tilde{f}) \leq h(f) + \sup\{h(\tilde{f}; p^{-1}(x)) \mid x \in X\}.$$

Since $p^{-1}(x)$ is a finite set, hence $h(\tilde{f}; p^{-1}(x)) = 0$. This implies $h(f) = h(\tilde{f})$. □

Remark 1.

(1) In Theorem 3.15, the desired dynamical system $(Z, \tilde{f})$ is obtained by the "inverse sequence" of symbolic dynamics

$$\{(\Sigma_j, \sigma_j) \mid j \in \mathbb{N}\}.$$

(2) For any dynamical system (X, f), there are finite regular closed partitions $\mathcal{D}(j)$ ($j \in \mathbb{N}$) of X such that (i) $\mathrm{mesh}(\mathcal{D}(j)) \leq 1/j$, and (ii) $\mathcal{D}(j+1)$ is a refinement of $\mathcal{D}(j)$. By the proof of Lemma 3.14, there exists a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via a map p, where p is not necessarily finite-to-one. Moreover, if f is a homeomorphism, then by a small modification of the proof of Lemma 3.14, we can take $\tilde{f}$ as a homeomorphism. This result was proved by Anderson [**1**].

(3) For any onto map $f : X \to X$ on a compactum X, the dynamical system (X, f) has a zero-dimensional cover (Y, g) such that $g : Y \to Y$ is a homeomorphism. We consider the inverse sequence:

$$\{X, f\} = \{X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \cdots\}$$

of f. Then there is the shift homeomorphism $\sigma_f : \varprojlim\{X, f\} \to \varprojlim\{X, f\}$ defined by $\sigma_f(x_1, x_2, \ldots) = (f(x_1), x_1, x_2, \ldots)$ such that for any $n \in \mathbb{N}$, the following diagram is commutative:

$$\begin{array}{ccc} \varprojlim\{X, f\} & \xrightarrow{\sigma_f} & \varprojlim\{X, f\} \\ \downarrow q_n & & \downarrow q_n \\ X & \xrightarrow{f} & X \end{array}$$

where $q_n : \varprojlim\{X, f\} \to X_n = X$ is the n-th coordinate projection. Note that $h(f) = h(\sigma_f)$, $\lim_{n\to\infty}\mathrm{diam}\ (q_n^{-1}(x)) = 0$ for each $x \in X$, but $q_n^{-1}(x)$ may be uncountable. Also the dynamical system $(\varprojlim\{X, f\}, \sigma_f)$ has a zero-dimensional cover (Y, g) such that $g : Y \to Y$ is a homeomorphism.

Now, we consider the case that $f : X \to X$ is a positively expansive map of a compactum X. A map $f : X \to X$ of a compactum X is *positively expansive* if there is $\epsilon > 0$ such that for any $x, y \in X$ with $x \neq y$, there is $k \in \mathbb{Z}_+$ such that $d(f^k(x), f^k(y)) \geq \epsilon$. Similarly, a map $f : X \to X$ of a compactum X is *positively continuum-wise expansive* if there is $\epsilon > 0$ such that for any nondegenarate subcontinuum A of X, there is a $k \in \mathbb{Z}_+$ such that $\mathrm{diam}(f^k(A)) \geq \epsilon$ (see [**12**]). Such an $\epsilon > 0$ is called an *expansive constant* for f. Note that any positively expansive map is two-sided zero-dimensional and positively continuum-wise expansive. In [**12**, Theorem 5.3], we know that if a compactum X admits an positively continuum-wise expansive map f on X, then $\dim X < \infty$ and every minimal set of f is zero-dimensional.

For a map $f : X \to X$, we consider the following subset of X;

$$I_0(f) = \bigcup\{M \mid M \text{ is a zero-dimensional } f\text{-invariant closed set of } X\}.$$

Proposition 3.16 (cf. [**13**, Proposition 2.5]). *Let $f : X \to X$ be a positively continuum-wise expansive map of a compact metric space X. Then $I_0(f)$ is a zero-dimensional F_σ-set of X. In particular,* $\dim P(f) \leq 0$.

Proof. The proof is similar to the proof of [**13**, Proposition 2.5]. Let $\epsilon > 0$ be an expansive constant for f. We choose a countable open base $\mathcal{B}$ of X such that if $U \in \mathcal{B}$, then $\mathrm{diam}(U) \leq \epsilon$. Put

$$\tilde{\mathcal{B}} = \{\cup\mathcal{B}' \mid \mathcal{B}' \subset \mathcal{B},\ |\mathcal{B}'| < \infty, \mathrm{diam}(\cup\mathcal{B}') < \epsilon\}.$$

For each $n \in \mathbb{N}$, we put

$$\mathcal{W}_n = \{(U_1, \ldots, U_n) \mid U_i \in \tilde{\mathcal{B}}, \mathrm{cl}(U_i) \cap \mathrm{cl}(U_j) = \emptyset\ (i \neq j)\}$$

and $\mathcal{W} = \bigcup_{n=1}^\infty \mathcal{W}_n$. For each $(U_1, \ldots, U_n) \in \mathcal{W}$, we consider the set

$$W(U_1, \ldots, U_n) = \left\{ x \in X \;\middle|\; \{f^p(x) |\ p \in \mathbb{Z}_+\}(= O(x)) \subset \bigcup_{i=1}^n \mathrm{cl}(U_i) \right\}.$$

Then $W(U_1, \ldots, U_n)$ is an f-invariant closed subset of X. Since f is a positively continuum-wise expansive map, each component of $W(U_1, \ldots, U_n)$ is a one point set and hence we see $\dim W(U_1, \ldots, U_n) \leq 0$. Note that if A is f-invariant closed set of X with $\dim A \leq 0$, then we can find $(U_1, \ldots, U_n) \in \mathcal{W}$ such that $A \subset \bigcup_{i=1}^n U_i$. By use of this fact, we see that

$$I_0(f) = \bigcup\{W(U_1, \ldots, U_n) \mid (U_1, \ldots, U_n) \in \mathcal{W}\}.$$

By Proposition 3.1, $I_0(f)$ is a zero-dimensional F_σ-set of X. Note that

$$P(f) \subset I_0(f). \qquad \square$$

Recall $Y_k = \{1, 2, \ldots, k\}$ and the shift map $\sigma : Y_k^{\mathbb{Z}_+} \to Y_k^{\mathbb{Z}_+}$ defined by $\sigma(x)_j = x_{j+1}$. Note that σ is the typical positively expansive map.

The following theorem is a more precise result than [**16**, Proposition 20].

Theorem 3.17. *Let $f : X \to X$ be a positively expansive map of a compactum X with $\dim X = n < \infty$. Then there exist $k \in \mathbb{N}$ and a closed σ-invariant set Σ of $\sigma : Y_k^{\mathbb{Z}_+} \to Y_k^{\mathbb{Z}_+}$ such that (Σ, σ) is a zero-dimensional cover (= symbolic extension) of (X, f) via an at most 2^n-to-one map $p : \Sigma \to X$ satisfying that $|p^{-1}(x)| = 1$ for any $x \in I_0(f)$.*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{f} & X \end{array}$$

Proof. Note that f is a two-sided zero-dimensional map. Let $\epsilon > 0$ be an expansive constant for f. Since $\dim I_0(f) \leq 0$, by Lemma 3.12 there is a finite open cover $\mathcal{C}(\epsilon)$ of X such that

(1) $\mathrm{mesh}(\mathcal{C}(\epsilon)) < \epsilon$,
(2) $\mathrm{ord}(\mathcal{G}) \leq n$, where $\mathcal{G} = \{f^{-p}(\mathrm{bd}(C)) \mid C \in \mathcal{C}(\epsilon), p \in \mathbb{Z}_+\}$,
(3) $\mathrm{bd}(C) \cap I_0(f) = \emptyset$ for each $C \in \mathcal{C}(\epsilon)$.

Let $\mathcal{C}(\epsilon) = \{C_1, C_2, \ldots, C_k\}$ and we consider the following partition

$$c_1 = \mathrm{cl}(C_1), c_i = \mathrm{cl}(C_i - (C_1 \cup C_2 \cup \cdots \cup C_{i-1}))\ (i \geq 2).$$

Consider the set

$$\Sigma = \left\{ (i_p)_{p=0}^{\infty} \in Y_k^{\mathbb{Z}_+} \;\middle|\; \bigcap_{p=0}^{\infty} f^{-p}(c_{i_p}) \neq \emptyset \right\}.$$

Note that $\bigcap_{p=0}^{\infty} f^{-p}(c_{i_p})$ is a one point set for each $(i_p)_p \in \Sigma$, because f is a positively expansive map. Define a map $p : \Sigma \to X$ by

$$p((i_p)_p) = \bigcap_{p=0}^{\infty} f^{-p}(c_{i_p}).$$

By (2) and the proof of Lemma 3.13, we see that $|p^{-1}(x)| \leq 2^n$ for each $x \in X$. Also, by (3), we see that $p : \Sigma \to X$ is the desired map such that $|p^{-1}(x)| = 1$ for any $x \in I_0(f)$. $\square$

Remark 2. For the case that $f : X \to X$ is an expansive homeomorphism of a compactum X with $\dim X = n < \infty$ (see [**12**] for the definition of expansive homeomorphism), there exist $k \in \mathbb{N}$ and a closed σ-invariant set Σ of $\sigma : Y_k^{\mathbb{Z}} \to Y_k^{\mathbb{Z}}$ such that (Σ, σ) is a zero-dimensional cover (= symbolic extension) of (X, f) via an at most 2^n-to-one map $p : \Sigma \to X$, where $\sigma : Y_k^{\mathbb{Z}} \to Y_k^{\mathbb{Z}}$ is the shift homeomorphism (see [**10**] and [**15**]).

Now we consider a generalization of Theorem 3.15. For a map $f : X \to X$ on a compactum X, let

$$D_0(f) = \{x \in X \mid \dim f^{-1}(x) \leq 0\}$$

and

$$D_+(f) = \{x \in X \mid \dim f^{-1}(x) \geq 1\} \ (= X - D_0(f)).$$

Note that a map $f : X \to X$ is a zero-dimensional map if and only if $D_+(f) = \emptyset$. The following theorem is a generalization of Theorem 3.15.

Main Theorem 3.18 (a generalization of Theorem 3.15). *Let $f : X \to X$ be a map on an n-dimensional compactum X ($n < \infty$). Suppose that f is a zero-dimension preserving map,* $\dim D_+(f) \leq 0$ *and* $\dim EP(f) \leq 0$*. Then there exist a dense G_δ-set H of X and a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most 2^n-to-one onto map p such that $EP(f) \subset H$ and $|p^{-1}(x)| = 1$ for $x \in H$. Moreover, if X is perfect, then Z can be chosen as a Cantor set. In particular, $h(f) = h(\tilde{f})$.*

Proof. Note that for $p \in \mathbb{N}$, $D_+(f^p)$ is an F_σ-set of X because that

$$D_+(f^p) = \bigcup\{D(1/k) \mid k \in \mathbb{N}\}$$

and $D(1/k)$ is a closed set of X, where $D(1/k)$ is the set of all points x of X such that $f^{-p}(x)$ contains a continuum whose diameter $\geq 1/k$. By induction on p ($p \in \mathbb{N}$), we will prove that for any $p \in \mathbb{N}$, $\dim D_+(f^p) \leq 0$. By the assumption, if $p = 1$ then $\dim D_+(f^p) = \dim D_+(f) \leq 0$. Assume that $\dim D_+(f^p) \leq 0$. We will prove $\dim D_+(f^{p+1}) \leq 0$. Suppose on the contrary that $\dim D_+(f^{p+1}) \geq 1$. Since $D_+(f^{p+1})$ is an F_σ-set of X, by Proposition 3.1 $D_+(f^{p+1})$ contains a closed subset A of X with $\dim A \geq 1$. Let

$$B = D_+(f^p) \cap f^{-1}(A).$$

Then $\dim B \leq 0$ and $\dim f(B) \leq 0$ because that B is an F_σ-set of X and f is a zero-dimension preserving map. We will prove $A = f(B) \cup A_1$, where $A_1 = A \cap D_+(f)$. Let $x \in A$. Since $\dim f^{-(p+1)}(x) \geq 1$, there is a nondegenerate continuum K in $f^{-(p+1)}(x)$. If $f^p(K)$ is a one point y, then $y \in B$ and hence $x = f(y) \in f(B)$. If $f^p(K)$ is nondegenerate, then $x = f(f^p(K)) \in A_1$. Hence we see $A = f(B) \cup A_1$. Since $f(B)$ and A_1 are zero-dimensional F_σ-sets of X, we see $\dim A \leq 0$. This is a contradiction. Hence $\dim D_+(f^{p+1}) \leq 0$.

Put

$$F = \bigcup\{D_+(f^p) \mid p \in \mathbb{N}\}.$$

Then F is a zero-dimensional F_σ-set of X. Note that for any $q \in \mathbb{N}$,

$$f^{-q}(X - F) \subset X - F. \tag{b}$$

In fact, suppose on the contrary that there is $x \in X - F$ with $f^{-q}(x) \cap F \neq \emptyset$. Let $y \in F \cap f^{-q}(x)$. Then $y \in D_+(p)$ for some $p \in \mathbb{N}$ and hence $x \in D_+(f^{p+q}) \subset F$. This is a contradiction.

Recall the proof of Lemma 3.11. Under the condition of this theorem (Theorem 3.18), if B is a closed subset of X contained in an open set C of X, then we can choose an open set C' of X such that $B \subset C' \subset C$ and $\mathrm{bd}(C') \cap (EP(f) \cup F) = \emptyset$. By (b), we see that if $p \in \mathbb{N}$, $f^{-p}(\mathrm{bd}(C')) \cap (EP(f) \cup F) = \emptyset$. Also note that if $S \subset X - F$, $\dim f^{-p}(S) \leq \dim S$ because that $S \subset D_0(f^p)$.

If we use the above facts and observe the proofs of Lemmas 3.11, 3.12 and 3.13 for Theorem 3.15, we can also construct a sequence of finite regular closed partitions $\mathcal{D}(j)(j \in \mathbb{N})$ of X as in Lemma 3.14. This completes the proof. □

In the special case that X is a graph G (= compact connected 1-dimensional polyhedron) and $f : G \to G$ is a piece-wise monotone map, we can omit the condition $\dim P(f) \leq 0$. A map $f : G \to G$ is *piece-wise monotone* (with respect to some triangulation K) if for any edge E of K (i.e., $E \in K^1$), the restriction $f|E : E \to G$ of f to the edge E is injective. We need the following result.

Lemma 3.19. *Suppose that $f : X \to X$ is a semi-open map of a compactum X and $\{\mathcal{C}(j) \mid j \in \mathbb{N}\}$ is a sequence of finite regular closed partitions of X such that*

(i) *there is $m \in \mathbb{N}$ such that $\mathrm{ord}(\mathcal{C}(j)) \leq m$ for each $j \in \mathbb{N}$,*

(ii) *$\mathcal{C}(j+1)$ refines $f^{-1}(\mathcal{C}(j))@\mathcal{C}(j)$,*

(iii) *$\lim_{j\to\infty} \mathrm{mesh}\, \mathcal{C}(j) = 0$.*

Then there is a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most m-to-one map $p : Z \to X$. Moreover, if X is a perfect, then Z can be taken as a Cantor set C.

Proof. We may assume that each element of each $\mathcal{C}(j)$ is nonempty. Put

$$C = \left\{ (c_1, c_2, \dots) \in \prod_{j=1}^{\infty} \mathcal{C}(j) \;\middle|\; c_1 \supset c_2 \supset \cdots \text{ and } \bigcap_{j=1}^{\infty} c_j \neq \emptyset \right\}$$

and suppose that each $\mathcal{C}(j)$ is a discrete space, i.e., $\{c\}$ is open in $\mathcal{C}(j)$ for each $c \in \mathcal{C}(j)$. By (iii), we see that $\bigcap_{j=1}^{\infty} c_j$ is a one point set for each $(c_1, c_2, \dots) \in C$. Since the product space $\prod_{j=1}^{\infty} \mathcal{C}(j)$ is a zero-dimensional compactum and so is C. Moreover, if X is perfect, we see that C is perfect and hence C is a Cantor set.

Define $p : C \to X$ by $p(c_1, c_2, \dots) = \bigcap_{j=1}^{\infty} c_j$. It is easy to see that p is a continuous onto map. First, we will show that p is at most m-to-one. Suppose, on the contrary, that there is $x \in X$ and pairwise distinct $m+1$ elements

$$(c(1)_1, c(1)_2, \dots), (c(2)_1, c(2)_2, \dots), \dots, (c(m+1)_1, c(m+1)_2, \dots)$$

of C such that

$$\bigcap_{j=1}^{\infty} c(1)_j = \cdots = \bigcap_{j=1}^{\infty} c(m+1)_j = \{x\}.$$

For each $1 \leq i < i' \leq m+1$, let $r_{i,i'} = \min\{j \in \mathbb{N} \mid c(i)_j \neq c(i')_j\}$. Note that $c(i)_j \neq c(i')_j$ for each $j \geq r_{i,i'}$. Put

$$r = \max\{r_{i,i'} \mid 1 \leq i < i' \leq m+1\}.$$

Then $c(1)_r, \dots, c(m+1)_r$ are pairwise distinct elements of $\mathcal{C}(r)$ satisfying $x \in c(1)_r \cap \cdots \cap c(m+1)_r$. Thus, $\mathrm{ord}(\mathcal{C}(r)) \geq m+1$. This is a contradiction.

Next, we will construct a desired map $\tilde{f} : C \to C$. By (ii), we see that $f(\mathcal{C}(j+1))$ is a refinement of $\mathcal{C}(j)$ and since f is a semi-open map, for each $c_{j+1} \in \mathcal{C}(j+1)$, $f(c_{j+1})$ contains a nonempty open set. Thus there is a unique map $\tilde{f}_j : \mathcal{C}(j+1) \to \mathcal{C}(j)$ given by $\tilde{f}_j(c_{j+1}) = c_{j,f}$ if $f(c_{j+1}) \subset c_{j,f}$. Now define $\tilde{f} : C \to C$ by

$$\tilde{f}(c_1, c_2, \dots) = (\tilde{f}_1(c_2), \tilde{f}_2(c_3), \dots).$$

We show that the following conditions (a) and (b) are satisfied: (a) $\tilde{f}$ is continuous and (b) $p\tilde{f} = fp$. (a) is obvious since each $\tilde{f}_j$ is continuous. We will prove (b). Let $(c_1, c_2, \dots) \in C$. Then

$$p\tilde{f}(c_1, c_2, \dots) = p(\tilde{f}_1(c_2), \tilde{f}_2(c_3), \dots) = \bigcap_{j=1}^{\infty} \tilde{f}_j(c_{j+1}) \supset \bigcap_{j=1}^{\infty} f(c_{j+1}),$$

and

$$fp(c_1, c_2, \dots) = f\left(\bigcap_{j=1}^{\infty} c_j\right) \subset \bigcap_{j=1}^{\infty} f(c_j) = \bigcap_{j=1}^{\infty} f(c_{j+1}).$$

Therefore, $p\tilde{f}(c_1, c_2, \dots) \supset fp(c_1, c_2, \dots)$. Note that $p\tilde{f}(c_1, c_2, \dots)$ and $fp(c_1, c_2, \dots)$ are one point sets in X. Thus $p\tilde{f} = fp$. □

THEOREM 3.20. *If $f : G \to G$ is a piece-wise monotone map on a graph G, then there is a zero-dimensional cover $(C, \tilde{f})$ of (G, f) via an at most 2-to-one map, where C is a Cantor set.*

PROOF. The proof is similar to a proof of the theorem of Misiurewics and Szlenk (e.g., see [**18**, Theorem 7.2]). Let K be a triangulation of G such that for any edge E of K, the restriction $f|E : E \to G$ of f to the edge E is injective. Let $B(G)$ be the set of all branch points of G, i.e., $B(G) = \{v \in K^0 \mid \mathrm{ord}_v\{E \mid E \in K^1\} \geq 3\}$. Consider the set

$$EO(B(G)) = \bigcup \{EO(v) \mid v \in B(G)\},$$

where $EO(v)$ denotes the eventual orbit of v. Note that the set $EO(B(G))$ is a countable set. Since f is piece-wise monotone, $f^{-1}(x)$ is a finite set for any $x \in G$. By use of this fact and induction on $j \in \mathbb{N}$, we can find a sequence of finite regular closed partitions $\mathcal{C}(j)$ $(j \in \mathbb{N})$ of G such that

(1) each element c of $\mathcal{C}(j)$ $(j \in \mathbb{N})$ is a closed connected set,

$$\mathrm{bd}(c) \cap EO(B(G)) = \emptyset,$$

and in particular, if $v \in B(G)$, then there is the unique $c_v \in \mathcal{C}(j)$ with $v \in \mathrm{int}(c_v)$,

(2) $$\bigcup\{\mathrm{bd}(c) \mid c \in \mathcal{C}(j+1)\} \supset f^{-1}\left(\bigcup\{\mathrm{bd}(c') \mid c' \in \mathcal{C}(j)\}\right)$$

and $\mathcal{C}(j)$ is a family of the elements c_v ($v \in B(G)$) and closed subintervals contained in some edges $E \in K^1$, and hence $\mathrm{ord}(\mathcal{C}(j)) \leq 2$,

(3) $\mathrm{mesh}(\mathcal{C}(j)) \leq 1/j$ for $j \in \mathbb{N}$,

(4) $\mathcal{C}(j+1)$ is a refinement of $f^{-1}(\mathcal{C}(j))@\mathcal{C}(j)$.

Also, since f is piece-wise monotone, we see that f is semi-open. By Lemma 3.19, we have the desired zero-dimensional cover $(C, \tilde{f})$ of (G, f). □

REMARK 3.

(1) If X is an n-dimensional simplicial manifold and $f : X \to X$ is a map such that f is injective on each simplex, then f is a two-sided zero-dimensional and semi-open map of X.

(2) Theorem 3.20 is not true for 2-dimensional polyhedra. Recall that there is a dynamical system (X, f) such that $f : X = I^2 \to X$ is a homeomorphism on X with $\dim P(f) = 1$ and (X, f) has no zero-dimensional cover via a finite-to-one map (see Example 2.2 of [**15**]).

(3) In Lemma 3.19, it can occur that $\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)}\mathcal{C}(j) = \infty$ for some $x \in X$.

4. Zero-dimensional decompositions of dynamical systems.

In dimension theory, the following decomposition theorem is well-known [**6**, Theorem 1.5.8]: A separable metric space X is $\dim X \leq n$ ($n \in \mathbb{Z}_+$) if and only if X can be represented as the union of $n+1$ subspaces $Z_0, Z_1, \ldots, Z_n$ of X such that $\dim Z_i \leq 0$ for each $i = 0, 1, \ldots, n$. In this section, we study the similar dynamical decomposition theorems of two-sided zero-dimensional maps (cf. [**7**]). We consider bright spaces and dark spaces of maps except n times, and by use of these spaces we prove some dynamical decomposition theorems of spaces related to given maps.

Let $f : X \to X$ be a map. A subset Z of X is a *bright space* of f except n times ($n \in \mathbb{Z}_+$) if for any $x \in X$,

$$|\{p \in \mathbb{Z}_+ \mid f^p(x) \notin Z\}| \leq n.$$

Also we say that $L = X - Z$ is a *dark space* of f except n times. Note that for any $x \in X$, $|O(x) \cap L| \leq n$ and $L \cap P(f) = \emptyset$. For each $z \in X$, put

$$t(z) = |\{p \in \mathbb{Z}_+ \mid f^p(z) \in L\}|.$$

Also we put

$$T(x) = \max\{t(z) \mid z \in EO(x)\}$$

for each $x \in X$. For a dark space L of f except n times and $0 \leq j \leq n$, we put

$$A_f(L, j) = \{x \in X \mid T(x) = j\}.$$

Note that $A_f(L, j)$ is f-invariant, i.e., $f(A_f(L, j)) \subset A_f(L, j)$ and $A_f(L, i) \cap A_f(L, j) = \emptyset$ if $i \neq j$. Hence we have the f-invariant decomposition related to the dark space L as follows;

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

The following theorem is an extension of [**7**, Theorem 2.4].

THEOREM 4.1. *Suppose that $f : X \to X$ is a two-sided zero-dimensional map of a compactum X with $\dim X = n < \infty$. Then there is a bright space Z of f except n times such that Z is a zero-dimensional dense G_δ-set of X and the dark space $L = X - Z$ of f is an $(n-1)$-dimensional F_σ-set of X if and only if $\dim P(f) \leq 0$.*

PROOF. Suppose $\dim P(f) \leq 0$. Then $\dim EP(f) \leq 0$. Since X is separable, there is a dense countable set D of X. Also we choose a zero-dimensional F_σ-set H of X with $\dim\,(X - H) \leq n - 1$ (see Proposition 3.2). Then the set $F = D \cup H$ is also a zero-dimensional F_σ-set of X. By Lemma 3.12, we have a countable base $\{B_i \mid i \in \mathbb{N}\}$ of X such that $\mathrm{ord}(\mathcal{G}) \leq n$ and $L \cap F = \emptyset$, where $\mathcal{G} = \{f^{-p}(\mathrm{bd}(B_i)) \mid i \in \mathbb{N}, p \in \mathbb{Z}_+\}$ and $L = \bigcup\{\mathrm{bd}(B_i) \mid i \in \mathbb{N}\}$. Put $Z = X - L$. Note that $D \subset Z$ and $L \subset X - F$. Then Z is dense in X and $\dim L \leq n - 1$ and hence Z and L are the desired spaces. Conversely, we assume that there exists a zero-dimensional bright space Z of f except n times. Then we see $P(f) \subset Z$, which implies that $\dim P(f) \leq 0$. □

The following corollary is an extension of [**7**, Corollary 2.5].

COROLLARY 4.2. *Suppose that X is a compactum with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a two-sided zero-dimensional onto map. Then there exists a zero-dimensional G_δ-dense set Z of X such that for any $n+1$ integers $k_0 < k_1 < \cdots < k_n$ $(k_i \in \mathbb{Z})$,*

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z)$$

if and only if $\dim P(f) \leq 0$.

PROOF. First, we assume $\dim P(f) \leq 0$. By Theorem 4.1, there is a bright space Z of f except n times such that Z is a zero-dimensional dense G_δ-set of X. Let $k_0 < k_1 < \cdots < k_n$ $(k_i \in \mathbb{Z})$ be any integers and let $x \in X$. Consider three cases as follows.

- Case (i): $0 \leq k_0$. Since f is onto, we can find $z \in X$ with $f^{k_n}(z) = x$. Since

$$|\{i \in \{0, 1, 2, \ldots, n\} \mid f^{k_n - k_i}(z) \notin Z\}| \leq n,$$

  there is an i such that $f^{k_n - k_i}(z)$ $(= y) \in Z$ and hence $f^{k_i}(y) = x$. This implies $x \in f^{k_i}(Z)$ and hence

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z).$$

- Case (ii): $k_n \leq 0$. Note that for each i, $-k_i \geq 0$. Since

$$|\{i \in \{0, 1, 2, \ldots, n\} \mid f^{-k_i}(x) \notin Z\}| \leq n,$$

there is an i such that $f^{-k_i}(x) \in Z$ and hence $x \in f^{k_i}(Z)$. This implies

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z).$$

- Case (iii): There is some j $(0 < j \leq n)$ with $k_{j-1} < 0 \leq k_j$. Since f is onto, we can find $z \in X$ with $f^{k_n}(z) = x$. Since

$$|\{i \in \{0, 1, 2, \ldots, n\} \mid f^{k_n - k_i}(z) \notin Z\}| \leq n,$$

there is an i such that $f^{k_n - k_i}(z) = y \in Z$. If $j \leq i$, $k_i \geq 0$ and hence $x = f^{k_n}(z) = f^{k_i}(y) \in f^{k_i}(Z)$. If $i < j$, we see that $k_i < 0$ and $y = f^{k_n - k_i}(z) = f^{-k_i}(x)$. Then $x \in f^{k_i}(y) \subset f^{k_i}(Z)$. Consequently we see

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z).$$

Conversely, we assume that there is such Z satisfying the condition of this corollary. We will show $P(f) \subset Z$. Let $x \in P(f)$ and let $f^k(x) = x$ for some $k \in \mathbb{N}$. Put $k_i = i \cdot k$ $(i = 0, 1, \ldots, n)$. Since $X = f^{-k_0}(Z) \cup f^{-k_1}(Z) \cup \cdots \cup f^{-k_n}(Z)$, we can find i such that $x \in f^{-k_i}(Z)$. Then $x = f^{k_i}(x) \in Z$ and hence $P(f) \subset Z$. Since $\dim Z = 0$, we see $\dim P(f) \leq 0$. □

By use of F_σ-dark spaces, we have the following decomposition theorem which is an extension of [**7**, Theorem 2.6].

Theorem 4.3. *Suppose that X is a compactum with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a two-sided zero-dimensional map on X with $\dim P(f) \leq 0$. If L is a dark space of f except n times such that L is an F_σ-set of X and $\dim (X - L) \leq 0$, then $\dim A_f(L, j) = 0$ for each $j = 0, 1, 2, \ldots, n$. In particular, we have the f-invariant zero-dimensional decomposition of X related to the dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Proof. Note that $A_f(L, 0) \subset X - L$ $(= Z)$ and hence $A_f(L, 0)$ is an f-invariant zero-dimensional subset of X. We will prove that $\dim A_f(L, j) = 0$ for each $j = 1, 2, \ldots, n$. Since L is an F_σ-set of X, we can put $L = \bigcup_{i=1}^{\infty} L_i$, where L_i is a closed subset of X. Let $1 \leq j \leq n$. For any j nonnegative integers $0 \leq k_1 < k_2 < \cdots < k_j$ and natural numbers $i_1, i_2, \ldots, i_j \in \mathbb{N}$, we consider the set

$$A(k_1, k_2, \ldots, k_j : L_{i_1}, L_{i_2}, \ldots, L_{i_j}) = \{x \in X \mid f^{k_p}(x) \in L_{i_p} \ (p = 1, 2, \ldots, j)\}.$$

Then we can easily see that $A(k_1, k_2, \ldots, k_j : L_{i_1}, L_{i_2}, \ldots, L_{i_j})$ is closed in X and $A(L, j)$ is represented as the following countable union of closed sets of $A(L, j)$:

$$\bigcup_{\alpha\in\Lambda} f^{-q}[f^p(A(k_1,\dots,k_j : L_{i_1},\dots,L_{i_j}))\cap A(L,j)],$$

where $\Lambda = \{(k_1,\dots,k_j;i_1,\dots,i_j;p,q) \mid 0 \le k_1 < \cdots < k_j, i_1,\dots,i_j \in \mathbb{N}, p,q \in \mathbb{Z}_+\}$. Note that for any $p \in \mathbb{Z}_+$,

$$f^{k_j+1}[f^p(A(k_1,\dots,k_j : L_{i_1},\dots,L_{i_j}))\cap A(L,j)] \subset Z.$$

Since f^{k_j+1} is a zero-dimensional map, by Theorem 3.4 we see that

$$f^p(A(k_1,\dots,k_j : L_{i_1},\dots,L_{i_j}))\cap A(L,j)$$

is zero-dimensional. By Proposition 3.1, we see that

$$\dim A_f(L,j) = 0. \qquad \square$$

In the case of positively expansive maps, we obtain decomposition theorem for a compact dark space L. We need the following lemma.

LEMMA 4.4 ([**12**, lemma 5.6]). *Suppose that $f : X \to X$ is a positively continuum-wise expansive map on a compactum X. Then there exists a $\delta > 0$ satisfying the condition: for any $\gamma > 0$ there is $N \in \mathbb{N}$ such that if A is a subcontinuum of X with* $\operatorname{diam} A \ge \gamma$, *then* $\operatorname{diam} f^n(A) \ge \delta$ *for all $n \ge N$.*

The following theorem is an extension of [**7**, Theorem 2.8].

THEOREM 4.5. *Suppose that X is a compactum with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a positively expansive map. Then there exists a compact $(n-1)$-dimensional dark space L of f except n times such that $\dim A_f(L,j) = 0$ for each $j = 0,1,2,\dots,n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the compact dark space L:*

$$X = A_f(L,0)\cup A_f(L,1)\cup\cdots\cup A_f(L,n).$$

PROOF. Note that f is a two-sided zero-dimensional map. Also, since f is a positively continuum-wise expansive map, we have a positive number δ as in Lemma 4.4. Since $\dim P(f) \le 0$, by Lemma 3.12 there is a finite open cover $\mathcal{C}$ of X such that

(1) $\operatorname{mesh}(\mathcal{C}) < \delta$,
(2) $\operatorname{ord}(\mathcal{G}) \le n$, where $\mathcal{G} = \{f^{-p}(\operatorname{bd}(C)) \mid C \in \mathcal{C}, p \in \mathbb{Z}_+\}$,
(3) $\operatorname{bd}(C)\cap EP(f) = \emptyset$ for each $C \in \mathcal{C}$ and
(4) $\dim H \le n-1$, where $H = \bigcup\{\operatorname{bd}(C) \mid C \in \mathcal{C}\}$.

Let $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ and put

$$c_1 = \operatorname{cl}(C_1), c_{i+1} = \operatorname{cl}(\operatorname{int}[C_{i+1} - \cup_{k\le i} C_k])\ (1 \le i \le m-1).$$

Then $\mathcal{C}' = \{c_1, c_2, \dots, c_m\}$ is a finite partition of X. Let $L = \bigcup\{\operatorname{bd}(c) \mid c \in \mathcal{C}'\}$. Then $L \subset H$ and we can easily see that L is a compact $(n-1)$-dimensional dark space of f except n times. We will show that $\dim A_f(L,j) = 0$ for each $j = 0,1,2,\dots,n$. Let

$1 \leq j \leq n$. For any j nonnegative integers $k_1 < k_2 < \cdots < k_j$, we consider the set

$$A(k_1, k_2, \ldots, k_j) = \{x \in A_f(L, j) \mid f^{k_p}(x) \in L \ (p = 1, 2, \ldots, j)\}.$$

Then we see that $A(k_1, k_2, \ldots, k_j)$ is closed in the subspace $A_f(L, j)$. We will show that $\dim A(k_1, k_2, \ldots, k_j) = 0$. Let $x \in A(k_1, k_2, \ldots, k_j)$ and let $\gamma > 0$ be any positive number. Then there is a sufficiently large natural number N such that $N > |k_i|$ $(i = 1, 2, \ldots, j)$ and N satisfies the condition of Lemma 4.4. Note that

$$A(k_1, k_2, \ldots, k_j) \subset \bigcup_{i=1}^{m} f^{-N}(\mathrm{int}(c_i))$$

and $f^{-N}(\mathrm{int}(c_i))$ $(1 \leq i \leq m)$ are mutually disjoint open sets of X. We can choose $1 \leq i \leq m$ such that $f^N(x) \in \mathrm{int}(c_i)$. Then the diameters of components of the compactum $f^{-N}(c_i)$ are less than γ. Since $f^{-N}(c_i)$ can be covered by finite mutually disjoint open sets of X whose diameters are less than γ, there is a closed and open neighborhood V of x in the subspace $A(k_1, k_2, \ldots, k_j)$ such that $V \subset f^{-N}(\mathrm{int}(c_i))$ and $\mathrm{diam}\ V < \gamma$. This implies that $\dim A(k_1, k_2, \ldots, k_j) = 0$. By the proof of Theorem 4.3, we see that $\dim A_f(L, j) = 0$. By the similar arguments to the case $j \geq 1$, we see that the case $j = 0$ is true, i.e., $\dim A_f(L, 0) = 0$. □

Acknowledgements. The authors wish to thank the referee for careful reading and useful suggestions that helped to improve the paper.

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