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Generalizations of the Conway–Gordon theorems and intrinsic knotting on complete graphs

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Abstract. In 1983, Conway and Gordon proved that for every spatial complete graph on six vertices, the sum of the linking numbers over all of the constituent two-component links is odd, and that for every spatial complete graph on seven vertices, the sum of the Arf invariants over all of the Hamiltonian knots is odd. In 2009, the second author gave integral lifts of the Conway–Gordon theorems in terms of the square of the linking number and the second coefficient of the Conway polynomial. In this paper, we generalize the integral Conway–Gordon theorems to complete graphs with arbitrary number of vertices greater than or equal to six. As an application, we show that for every rectilinear spatial complete graph whose number of vertices is greater than or equal to six, the sum of the second coefficients of the Conway polynomials over all of the Hamiltonian knots is determined explicitly in terms of the number of triangle-triangle Hopf links.

1. Introduction.

Throughout this paper we work in the piecewise linear category. Let G be a finite simple graph. An embedding f of G into the 3-dimensional Euclidean space $\mathbb{R}^3$ is called a *spatial embedding* of G, and the image $f(G)$ is called a *spatial graph* of G. Two spatial embeddings f and g of G are said to be *equivalent* if there exists a self homeomorphism Φ on $\mathbb{R}^3$ such that $\Phi(f(G)) = g(G)$. We call a subgraph γ of G homeomorphic to the circle a *cycle* of G, and a cycle of G containing exactly k edges a *k-cycle* of G. In particular, a k-cycle is also called a *Hamiltonian cycle* if k equals the number of vertices of G. We denote the set of all k-cycles of G by $\Gamma_k(G)$. Moreover, we denote the set of all pairs of two disjoint cycles of G consisting of a k-cycle and an l-cycle by $\Gamma_{k,l}(G)$. For a cycle γ (resp. a pair of disjoint cycles λ) and a spatial embedding f of G, $f(\gamma)$ (resp. $f(\lambda)$) is none other than a knot (resp. a 2-component link) in $f(G)$. In particular for a Hamiltonian cycle γ of G, we also call $f(\gamma)$ a *Hamiltonian knot* in $f(G)$.

Let K_n be the *complete graph* on n vertices, that is the graph consisting of n vertices such that each pair of its distinct vertices is connected by exactly one edge. Then the following fact is well-known as the *Conway–Gordon theorem*.

THEOREM 1.1 (Conway–Gordon [**8**]).

(1) *For any spatial embedding f of K_6, we have*

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$$\sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}\,(f(\lambda)) \equiv 1 \pmod 2,$$

where lk *denotes the linking number in* $\mathbb{R}^3$.

(2) *For any spatial embedding* f *of* K_7, *we have*

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2\,(f(\gamma)) \equiv 1 \pmod 2,$$

where a_2 *denotes the second coefficient of the Conway polynomial.*

The second coefficient of the Conway polynomial of a knot is also congruent with the *Arf invariant* of the knot modulo two [**21**, Corollary 10.8]. Theorem 1.1 implies that K_6 is *intrinsically linked*, that is, every spatial graph of K_6 contains a nonsplittable 2-component link, and K_7 is *intrinsically knotted*, that is, every spatial graph of K_7 contains a nontrivial knot. The Conway–Gordon theorem made a beginning of the study of intrinsic linking and knotting of graphs and has motivated a lot of studies of intrinsic properties of graphs (see for example [**11**, Sections 2–6]). On the other hand, as far as the authors know, there have been little results about a generalization of the Conway–Gordon type congruences for complete graphs on eight or more vertices. Our purposes in this paper are to generalize the Conway–Gordon theorems for complete graphs with arbitrary number of vertices greater than or equal to six and to investigate the behavior of the nontrivial Hamiltonian knots in a spatial complete graph. First of all, we recall an integral Conway–Gordon theorem for K_6 which was proven by the second author as follows.

Theorem 1.2 (Nikkuni [**24**]). *For any spatial embedding* f *of* K_6, *we have*

$$2\sum_{\gamma \in \Gamma_6(K_6)} a_2\,(f(\gamma)) - 2\sum_{\gamma \in \Gamma_5(K_6)} a_2\,(f(\gamma)) = \sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}\,(f(\lambda))^2 - 1. \tag{1.1}$$

Note that Theorem 1.1 (1) can be recovered by taking the modulo two reduction of (1.1), namely Theorem 1.2 is an integral lift of Theorem 1.1 (1). In [**24**], an integral lift of Theorem 1.1 (2) was also given (see Theorem 2.2 (1) of the present paper). In this paper, we generalize Theorem 1.2 for complete graphs with arbitrary number of vertices greater than or equal to six as follows.

Theorem 1.3. *Let* $n \geq 6$ *be an integer. For any spatial embedding* f *of* K_n, *we have*

$$\begin{aligned}\sum_{\gamma \in \Gamma_n(K_n)} a_2\,(f(\gamma)) &- (n-5)!\sum_{\gamma \in \Gamma_5(K_n)} a_2\,(f(\gamma)) \\ &= \frac{(n-5)!}{2}\Bigg(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f(\lambda))^2 - \binom{n-1}{5}\Bigg).\end{aligned} \tag{1.2}$$

By Theorem 1.3, we also obtain formulae of two types. First we have the following inequality, where the case of $n = 7$ has already been observed in [**24**, Lemma 4.2].

Corollary 1.4. *Let $n \geq 6$ be an integer. For any spatial embedding f of K_n, we have*

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2(f(\gamma)) \geq \frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!}.$$

The lower bound of Corollary 1.4 is sharp, see Remark 2.5. Next we also have the following congruence, that is a generalization of Theorem 1.1 (2).

Corollary 1.5. *Let $n \geq 7$ be an integer. For any spatial embedding f of K_n, we have the following congruence modulo $(n-5)!$:*

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv \begin{cases} -\dfrac{(n-5)!}{2}\dbinom{n-1}{5} & (n \equiv 0 \pmod 8) \\ 0 & (n \not\equiv 0, 7 \pmod 8) \\ \dfrac{(n-5)!}{2}\dbinom{n}{6} & (n \equiv 7 \pmod 8). \end{cases}$$

Corollary 1.5 contains the preceding results concerning Conway–Gordon type congruences on the sum of a_2, see Remark 2.6.

Theorem 1.3 (and Corollary 1.4) is also useful for investigating the behavior of the nontrivial Hamiltonian knots in rectilinear spatial complete graphs. Here, a spatial embedding f_{r} of a graph G is said to be *rectilinear* if for any edge e of G, $f_{\mathrm{r}}(e)$ is a straight line segment in $\mathbb{R}^3$. A rectilinear spatial graph appears in polymer chemistry as a mathematical model for chemical compounds (see [**3**, Section 7], for example), and the range of rectilinear spatial graph types is much narrower than the general spatial graphs. So we are interested in the behavior of the nontrivial Hamiltonian knots in a rectilinear spatial complete graph. Note that every knot (resp. link) contained in a rectilinear spatial graph of K_n is a "polygonal" knot (resp. link) with less than or equal to n sticks. It is well-known that every polygonal knot with less than or equal to five sticks is trivial (Proposition 3.1 (1)). Thus for rectilinear spatial complete graphs, by Theorem 1.3 we have the following immediately.

Theorem 1.6. *Let $n \geq 6$ be an integer. For any rectilinear spatial embedding f_{r} of K_n, we have*

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f_{\mathrm{r}}(\gamma)) = \frac{(n-5)!}{2}\left(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f_{\mathrm{r}}(\lambda))^2 - \binom{n-1}{5}\right).$$

Also note that a 2-component link with exactly six sticks is either a trivial link or a Hopf link (Proposition 3.1 (2)). Thus for any rectilinear spatial embedding f_{r} of K_n, $\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f_{\mathrm{r}}(\lambda))^2$ is equal to the number of "triangle-triangle" Hopf links in $f_{\mathrm{r}}(K_n)$. Then, by using Corollary 1.4 and Theorem 1.6, we can obtain the following upper and lower bounds of $\sum_{\gamma \in \Gamma_n(K_n)} a_2(f_{\mathrm{r}}(\gamma))$.

COROLLARY 1.7. *Let $n \geq 6$ be an integer. For any rectilinear spatial embedding f_{r} of K_n, we have*

$$\frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!} \leq \sum_{\gamma \in \Gamma_n(K_n)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \leq \frac{3(n-2)(n-5)(n-1)!}{2 \cdot 6!}.$$

The lower bound in Corollary 1.7 is also sharp, see Remark 2.7. However, the authors expect that the upper bound is not sharp if $n \geq 7$, see Example 3.3.

For every spatial embedding f of K_n (which does not need to be rectilinear), Hirano showed that there exist at least three nontrivial Hamiltonian knots with an odd value of a_2 in $f(K_8)$ [**16**], and Foisy showed that there exist at least $(n-1)(n-2)\cdots 9 \cdot 8$ nontrivial Hamiltonian knots with an odd value of a_2 in $f(K_n)$ if $n \geq 9$ [**5**]. On the other hand, Corollary 1.7 makes us possible to evaluate the number of nontrivial Hamiltonian knots with a positive value of a_2 in a rectilinear spatial graph of K_n as follows.

COROLLARY 1.8. *Let $n \geq 7$ be an integer. The minimum number of nontrivial Hamiltonian knots with a positive value of a_2 in every rectilinear spatial graph of K_n is at least*

$$r_n = \left\lceil \frac{(n-5)(n-6)(n-1)!/(2 \cdot 6!)}{\lfloor (n-3)^2(n-4)^2/32 \rfloor} \right\rceil,$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling function and the floor function, respectively.

We see that r_n is greater than Foisy's lower bound of the minimum number of nontrivial Hamiltonian knots with an odd value of a_2 if $n = 9, 10, 11$, see Remark 2.8.

The paper is organized as follows. We shall devote Section 2 to proofs of Theorem 1.3 and Corollaries 1.4, 1.5, 1.7 and 1.8. In Section 3, we give examples and present some open problems.

2. Proofs of Theorem 1.3 and its corollaries.

We show some lemmas which are needed to prove Theorem 1.3.

LEMMA 2.1. (1) *Let $n \geq 6$ be an integer. For any spatial embedding f of K_n, we have*

$$2 \sum_{\gamma \in \Gamma_6(K_n)} a_2\left(f(\gamma)\right) - 2(n-5) \sum_{\gamma \in \Gamma_5(K_n)} a_2\left(f(\gamma)\right) = \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2 - \binom{n}{6}.$$

(2) *Let $n \geq 7$ be an integer. For any spatial embedding f of K_n, we have*

$$\sum_{\lambda \in \Gamma_{3,4}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2 = 2(n-6) \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2.$$

PROOF OF LEMMA 2.1 (1). Note that each 5-cycle of K_n is shared by exactly $n-5$ subgraphs isomorphic to K_6 if $n \geq 6$. Then by applying Theorem 1.2 to the embedding f restricted to each of the subgraphs of K_n isomorphic to K_6 and taking the sum of both sides of (1.1) over all of them, we have the result. □

In order to prove Lemma 2.1 (2), we recall integral Conway–Gordon type theorems for spatial embeddings of K_7 and $K_{3,3,1}$ which were proven by the second author [**24**] and O'Donnol [**25**], respectively. Here, the *complete k-partite graph* $K_{n_1,n_2,\ldots,n_k}$ is the graph whose vertex set can be decomposed into k mutually disjoint nonempty sets $V_1, V_2, \ldots, V_k$ where the number of elements in V_i equals n_i such that no two vertices in V_i are connected by an edge and every pair of vertices in distinct sets V_i and V_j is connected by exactly one edge. See Figure 2.1 which illustrates $K_{3,3}$ and $K_{3,3,1}$. In particular for $K_{3,3,1}$, let us denote the subgraph of $K_{3,3,1}$ which is isomorphic to $K_{3,3}$ and does not contain the vertex u by H.

Theorem 2.2. (1) (Nikkuni [**24**]) *For any spatial embedding f of K_7, we have*

$$7 \sum_{\gamma\in\Gamma_7(K_7)} a_2(f(\gamma)) - 6 \sum_{\gamma\in\Gamma_6(K_7)} a_2(f(\gamma)) - 2 \sum_{\gamma\in\Gamma_5(K_7)} a_2(f(\gamma)) = 2 \sum_{\lambda\in\Gamma_{3,4}(K_7)} \mathrm{lk}(f(\lambda))^2 - 21.$$

(2) (O'Donnol [**25**]) *For any spatial embedding f of $K_{3,3,1}$, we have*

$$2 \sum_{\gamma\in\Gamma_7(K_{3,3,1})} a_2(f(\gamma)) - 4 \sum_{\gamma\in\Gamma_6(H)} a_2(f(\gamma)) - 2 \sum_{\gamma\in\Gamma_5(K_{3,3,1})} a_2(f(\gamma)) = \sum_{\lambda\in\Gamma_{3,4}(K_{3,3,1})} \mathrm{lk}(f(\lambda))^2 - 1.$$

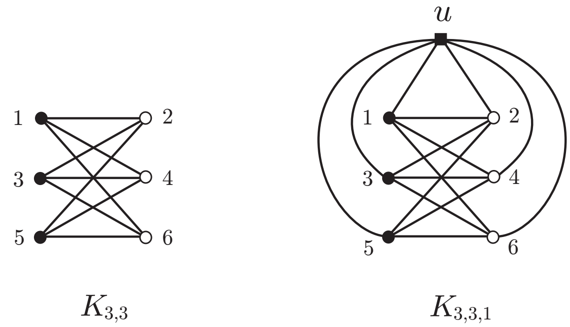


Figure 2.1. $K_{3,3}$ and $K_{3,3,1}$.

Then by applying Theorem 2.2 (2) to each of the subgraphs of K_7 isomorphic to $K_{3,3,1}$ and combining with Theorem 2.2 (1), we also have the following equation for every spatial embedding of K_7.

Theorem 2.3. *For any spatial embedding f of K_7, we have*

$$\sum_{\lambda\in\Gamma_{3,4}(K_7)} \mathrm{lk}\,(f(\lambda))^2 = 2\sum_{\lambda\in\Gamma_{3,3}(K_7)} \mathrm{lk}\,(f(\lambda))^2. \tag{2.1}$$

Proof of Theorem 2.3. For vertices $1,2,\ldots,6$ and u of $K_{3,3,1}$, we call the vertices $1,3,5$ the *black vertices*, the vertices $2,4,6$ the *white vertices* and the vertex u the *square vertex.* Note that a k-cycle of $K_{3,3,1}$ contains the square vertex if k is odd. There are exactly seventy subgraphs G_i $(i=1,2,\ldots,70)$ of K_7 isomorphic to $K_{3,3,1}$, because there are seven ways to choose the square vertex and $\binom{6}{3}/2$ ways to choose the remaining black and white vertices. Then for a spatial embedding f of K_7, by applying Theorem 2.2 (2) to the embedding f restricted to G_i, we have

$$\begin{aligned} 2\sum_{\gamma\in\Gamma_7(G_i)} a_2\,(f(\gamma)) - 4\sum_{\gamma\in\Gamma_6(H_i)} a_2\,(f(\gamma)) - 2\sum_{\gamma\in\Gamma_5(G_i)} a_2\,(f(\gamma)) \\ = \sum_{\lambda\in\Gamma_{3,4}(G_i)} \mathrm{lk}\,(f(\lambda))^2 - 1, \end{aligned} \tag{2.2}$$

where H_i is the subgraph of G_i isomorphic to $K_{3,3}$ not containing the square vertex $(i=1,2,\ldots,70)$. Let us take the sum of both sides of (2.2) for all i. Since each 7-cycle γ of K_7 is shared by exactly seven G_i's (there are seven ways to choose the square vertex from the vertices of γ and then the assignment of the black and white vertices is uniquely determined), we have

$$\sum_{i=1}^{70}\sum_{\gamma\in\Gamma_7(G_i)} a_2\,(f(\gamma)) = 7\sum_{\gamma\in\Gamma_7(K_7)} a_2\,(f(\gamma))\,. \tag{2.3}$$

Since for each 6-cycle γ of K_7 there exists the unique G_i such that H_i contains γ (the assignment of the black and white vertices is uniquely determined), we have

$$\sum_{i=1}^{70}\sum_{\gamma\in\Gamma_6(H_i)} a_2\,(f(\gamma)) = \sum_{\gamma\in\Gamma_6(K_7)} a_2\,(f(\gamma))\,. \tag{2.4}$$

Since each 5-cycle γ of K_7 is shared by exactly ten G_i's (there are five ways to choose the square vertex from the vertices of γ and two ways to choose the remaining black and white vertices), we have

$$\sum_{i=1}^{70}\sum_{\gamma\in\Gamma_5(G_i)} a_2\,(f(\gamma)) = 10\sum_{\gamma\in\Gamma_5(K_7)} a_2\,(f(\gamma))\,. \tag{2.5}$$

Since each pair of two disjoint cycles λ in $\Gamma_{3,4}\,(K_7)$ is shared by exactly six G_i's (there are three ways to choose the square vertex from the 3-cycle in λ and two ways to choose the remaining black and white vertices), we have

$$\sum_{i=1}^{70}\sum_{\lambda\in\Gamma_{3,4}(G_i)} \mathrm{lk}\,(f(\lambda))^2 = 6\sum_{\lambda\in\Gamma_{3,4}(K_7)} \mathrm{lk}\,(f(\lambda))^2\,. \tag{2.6}$$

Thus by combining (2.3), (2.4), (2.5) and (2.6) with (2.2), we have

$$7\sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f(\gamma)\right) - 2\sum_{\gamma\in\Gamma_6(K_7)} a_2\left(f(\gamma)\right) - 10\sum_{\gamma\in\Gamma_5(K_7)} a_2\left(f(\gamma)\right)$$
$$= 3\sum_{\lambda\in\Gamma_{3,4}(K_7)} \mathrm{lk}\left(f(\lambda)\right)^2 - 35. \tag{2.7}$$

Then by (2.7) and Theorem 2.2 (1), we have

$$4\sum_{\gamma\in\Gamma_6(K_7)} a_2\left(f(\gamma)\right) - 8\sum_{\gamma\in\Gamma_5(K_7)} a_2\left(f(\gamma)\right) = \sum_{\lambda\in\Gamma_{3,4}(K_7)} \mathrm{lk}\left(f(\lambda)\right)^2 - 14. \tag{2.8}$$

On the other hand, by Lemma 2.1 (1) we have

$$2\sum_{\gamma\in\Gamma_6(K_7)} a_2\left(f(\gamma)\right) - 4\sum_{\gamma\in\Gamma_5(K_7)} a_2\left(f(\gamma)\right) = \sum_{\lambda\in\Gamma_{3,3}(K_7)} \mathrm{lk}\left(f(\lambda)\right)^2 - 7. \tag{2.9}$$

By (2.8) and (2.9), we have the desired conclusion. □

Proof of Lemma 2.1 (2). Note that each pair of two disjoint 3-cycles of K_n is shared by exactly $n-6$ subgraphs isomorphic to K_7 if $n \geq 7$. Then by applying Theorem 2.3 to the embedding f restricted to each of the subgraphs of K_n isomorphic to K_7 and taking the sum of both sides of (2.1) over all of them, we have the result. □

Now we show a lemma which plays a major role in the proof of Theorem 1.3. The proof is in the same spirit as that of Theorem 2.2 (1) in [**24**].

Lemma 2.4. *Let $n \geq 7$ be an integer. Assume that there exist three constants b, c and d such that*

$$\sum_{\gamma\in\Gamma_{n-1}(K_{n-1})} a_2\left(g(\gamma)\right) + b\sum_{\gamma\in\Gamma_5(K_{n-1})} a_2\left(g(\gamma)\right) = c\sum_{\lambda\in\Gamma_{3,3}(K_{n-1})} \mathrm{lk}\left(g(\lambda)\right)^2 + d$$

for any spatial embedding g of K_{n-1}. Then we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) + b(n-5)\sum_{\gamma\in\Gamma_5(K_n)} a_2\left(f(\gamma)\right)$$
$$= \frac{c(n-6)(n+1)-3b}{n}\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2 + d(n-1) + \frac{b}{2}\binom{n-1}{5}$$

for any spatial embedding f of K_n.

Proof. In the following, we denote the edge of K_n connecting two distinct vertices i and j by $\overline{ij}$, and denote a path of length 2 of K_n consisting of two edges $\overline{ij}$ and $\overline{jk}$ by $\overline{ijk}$. We denote the subgraph of K_n obtained from K_n by deleting the vertex m and all of the edges incident to m by $K_{n-1}^{(m)}$ ($m = 1, 2, \ldots, n$). Actually $K_{n-1}^{(m)}$ is isomorphic to K_{n-1} for any m. For $1 \leq i < j \leq n$ and $i, j \neq m$, let $F_{ij}^{(m)}$ be the subgraph of K_n obtained from K_n by deleting the edges $\overline{ij}$ and $\overline{mk}$ for all k with $1 \le k \le n,\ k \ne i,j$. Note that $F_{ij}^{(m)}$ is homeomorphic to K_{n-1}, namely $F_{ij}^{(m)}$ is obtained from $K_{n-1}^{(m)}$ by subdividing the edge $\overline{ij}$ by the vertex m, see Figure 2.2.

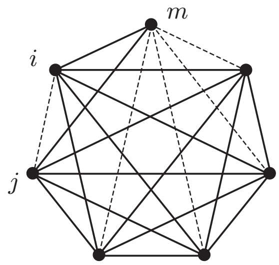


Figure 2.2. $F_{ij}^{(m)}$ $(n = 7)$.

Let f be a spatial embedding of K_n. Then for the embedding f restricted to $F_{ij}^{(m)}$, by the assumption we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_n(F_{ij}^{(m)})} a_2\left(f(\gamma)\right) + \sum_{\substack{\gamma\in\Gamma_{n-1}(K_{n-1}^{(m)})\\ \overline{ij}\not\subset\gamma}} a_2\left(f(\gamma)\right) \\
&\qquad + b\Bigg(\sum_{\substack{\gamma\in\Gamma_6(F_{ij}^{(m)})\\ \overline{imj}\subset\gamma}} a_2\left(f(\gamma)\right) + \sum_{\substack{\gamma\in\Gamma_5(K_{n-1}^{(m)})\\ \overline{ij}\not\subset\gamma}} a_2\left(f(\gamma)\right) \Bigg) \\
&\quad = c\Bigg(\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,4}(F_{ij}^{(m)})\\ \gamma\in\Gamma_4(F_{ij}^{(m)}),\ \gamma'\in\Gamma_3(F_{ij}^{(m)})\\ \overline{imj}\subset\gamma}} \mathrm{lk}\left(f(\lambda)\right)^2 + \sum_{\substack{\lambda\in\Gamma_{3,3}(K_{n-1}^{(m)})\\ \overline{ij}\not\subset\lambda}} \mathrm{lk}\left(f(\lambda)\right)^2 \Bigg) + d. \qquad (2.10)
\end{aligned}$$

Let us take the sum of both sides of (2.10) over $1 \le i < j \le n$ and $i,j \ne m$. For an n-cycle γ of K_n, let i and j be the two vertices of K_n which are adjacent to m in γ $(1 \le i < j \le n$ and $i,j \ne m)$. Then γ is an n-cycle of $F_{ij}^{(m)}$. This implies that

$$\sum_{\substack{1\le i<j\le n\\ i,j\ne m}} \sum_{\gamma\in\Gamma_n(F_{ij}^{(m)})} a_2\left(f(\gamma)\right) = \sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right). \qquad (2.11)$$

For an $(n-1)$-cycle γ of $K_{n-1}^{(m)}$, let $\overline{ij}$ be an edge of $K_{n-1}^{(m)}$ which is not contained in γ. Note that there are $\binom{n-1}{2} - (n-1) = (n^2-5n+4)/2$ ways to choose such a pair of i and j. This implies that

$$\sum_{\substack{1\le i<j\le n\\ i,j\ne m}} \sum_{\substack{\gamma\in\Gamma_{n-1}(K_{n-1}^{(m)})\\ \overline{ij}\not\subset\gamma}} a_2\left(f(\gamma)\right) = \frac{n^2-5n+4}{2} \sum_{\gamma\in\Gamma_{n-1}(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right). \qquad (2.12)$$

For a 6-cycle γ of K_n which contains the vertex m, let i and j be the two vertices of K_n which are adjacent to m in γ. Then γ is a 6-cycle of $F_{ij}^{(m)}$ which contains $\overline{imj}$. This implies that

$$\sum_{\substack{1\le i<j\le n\\ i,j\neq m}}\ \sum_{\substack{\gamma\in\Gamma_6(F_{ij}^{(m)})\\ \overline{imj}\subset\gamma}} a_2\left(f(\gamma)\right) = \sum_{\substack{\gamma\in\Gamma_6(K_n)\\ m\subset\gamma}} a_2\left(f(\gamma)\right). \tag{2.13}$$

For a 5-cycle γ of $K_{n-1}^{(m)}$, let $\overline{ij}$ be an edge of $K_{n-1}^{(m)}$ which is not contained in γ. Note that there are $\binom{n-1}{2}-5=(n^2-3n-8)/2$ ways to choose such a pair of i and j. This implies that

$$\sum_{\substack{1\le i<j\le n\\ i,j\neq m}}\ \sum_{\substack{\gamma\in\Gamma_5(K_{n-1}^{(m)})\\ \overline{ij}\not\subset\gamma}} a_2\left(f(\gamma)\right) = \frac{n^2-3n-8}{2}\sum_{\gamma\in\Gamma_5(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right). \tag{2.14}$$

For a pair of disjoint cycles λ of K_n consisting of a 4-cycle γ which contains the vertex m and a 3-cycle γ', let i and j be the two vertices of K_n which are adjacent to m in γ. Then λ is a pair of disjoint cycles of K_n consisting of a 4-cycle γ which contains $\overline{imj}$ and a 3-cycle γ'. This implies that

$$\sum_{\substack{1\le i<j\le n\\ i,j\neq m}}\ \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,4}(F_{ij}^{(m)})\\ \gamma\in\Gamma_4(F_{ij}^{(m)}),\ \gamma'\in\Gamma_3(F_{ij}^{(m)})\\ \overline{imj}\subset\gamma}} \mathrm{lk}\left(f(\lambda)\right)^2 = \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,4}(K_n)\\ \gamma\in\Gamma_4(K_n),\ \gamma'\in\Gamma_3(K_n)\\ m\subset\gamma}} \mathrm{lk}\left(f(\lambda)\right)^2. \tag{2.15}$$

For a pair of disjoint 3-cycles λ of $K_{n-1}^{(m)}$, let $\overline{ij}$ be an edge of $K_{n-1}^{(m)}$ which is not contained in λ. Note that there are $\binom{n-1}{2}-6=(n^2-3n-10)/2$ ways to choose such a pair of i and j. This implies that

$$\sum_{\substack{1\le i<j\le n\\ i,j\neq m}}\ \sum_{\substack{\lambda\in\Gamma_{3,3}(K_{n-1}^{(m)})\\ \overline{ij}\not\subset\lambda}} \mathrm{lk}\left(f(\lambda)\right)^2 = \frac{n^2-3n-10}{2}\sum_{\lambda\in\Gamma_{3,3}(K_{n-1}^{(m)})} \mathrm{lk}\left(f(\lambda)\right)^2. \tag{2.16}$$

By combining (2.11), (2.12), (2.13), (2.14), (2.15) and (2.16) with (2.10), we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) + \frac{n^2-5n+4}{2}\sum_{\gamma\in\Gamma_{n-1}(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right)\\
&\quad + b\left(\sum_{\substack{\gamma\in\Gamma_6(K_n)\\ m\subset\gamma}} a_2\left(f(\gamma)\right) + \frac{n^2-3n-8}{2}\sum_{\gamma\in\Gamma_5(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right)\right)\\
&\quad = c\left(\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,4}(K_n)\\ \gamma\in\Gamma_4(K_n),\ \gamma'\in\Gamma_3(K_n)\\ m\subset\gamma}} \mathrm{lk}\left(f(\lambda)\right)^2 + \frac{n^2-3n-10}{2}\sum_{\lambda\in\Gamma_{3,3}(K_{n-1}^{(m)})} \mathrm{lk}\left(f(\lambda)\right)^2\right)
\end{aligned}$$

$$+\frac{d(n^2-3n+2)}{2}. \tag{2.17}$$

Then for the embedding f restricted to $K_{n-1}^{(m)}$, by the assumption we have

$$\sum_{\gamma\in\Gamma_{n-1}(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right) = -b \sum_{\gamma\in\Gamma_5(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right) + c \sum_{\lambda\in\Gamma_{3,3}(K_{n-1}^{(m)})} \operatorname{lk}\left(f(\lambda)\right)^2 + d. \tag{2.18}$$

By combining (2.17) and (2.18), we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) + b \sum_{\substack{\gamma\in\Gamma_6(K_n)\\ m\subset\gamma}} a_2\left(f(\gamma)\right) + b(n-6) \sum_{\gamma\in\Gamma_5(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right)\\
&\quad = c \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,4}(K_n)\\ \gamma\in\Gamma_4(K_n),\ \gamma'\in\Gamma_3(K_n)\\ m\subset\gamma}} \operatorname{lk}\left(f(\lambda)\right)^2 + c(n-7) \sum_{\lambda\in\Gamma_{3,3}(K_{n-1}^{(m)})} \operatorname{lk}\left(f(\lambda)\right)^2 + d(n-1).
\end{aligned} \tag{2.19}$$

Now we take the sum of both sides of (2.19) over $m = 1, 2, \ldots, n$. For a 6-cycle γ of K_n, let m be a vertex of K_n which is contained in γ. Note that there are six ways to choose such a vertex m. This implies that

$$\sum_{m=1}^{n} \sum_{\substack{\gamma\in\Gamma_6(K_n)\\ m\subset\gamma}} a_2\left(f(\gamma)\right) = 6 \sum_{\gamma\in\Gamma_6(K_n)} a_2\left(f(\gamma)\right). \tag{2.20}$$

For a 5-cycle γ of K_n, let m be a vertex of K_n which is not contained in γ. Then γ is a 5-cycle of $K_{n-1}^{(m)}$. Note that there are $n-5$ ways to choose such a vertex m. This implies that

$$\sum_{m=1}^{n} \sum_{\gamma\in\Gamma_5(K_{n-1}^{(m)})} a_2\left(f(\gamma)\right) = (n-5) \sum_{\gamma\in\Gamma_5(K_n)} a_2\left(f(\gamma)\right). \tag{2.21}$$

For a pair of disjoint cycles λ of K_n consisting of a 4-cycle γ and a 3-cycle γ', let m be a vertex of K_n which is contained in γ. Note that there are four ways to choose such a vertex m. This implies that

$$\sum_{m=1}^{n} \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,4}(K_n)\\ \gamma\in\Gamma_4(K_n),\ \gamma'\in\Gamma_3(K_n)\\ m\subset\gamma}} \operatorname{lk}\left(f(\lambda)\right)^2 = 4 \sum_{\lambda\in\Gamma_{3,4}(K_n)} \operatorname{lk}\left(f(\lambda)\right)^2. \tag{2.22}$$

For a pair of two disjoint 3-cycles λ of K_n, let m be a vertex of K_n which is not contained in λ. Then λ is a pair of two disjoint 3-cycles of $K_{n-1}^{(m)}$. Note that there are $n-6$ ways to choose such a vertex m. This implies that

$$\sum_{m=1}^{n} \sum_{\lambda\in\Gamma_{3,3}(K_{n-1}^{(m)})} \operatorname{lk}\left(f(\lambda)\right)^2 = (n-6) \sum_{\gamma\in\Gamma_{3,3}(K_n)} \operatorname{lk}\left(f(\lambda)\right)^2. \tag{2.23}$$

By combining (2.20), (2.21), (2.22) and (2.23) with (2.19), we have

$$n \sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) + 6b \sum_{\gamma\in\Gamma_6(K_n)} a_2\left(f(\gamma)\right) + b(n-5)(n-6) \sum_{\gamma\in\Gamma_5(K_n)} a_2\left(f(\gamma)\right)$$
$$= 4c \sum_{\lambda\in\Gamma_{3,4}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2 + c(n-6)(n-7) \sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2 + dn(n-1). \quad (2.24)$$

Then by (2.24) and Lemma 2.1 (1) and (2), we have the desired conclusion. □

PROOF OF THEOREM 1.3. We prove this by induction on n. In the case of $n = 6$, by Theorem 1.2 we have the result. Assume that $n \geq 7$, then we have

$$\sum_{\gamma\in\Gamma_n(K_{n-1})} a_2\left(g(\gamma)\right) - (n-6)! \sum_{\gamma\in\Gamma_5(K_{n-1})} a_2\left(g(\gamma)\right)$$
$$= \frac{(n-6)!}{2} \sum_{\lambda\in\Gamma_{3,3}(K_{n-1})} \mathrm{lk}\left(g(\lambda)\right)^2 - \frac{(n-6)!}{2}\binom{n-2}{5} \quad (2.25)$$

for any spatial embedding g of K_{n-1}. Then by (2.25) and Lemma 2.4, we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) - (n-5)! \sum_{\gamma\in\Gamma_5(K_n)} a_2\left(f(\gamma)\right)$$
$$= \frac{1}{n}\left(\frac{(n-6)!}{2}(n-6)(n+1) + 3(n-6)!\right) \sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2$$
$$- \frac{(n-6)!}{2}\binom{n-2}{5}(n-1) - \frac{(n-6)!}{2}\binom{n-1}{5}$$
$$= \frac{(n-5)!}{2} \sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2 - \frac{(n-5)!}{2}\binom{n-1}{5}$$

for any spatial embedding f of K_n. This completes the proof. □

PROOF OF COROLLARY 1.4. Note that no pair of two disjoint 3-cycles λ of K_n is shared by two distinct subgraphs of K_n isomorphic to K_6. Then Theorem 1.1 (1) implies that $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2$ is greater than or equal to the number of subgraphs of K_n isomorphic to K_6, that is equal to $\binom{n}{6}$, and by a direct calculation we have

$$\frac{(n-5)!}{2}\left(\binom{n}{6} - \binom{n-1}{5}\right) = \frac{(n-5)(n-6)(n-1)!}{2\cdot 6!}. \quad (2.26)$$

Thus by (2.26) and Theorem 1.3, we have the result. □

REMARK 2.5. Endo–Otsuki introduced a certain special spatial embedding f_{b} of K_n, a *canonical book presentation* of K_n [**9**], and Otsuki also showed that $f_{\mathrm{b}}\left(K_n\right)$ contains exactly $\binom{n}{6}$ Hopf links corresponding to all the pairs of two disjoint 3-cycles of K_n if $n \geq 6$ [**26**]. Thus the lower bound of Corollary 1.4 is sharp. Furthermore, for any 5-cycle γ of K_n, $f_{\mathrm{b}}(\gamma)$ is a trivial knot. Thus for an integer $n \geq 6$, we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f_{\mathrm{b}}(\gamma)\right) = \frac{(n-5)(n-6)(n-1)!}{2\cdot 6!}.$$

Proof of Corollary 1.5. For any two spatial embeddings f and g of K_n, by Theorem 1.3, we have

$$\begin{aligned}&\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) - \sum_{\gamma\in\Gamma_n(K_n)} a_2\left(g(\gamma)\right)\\ &\quad\equiv \frac{(n-5)!}{2}\left(\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2 - \sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(g(\lambda)\right)^2\right) \pmod{(n-5)!}. \end{aligned}\tag{2.27}$$

Since $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(f(\lambda)\right)^2$ and $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(g(\lambda)\right)^2$ have the same parity, that is also equal to the parity of $\binom{n}{6}$, by (2.27), we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) \equiv \sum_{\gamma\in\Gamma_n(K_n)} a_2\left(g(\gamma)\right) \pmod{(n-5)!}.\tag{2.28}$$

Note that there exists a spatial embedding g of K_n such that

$$\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}\left(g(\lambda)\right)^2 = \binom{n}{6},\tag{2.29}$$

see Remark 2.5 or Remark 2.7. Thus by (2.28) and (2.29), we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) \equiv \frac{(n-5)!}{2}\left(\binom{n}{6} - \binom{n-1}{5}\right) \pmod{(n-5)!}\tag{2.30}$$

for any spatial embedding f of K_n. Here, it can be seen that $\binom{n}{6}$ is odd if and only if $n\equiv 6,7 \pmod 8$, and $\binom{n-1}{5}$ is odd if and only if $n\equiv 0,6 \pmod 8$ by an application of Lucas's theorem for binomial coefficients (see [**10**] for example). If $n\not\equiv 0,7 \pmod 8$, then since $\binom{n}{6}-\binom{n-1}{5}$ is even, by (2.30), we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) \equiv 0 \pmod{(n-5)!}.$$

If $n\equiv 0 \pmod 8$, then since $\binom{n}{6}$ is even and $\binom{n-1}{5}$ is odd, by (2.30), we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) \equiv -\frac{(n-5)!}{2}\binom{n-1}{5} \pmod{(n-5)!}.$$

If $n\equiv 7 \pmod 8$, then since $\binom{n}{6}$ is odd and $\binom{n-1}{5}$ is even, by (2.30), we have

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f(\gamma)\right) \equiv \frac{(n-5)!}{2}\binom{n}{6} \pmod{(n-5)!}.$$

This completes the proof. $\square$

Remark 2.6. By applying the case of $n = 7$ in Corollary 1.5, we have

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2\left(f(\gamma)\right) \equiv \frac{2!}{2}\binom{7}{6} \equiv 1 \pmod{2},$$

that is, Theorem 1.1 (2). On the other hand, for any spatial embedding f of K_8, it was shown that $\sum_{\gamma \in \Gamma_8(K_8)} a_2\left(f(\gamma)\right) \equiv 0 \pmod{3}$ by Foisy [**13**] and $1 \pmod{2}$ by Hirano [**15**]. These results imply that $\sum_{\gamma \in \Gamma_8(K_8)} a_2\left(f(\gamma)\right) \equiv 3 \pmod{6}$, and it can also be shown by applying the case of $n = 8$ in Corollary 1.5:

$$\sum_{\gamma \in \Gamma_8(K_8)} a_2\left(f(\gamma)\right) \equiv -\frac{3!}{2}\binom{7}{5} = -63 \equiv 3 \pmod{6}.$$

Hirano also showed that $\sum_{\gamma \in \Gamma_n(K_n)} a_2\left(f(\gamma)\right) \equiv 0 \pmod{2}$ for any spatial embedding f of K_n if $n \geq 9$ [**15**]. Corollary 1.5 also generalizes it remarkably.

Proof of Corollary 1.7. We obtain the desired lower bound from Corollary 1.4 directly, since for every 5-cycle γ, $f_{\mathrm{r}}(\gamma)$ is trivial. On the other hand, it is known that every rectilinear spatial graph of K_6 contains at most three Hopf links (Hughes [**17**], Huh–Jeon [**18**], Nikkuni [**24**]). This implies that $\sum_{\lambda \in \Gamma_{3,3}(K_n)} \operatorname{lk}\left(f_{\mathrm{r}}(\lambda)\right)^2$ is less than or equal to $3\binom{n}{6}$, and by a direct calculation we have

$$\frac{(n-5)!}{2}\left(3\binom{n}{6} - \binom{n-1}{5}\right) = \frac{3(n-2)(n-5)(n-1)!}{2 \cdot 6!}. \tag{2.31}$$

Thus by (2.31) and Theorem 1.6, we get the desired upper bound. $\square$

Remark 2.7. A special rectilinear spatial embedding f_{sr} of K_n can be constructed by taking n vertices $1, 2, \ldots, n$ of K_n in order on the moment curve (t, t^2, t^3) in $\mathbb{R}^3$ and connecting every pair of two distinct vertices i and j by a straight line segment, see Figure 2.3 for $n = 6, 7, 8$. We call f_{sr} the *standard rectilinear spatial embedding* of K_n. For the standard rectilinear spatial embedding f_{sr} of K_n $(n \geq 6)$ and a subgraph F of K_n isomorphic to K_6, it can be easily seen that the embedding f_{sr} restricted to F is equivalent to the standard rectilinear spatial embedding of K_6. Since the standard rectilinear spatial graph of K_6 contains exactly one nonsplittable 2-component link which is a Hopf link, $f_{\mathrm{sr}}\left(K_n\right)$ contains exactly $\binom{n}{6}$ triangle-triangle Hopf links. Thus the lower bound in Corollary 1.7 is sharp.

Before proving Corollary 1.8, we recall two geometric invariants of knots and links. For a knot or link L, the *crossing number* of L is the minimum number of crossings in a regular diagram of L on the plane, denoted by $c(L)$, and the *stick number* of L is the minimum number of edges in a polygon which represents L, denoted by $s(L)$.

Proof of Corollary 1.8. For a knot K, it has been shown that

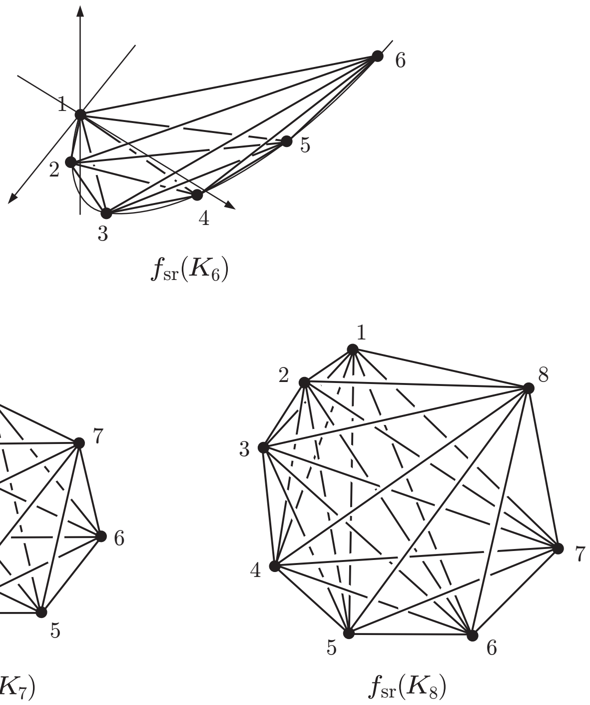


Figure 2.3. Standard rectilinear spatial embedding f_{sr} of K_n $(n = 6, 7, 8)$.

$$c(K) \leq \frac{(s(K)-3)\,(s(K)-4)}{2} \tag{2.32}$$

by Calvo [**7**, Theorem 4], and also has been shown that

$$a_2(K) \leq \frac{c(K)^2}{8} \tag{2.33}$$

by Polyak–Viro [**27**, Theorem 1.E]. By combining (2.32) and (2.33), for a polygonal knot K with less than or equal to n sticks, we have

$$a_2(K) \leq \left\lfloor \frac{(n-3)^2(n-4)^2}{32} \right\rfloor . \tag{2.34}$$

Then by the lower bound in Corollary 1.7 and (2.34), we have the desired estimation from below. □

Remark 2.8. The concrete values of r_n for $7 \leq n \leq 15$ are given in the following table. Note that in the case of $n = 8$, we can obtain an estimate from below better than r_8 of the number of nontrivial Hamiltonian knots with a positive value of a_2 in every rectilinear spatial graph of K_8, see Example 3.5 and Remark 3.6.

n	7	8	9	10	11	12	13	14	15	$\cdots$
r_n	1	2	12	92	772	7187	73628	823680	10015889	$\cdots$

3. Examples and problems.

In the following examples, we denote a k-cycle $\overline{i_1 i_2} \cup \overline{i_2 i_3} \cup \cdots \cup \overline{i_{k-1} i_k} \cup \overline{i_k i_1}$ of K_n by $[i_1 i_2 \cdots i_k]$. We also recall the following fundamental results on stick numbers for knots and links (see Adams [**3**, Section 1.6], Negami [**23**, Theorem 6], Adams–Brennan–Greilsheimer–Woo [**4**, Theorem 2.1] and Calvo [**7**, Theorem 1]), where we denote each of knots and links appearing in the statement by using its label in Rolfsen's table [**30**].

Proposition 3.1. *Let L be a link. Then the following statements hold.*

(1) *If L is a nontrivial knot, then $s(L) \geq 6$.*

(2) *$s(L) = 6$ if and only if L is equivalent to 3_1, 0_1^2 or 2_1^2.*

(3) *$s(L) = 7$ if and only if L is equivalent to 4_1 or 4_1^2.*

(4) *$s(L) = 8$ if and only if L is equivalent to 5_1, 5_2, 6_1, 6_2, 6_3, the granny knot $3_1 \# 3_1$, the square knot $3_1 \# 3_1^*$, 8_{19}, 8_{20} or 5_1^2.*

Example 3.2. Let f_{r} be a rectilinear spatial embedding of K_6. Then by Theorem 1.6 (Theorem 1.2) and Corollary 1.7, we have

$$\sum_{\gamma \in \Gamma_6(K_6)} a_2\left(f_{\mathrm{r}}(\gamma)\right) = \frac{1}{2} \sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}\left(f_{\mathrm{r}}(\lambda)\right)^2 - \frac{1}{2}, \tag{3.1}$$

$$0 \leq \sum_{\gamma \in \Gamma_6(K_6)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \leq 1. \tag{3.2}$$

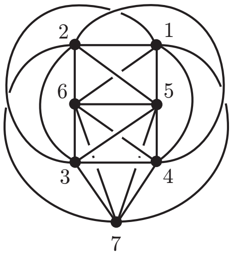
As it has been shown in [**24**, Section 4], (3.1) and (3.2) enable us to give an alternative topological proof of the fact that every rectilinear spatial graph $f_{\mathrm{r}}(K_6)$ contains at most one trefoil knot, in particular, $f_{\mathrm{r}}(K_6)$ does not contain a trefoil knot if and only if $f_{\mathrm{r}}(K_6)$ contains exactly one Hopf link, and $f_{\mathrm{r}}(K_6)$ contains a trefoil knot if and only if $f_{\mathrm{r}}(K_6)$ contains exactly three Hopf links, which was originally proven by Huh–Jeon [**18**] in combinatorial way. Actually, it follows from Proposition 3.1 (1) and (2) that $\sum_{\gamma \in \Gamma_6(K_6)} a_2\left(f_{\mathrm{r}}(\gamma)\right)$ equals the number of trefoil knots in $f_{\mathrm{r}}(K_6)$ because $a_2(3_1) = 1$, and $\sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}\left(f_{\mathrm{r}}(\lambda)\right)^2$ equals the number of Hopf links in $f_{\mathrm{r}}(K_6)$.

Example 3.3. For a spatial embedding f of K_7, by Theorem 1.3, we have

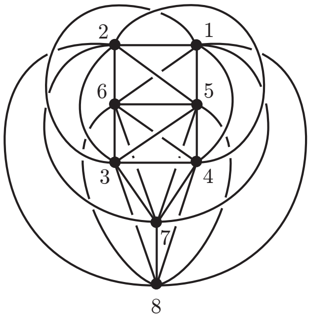
$$\sum_{\gamma \in \Gamma_7(K_7)} a_2\left(f(\gamma)\right) - 2 \sum_{\gamma \in \Gamma_5(K_7)} a_2\left(f(\gamma)\right) = \sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathrm{lk}\left(f(\lambda)\right)^2 - 6, \tag{3.3}$$

and we also have

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2\left(f(\gamma)\right) - 2 \sum_{\gamma \in \Gamma_5(K_7)} a_2\left(f(\gamma)\right) \geq 1, \tag{3.4}$$



$h(K_7)$



$h(K_8)$

Figure 3.1.

$$\sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f(\gamma)\right) \equiv 1 \pmod 2$$

by Corollaries 1.4 and 1.5. Let h be the spatial embedding of K_7 as illustrated in the left hand side of Figure 3.1. It is known that $h\left(K_7\right)$ contains exactly one nontrivial knot $h([1357246])$ which is a trefoil knot [**8**]. Since $a_2\left(3_1\right) = 1$, the embedding h realizes the lower bound in (3.4). In particular for a rectilinear spatial embedding f_{r} of K_7, by Theorem 1.6 and Corollary 1.7, we have

$$\sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f_{\mathrm{r}}(\gamma)\right) = \sum_{\lambda\in\Gamma_{3,3}(K_7)} \mathrm{lk}\left(f_{\mathrm{r}}(\lambda)\right)^2 - 6, \tag{3.5}$$

$$1 \le \sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \le 15. \tag{3.6}$$

As it has been shown in [**24**], the lower bound in (3.6) enables us to give much simpler topological proof of the fact that every rectilinear spatial graph of K_7 contains a trefoil knot, which was originally proven by Brown [**6**] and Ramírez Alfonsín [**28**] in combinatorial and computational way. Actually, by (3.6), there exists at least one Hamiltonian cycle γ_0 of K_7 such that $a_2\left(f_{\mathrm{r}}(\gamma_0)\right) > 0$. Then by Proposition 3.1 (2) and (3), $f_{\mathrm{r}}(\gamma_0)$ is either a trefoil knot or a figure eight knot. Since $a_2\left(4_1\right) = -1$, the knot $f_{\mathrm{r}}\left(\gamma_0\right)$ must be a trefoil knot. We also remark here that h is equivalent to the standard rectilinear spatial embedding f_{sr} of K_7 in Figure 2.3. We refer the reader to [**19**], [**22**] for related works on rectilinear spatial graphs of K_7 (especially in [**19**], a remarkable result is shown that the number of figure eight knots in a rectilinear spatial graph of K_7 is at most three). Moreover, according to a computer search in [**20**], there seems to be no rectilinear embedding f_{r} of K_7 such that $\sum_{\gamma\in\Gamma_7(K_7)} a_2\left(f_{\mathrm{r}}(\gamma)\right) = 13, 15$, or equivalently by (3.5), $\sum_{\lambda\in\Gamma_{3,3}(K_7)} \mathrm{lk}(f_{\mathrm{r}}(\gamma))^2 = 19, 21$. This strongly suggests that the upper bound in Corollary 1.7 is not sharp.

Problem 3.4. Determine the sharp upper bound of $\sum_{\gamma\in\Gamma_n(K_n)} a_2\left(f_{\mathrm{r}}(\gamma)\right)$ for all rectilinear spatial embeddings f_{r} of K_n for each $n \geq 7$.

Example 3.5. For a spatial embedding f of K_8, by Theorem 1.3, we have

$$\sum_{\gamma\in\Gamma_8(K_8)} a_2\left(f(\gamma)\right) - 6\sum_{\gamma\in\Gamma_5(K_8)} a_2\left(f(\gamma)\right) = 3\sum_{\lambda\in\Gamma_{3,3}(K_8)} \mathrm{lk}\left(f(\lambda)\right)^2 - 63, \tag{3.7}$$

and we also have

$$\sum_{\gamma\in\Gamma_8(K_8)} a_2\left(f(\gamma)\right) - 6\sum_{\gamma\in\Gamma_5(K_8)} a_2\left(f(\gamma)\right) \geq 21, \tag{3.8}$$
$$\sum_{\gamma\in\Gamma_8(K_8)} a_2\left(f(\gamma)\right) \equiv 3 \pmod 6$$

by Corollaries 1.4 and 1.5. Let h be the spatial embedding of K_8 as illustrated in the right hand side of Figure 3.1. It is known that $h\left(K_8\right)$ contains exactly twenty one nontrivial Hamiltonian knots, all of which are trefoil knots [**5**]. Since $h(\gamma)$ is a trivial knot for any 5-cycle γ of K_8, the embedding h realizes the lower bound in (3.8). In particular for a rectilinear spatial embedding f_{r} of K_8, by Theorem 1.6 and Corollary 1.7, we have

$$\sum_{\gamma\in\Gamma_8(K_8)} a_2\left(f_{\mathrm{r}}(\gamma)\right) = 3\sum_{\lambda\in\Gamma_{3,3}(K_8)} \mathrm{lk}\left(f_{\mathrm{r}}(\lambda)\right)^2 - 63, \tag{3.9}$$

$$21 \leq \sum_{\gamma\in\Gamma_8(K_8)} a_2\left(f_{\mathrm{r}}(\gamma)\right) \leq 189. \tag{3.10}$$

By Proposition 3.1, all of the polygonal knots with eight sticks are 0_1, 3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3, $3_1\#3_1$, $3_1\#3_1^*$, 8_{19} and 8_{20}. Moreover, the values of a_2 for them are as follows:

K	0_1	3_1	4_1	5_1	5_2	6_1	6_2	6_3	$3_1\#3_1$	$3_1\#3_1^*$	8_{19}	8_{20}
$a_2(K)$	0	1	-1	3	2	-2	-1	1	2	2	5	2

Thus it follows from (3.10) that every rectilinear spatial graph of K_8 always contains at least one of 3_1, 5_1, 5_2, 6_3, $3_1\#3_1$, $3_1\#3_1^*$, 8_{19} and 8_{20} as a Hamiltonian knot. Moreover, since the maximum value of a_2 in every polygonal knot with exactly eight sticks is equal to five, we can refine (2.34) if $n = 8$ and then we can also refine Corollary 1.8: the minimum number of nontrivial Hamiltonian knots with a positive value of a_2 in every rectilinear spatial graph of K_8 is at least $\lceil 21/5 \rceil = 5$. But this is not yet the sharp lower bound, see Remark 3.6.

As we mentioned in Remark 2.7, the standard rectilinear spatial embedding f_{sr} of K_8 in Figure 2.3 realizes the lower bound in (3.10). Moreover, it is known that all of the nontrivial Hamiltonian knots in $f_{\mathrm{sr}}\left(K_8\right)$ are trefoil knots [**29**]. This means that $f_{\mathrm{sr}}\left(K_8\right)$ also contains exactly twenty one nontrivial Hamiltonian knots, all of which are trefoil knots. We also remark here that h and f_{sr} are not equivalent because $h\left(K_8\right)$ contains a "triangle-pentagon" link with nonzero even linking number (actually $h([257] \cup [13846])$ is equivalent to 4_1^2), but $f_{\mathrm{sr}}\left(K_8\right)$ does not contain such a triangle-pentagon link. The authors do not know whether the embedding h is equivalent to a certain rectilinear spatial embedding of K_8 or not.

Remark 3.6. It is known that every rectilinear spatial graph of $K_{3,3,1,1}$ contains at least one nontrivial Hamiltonian knot with a positive value of a_2 (Hashimoto–Nikkuni [**14**, Corollary 1.10]). Since there are two hundred and eighty subgraphs of K_8 isomorphic to $K_{3,3,1,1}$ and for any 8-cycle γ of K_8 there exist thirty six subgraphs of K_8 isomorphic to $K_{3,3,1,1}$ containing γ, we have that there are at least $\lceil 280/36 \rceil = 8$ nontrivial Hamiltonian knots with a positive value of a_2 in every rectilinear spatial graph of K_8.

Problem 3.7. Determine the minimum number of nontrivial Hamiltonian knots (with a positive value of a_2) in every rectilinear spatial graph of K_n for each $n \geq 8$.

We also refer the reader to [**12**], [**1**] and [**2**] for a study of counting nontrivial knots and nonsplittable links in a spatial graph of $K_{n_1,n_2,\ldots,n_k}$. In particular, a computer program *Gordian* [**2**] is very useful, which enables us to calculate the values of a_2 for all constituent knots and lk for all constituent 2-component links in a spatial complete graph without difficulty.

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