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On delta invariants and indices of ideals

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Abstract. Let R be a Cohen–Macaulay local ring with a canonical module. We consider Auslander's (higher) delta invariants of powers of certain ideals of R. Firstly, we shall provide some conditions for an ideal to be a parameter ideal in terms of delta invariants. As an application of this result, we give upper bounds for orders of Ulrich ideals of R when R has Gorenstein punctured spectrum. Secondly, we extend the definition of indices to the ideal case, and generalize the result of Avramov–Buchweitz–Iyengar–Miller on the relationship between the index and regularity.

1. Introduction.

Let $(R,\mathfrak{m},k)$ be a Cohen–Macaulay local ring with a canonical module. The Auslander δ-invariant $\delta_R(M)$ for a finitely generated R-module M is defined to be the rank of maximal free summand of the minimal Cohen–Macaulay approximation of M. For an integer $n \geq 0$, the n-th δ-invariant is defined by Auslander, Ding and Solberg [**2**] as $\delta_R^n(M) = \delta_R(\Omega_R^n M)$, where $\Omega_R^n M$ denotes the n-th syzygy module of M in the minimal free resolution.

On these invariants, combining the Auslander's result (see [**2**, Corollary 5.7]) and Yoshino's one [**13**], we have the following theorem.

Theorem 1.1 (Auslander, Yoshino). *Let $d > 0$ be the Krull dimension of R. Consider the following conditions.*

(a) *R is a regular local ring.*

(b) *There exists $n \geq 0$ such that $\delta^n(R/\mathfrak{m}) > 0$.*

(c) *There exist $n > 0$ and $l > 0$ such that $\delta^n(R/\mathfrak{m}^l) > 0$.*

Then, the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) *hold. The implication* (c) $\Rightarrow$ (a) *holds if* $\operatorname{depth} \operatorname{gr}_{\mathfrak{m}}(R) \geq d-1$.

Here we denote by $\operatorname{gr}_I(R)$ the associated graded ring of R with respect to an ideal I of R. In this paper, we characterize parameter ideals in terms of (higher) δ-invariants as follows.

Theorem 1.2. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring with a canonical module ω, having infinite residue field k and Krull dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R such that I/I^2 is a free R/I-module. Consider the following conditions.*

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(a) $\delta(R/I) > 0$.

(b) *I is a parameter ideal of R.*

(c) *There exists $n \geq 0$ such that $\delta^n(R/I) > 0$.*

(d) *There exist $n > 0$ and $l > 0$ such that $\delta^n(R/I^l) > 0$.*

Then, the implications (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) *hold. The implication* (d) $\Rightarrow$ (c) *holds if* $\operatorname{depth} \operatorname{gr}_I(R) \geq d-1$ *and I^i/I^{i+1} is a free R/I-module for any $i > 0$. The implication* (b) $\Rightarrow$ (a) *holds if $I \subset \operatorname{tr}(\omega)$.*

Here $\operatorname{tr}(\omega)$ is the trace ideal of ω. that is, the image of the natural homomorphism $\omega \otimes_R \operatorname{Hom}_R(\omega, R) \to R$ mapping $x \otimes f$ to $f(x)$ for $x \in \omega$ and $f \in \operatorname{Hom}_R(\omega, R)$. This result recovers Theorem 1.1 by letting $I = \mathfrak{m}$.

On the other hand, Ding [**4**] studies the δ-invariant of $R/\mathfrak{m}^l$ with $l \geq 1$ and defines the index $\operatorname{index}(R)$ of R to be the smallest integer l such that $\delta(R/\mathfrak{m}^l) = 1$. Extending this, we define the index of an ideal.

Definition 1.3. For an ideal I of R, we define the *index* $\operatorname{index}(I)$ of I to be the infimum of integers $l \geq 1$ such that $\delta_R(R/I^l) = 1$.

For example, we have $\operatorname{index}(\mathfrak{m}) = \operatorname{index}(R)$.

Taking into account the argument of Ding [**5**] on indices of rings, Avramov, Buchweitz, Iyengar and Miller [**3**, Lemma 1.5] showed the following equality.

Theorem 1.4 (Avramov–Buchweitz–Iyengar–Miller). *Assume that R is a Gorenstein local ring and $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay. Then $\operatorname{index}(R) = \operatorname{reg}(\operatorname{gr}_{\mathfrak{m}}(R)) + 1$.*

The other main aim of this paper is to prove the following result.

Theorem 1.5. *Let R be a Cohen–Macaulay local ring having a canonical module and Krull dimension $d > 0$, and I be an $\mathfrak{m}$-primary ideal of R such that $\operatorname{gr}_I(R)$ is a Cohen–Macaulay graded ring and I^l/I^{l+1} is R/I-free for $1 \leq l \leq \operatorname{index} I$. Then we have $\operatorname{index} I \geq \operatorname{reg}(\operatorname{gr}_I(R)) + 1$. The equality holds if $I \subset \operatorname{tr}(\omega)$.*

Note that this theorem recovers Theorem 1.4 by letting $I = \mathfrak{m}$.

There are some examples of ideals which satisfy the whole conditions in Theorem 1.2 and 1.5. One of them is the maximal ideal $\mathfrak{m}$ in the case where $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay (for example, R is a hypersurface or a localization of a homogeneous graded Cohen–Macaulay ring.)

Other interesting examples are Ulrich ideals. These ideals are defined in [**6**] and many examples of Ulrich ideals are given in [**6**] and [**7**]. We shall show in Section 3 that Ulrich ideals satisfy the assumptions of Theorems 1.2 and 1.5. We have an application of Theorem 1.2 concerning Ulrich ideals as follows.

Corollary 1.6. *Let I be an Ulrich ideal of R that is not a parameter ideal. Assume that R is Gorenstein on the punctured spectrum. Then $I \not\subset \mathfrak{m}^{\operatorname{index}(R)}$. In particular,* *the supremum of set of integers n satisfying $I \subset \mathfrak{m}^n$ for any Ulrich ideal I that is not a parameter ideal is finite.*

We prove this result in Section 3.

2. Proofs.

Throughout this section, let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of dimension $d > 0$ with a canonical module ω, and assume that k is infinite. We recall some basic properties of the Auslander δ-invariant.

For a finitely generated R-module M, a short exact sequence

$$0 \to Y \to X \xrightarrow{p} M \to 0 \tag{2.0.1}$$

is called a Cohen–Macaulay approximation of M if X is a maximal Cohen–Macaulay R-module and Y has finite injective dimension over R. We say that the sequence (2.0.1) is minimal if each endomorphism ϕ of X with $p \circ \phi = p$ is an automorphism of X. It is known (see [**1**], [**8**]) that a minimal Cohen–Macaulay approximation of M exists and is unique up to isomorphism.

If the sequence (2.0.1) is a minimal Cohen–Macaulay approximation of M, then we define the (Auslander) δ-invariant $\delta(M)$ of M as the maximal rank of a free direct summand of X. We denote by $\delta^n(M)$ the δ-invariant of n-th syzygy $\Omega^n M$ of M in the minimal free resolution for $n \geq 0$.

We prepare some basic properties of delta invariants in the next Lemma; see [**10**, Corollary 11.28].

Lemma 2.1. *Let M and N be finitely generated R-modules.*

(1) *If there exists a surjective homomorphism $M \to N$, then $\delta(M) \geq \delta(N)$.*

(2) *The equality $\delta(M \oplus N) = \delta(M) + \delta(N)$ holds true.*

Lemma 2.2. *Let N be a maximal Cohen–Macaulay R-module. Then $\delta^1(N) = 0$. In particular, $\delta^n(M) = 0$ for $n \geq d+1$ and any finitely generated R-module M.*

Proof. Suppose that $\delta^1(N) > 0$. Then $\Omega^1 N$ has a free direct summand. Let $\Omega^1 N = X \oplus R$. There is a short exact sequence $0 \to X \oplus R \xrightarrow{(\sigma,\tau)^T} R^{\oplus m} \xrightarrow{\pi} N \to 0$. According to [**12**, Lemma 3.1], there exist exact sequences

$$0 \to R \xrightarrow{\tau} R^{\oplus m} \to B \to 0, \tag{2.2.1}$$

$$0 \to R^{\oplus m} \to A \oplus B \to N \to 0 \tag{2.2.2}$$

for some R-modules A, B. By the sequence (2.2.2), B is a maximal Cohen–Macaulay R-module. In view of (2.2.1), B is a free R-module provided that B has finite projective dimension. Then, the sequence (2.2.1) splits and τ has a left inverse map. This contradicts that the map π is minimal. $\square$

We now remark on δ-invariants under reduction by a regular element. The following lemma is shown in [**9**, Corollary 2.5].

Lemma 2.3. *Let M be a finitely generated R-module and $x \in \mathfrak{m}$ be a regular element on M and R. If $0 \to Y \to X \to M \to 0$ is a minimal Cohen–Macaulay approximation of M, then*

$$0 \to Y/xY \to X/xX \to M/xM \to 0$$

is a minimal Cohen–Macaulay approximation of M/xM over $R/(x)$. In particular, it holds that $\delta_R(M) \le \delta_{R/(x)}(M/xM)$.

In the proofs of our theorems, the following lemma plays a key role. We remark that in the case $I = \mathfrak{m}$, similar statements are shown in [**5**] and [**13**].

Lemma 2.4. *Let $l > 0$ be an integer, I be an $\mathfrak{m}$-primary ideal of R and $x \in I \setminus I^2$ be an R-regular element. Assume that I^i/I^{i+1} is a free R/I-module for any $1 \le i \le l$ and the multiplication map $x : I^{i-1}/I^i \to I^i/I^{i+1}$ is injective for any $1 \le i \le l$, where we set $I^0 = R$. Then the following hold.*

(1) *$xI^i = (x) \cap I^{i+1}$ for all $0 \le i \le l$.*

(2) *$I^i/I^{i+1} \cong I^{i-1}/I^i \oplus I^i/(xI^{i-1} + I^{i+1})$ for all $1 \le i \le l$.*

(3) *$I^i/xI^i \cong I^{i-1}/I^i \oplus I^i/xI^{i-1}$ for all $1 \le i \le l$.*

(4) *$(I^i + (x))/xI^i \cong R/I^i \oplus I^i/xI^{i-1}$ for all $1 \le i \le l$.*

(5) *$(I^i + (x))/x(I^i + (x)) \cong R/(I^i + (x)) \oplus I^i/xI^{i-1}$ for all $1 \le i \le l$.*

Proof. (1): We prove this by induction on i. If $i = 0$, there is nothing to prove. Let $i > 0$. The injectivity of $x : I^{i-1}/I^i \to I^i/I^{i+1}$ shows that $xI^{i-1} \cap I^{i+1} = xI^i$. By the induction hypothesis, $xI^{i-1} = (x) \cap I^i$. Thus it is seen that

$$\begin{aligned} xI^i &= (x) \cap I^i \\ &= (x) \cap I^i \cap I^{i+1} = (x) \cap I^{i+1}. \end{aligned}$$

(2): As R/I is an Artinian ring, the injective map $x : I^{i-1}/I^i \to I^i/I^{i+1}$ of free R/I-modules is split injective. We can also see that the cokernel of this map is $I^i/(xI^{i-1} + I^{i+1})$. Therefore we have an isomorphism $I^i/I^{i+1} \cong I^{i-1}/I^i \oplus I^i/(xI^{i-1} + I^{i+1})$.

(3): We have the following natural commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & I^{i-1}/I^i \arrow[r, "x"] \arrow[d, "="] & I^i/xI^i \arrow[r] \arrow[d] & I^i/xI^{i-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^{i-1}/I^i \arrow[r, "x"] & I^i/I^{i+1} \arrow[r] & I^i/(xI^{i-1} + I^{i+1}) \arrow[r] & 0 \end{tikzcd}$$

We have already seen in (2) that the second row is a split exact sequence, and thus the first row is also a split exact sequence. Therefore we have an isomorphism $I^i/xI^i \cong I^{i-1}/I^i \oplus I^i/xI^{i-1}$.

(4): The cokernel of the multiplication map $x : R/I^i \to (I^i + (x))/xI^i$ is $(I^i + (x))/(x) = I^i/((x) \cap I^i)$, which coincides with I^i/xI^{i-1} by (1). Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I^{i-1}/I^i \arrow[r, "x"] \arrow[d, "\iota_1"] & I^i/xI^i \arrow[r] \arrow[d, "\iota_2"] & I^i/xI^{i-1} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & R/I^i \arrow[r, "x"] & (I^i + (x))/xI^i \arrow[r] & I^i/xI^{i-1} \arrow[r] & 0
\end{tikzcd}$$

Here ι_1, ι_2 are the natural inclusions. The first row is a split exact sequence as in (3). Therefore the second row is also a split exact sequence and we have an isomorphism $(I^i + (x))/xI^i \cong R/I^i \oplus I^i/xI^{i-1}$.

(5): The cokernel of the multiplication map $x : R/(I^i + (x)) \to (I^i + (x))/x(I^i + (x))$ is $(I^i + (x))/(x) = I^i/xI^{i-1}$. We can get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R/I^i \arrow[r, "x"] \arrow[d, "\pi_1"] & (I^i + (x))/xI^i \arrow[r] \arrow[d, "\pi_2"] & I^i/xI^{i-1} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & R/(I^i + (x)) \arrow[r, "x"] & (I^i + (x))/x(I^i + (x)) \arrow[r] & I^i/xI^{i-1} \arrow[r] & 0
\end{tikzcd}$$

Here π_1, π_2 are the natural surjections. Then we can prove (5) in a manner similar to (4). $\square$

In the case that the dimension d is at most 1, the δ-invariants mostly vanish.

LEMMA 2.5. *Assume $d \leq 1$ and I is an $\mathfrak{m}$-primary ideal of R. If $\delta(I) > 0$, then I is a parameter ideal of R.*

PROOF. Since $d \leq 1$, the $\mathfrak{m}$-primary ideal I is a maximal Cohen–Macaulay R-module. Therefore the condition $\delta(I) > 0$ provides that I has a free direct summand. We have $I = J + (x)$ and $J \cap (x) = 0$ for some ideal J and R-regular element $x \in I$. Let $y \in J$. Then $xy \in J \cap (x) = 0$. Since x is R-regular, the equality $xy = 0$ implies $y = 0$. This shows that $J = 0$ and $I = (x)$. $\square$

Now we can prove Theorem 1.2.

PROOF OF THEOREM 1.2. (b) $\Rightarrow$ (c): If I is a parameter ideal, then $\Omega^d(R/I) = R$ and hence $\delta^d(R/I) = 1 > 0$.

(a), (c) $\Rightarrow$ (b): Assume that $\delta(R/I) > 0$. Then the inequality $\delta(I) > 0$ also holds because I/I^2 is a free R/I-module and thus there is a surjective homomorphism $I \to R/I$. Therefore we only need to prove the implication (c) $\Rightarrow$ (b) in the case $n > 0$. We show the implication by induction on the dimension d.

If $d = 1$, then $n = 1$ by Lemma 2.2. Using Lemma 2.5, it follows that I is a parameter ideal.

Now let $d > 1$. Take $x \in I \setminus \mathfrak{m}I$ to be an R-regular element. Then the image of x in the free R/I-module I/I^2 forms a part of a free basis over R/I. This provides that the map $x : R/I \to I/I^2$ is injective. We see from Lemma 2.3 that

$$\begin{aligned} \delta^{n-1}_{R/(x)}(I/xI) &= \delta_{R/(x)}(\Omega^{n-1}_{R/(x)}(I/xI)) \\ &= \delta_{R/(x)}(\Omega^{n-1}_{R}(I) \otimes_R R/(x)) \\ &\geq \delta_R(\Omega^{n-1}_R I) = \delta^n_R(R/I) > 0. \end{aligned} \tag{2.5.1}$$

Applying Lemma 2.4 (3) to $i = 1$, we have an isomorphism $I/xI \cong R/I \oplus I/(x)$ and hence we obtain an equality

$$\delta^{n-1}_{R/(x)}(I/xI) = \delta^{n-1}_{R/(x)}(R/I) + \delta^{n-1}_{R/(x)}(I/(x)).$$

It follows from (2.5.1) that $\delta^{n-1}_{R/(x)}(R/I) > 0$ or $\delta^{n-1}_{R/(x)}(I/(x)) > 0$. Note that the ideal $\overline{I} := I/(x)$ of $\overline{R} := R/(x)$ satisfies the same condition as (c), that is, the module $\overline{I}/\overline{I}^2$ is free over $\overline{R}/\overline{I} = R/I$, because $\overline{I}/\overline{I}^2 = I/((x) + I^2)$ is a direct summand of I/I^2 by Lemma 2.4 (2). By the induction hypothesis, the ideal $\overline{I}$ is a parameter ideal of $\overline{R}$. Then we see that I is also a parameter ideal of R.

(c) $\Rightarrow$ (d): This implication is trivial.

Next we prove by induction on d the implication (d) $\Rightarrow$ (b) when $\operatorname{depth} \operatorname{gr}_I(R) \geq d-1$ and I^i/I^{i+1} is a free R/I-module for any $i > 0$. If $d = 1$, then $\delta(I^l) > 0$ by Lemma 2.2. By Lemma 2.5, it follows that I^l is a parameter ideal. Set $(y) := I^l$. Taking a minimal reduction (t) of I, we have $I^{m+1} = tI^m$ for any $m \gg 0$. Setting $m = pl$, we obtain that $I \cong y^p I = I^{m+1} = tI^m = (ty^p)$. This shows that I is a parameter ideal.

Assume $d > 1$. Since k is infinite, there is an element $x \in I \setminus I^2$ such that the initial form $x^* \in G$ is a non-zerodivisor of G. The G-regularity of x^* yields that the map $x : I^{i-1}/I^i \to I^i/I^{i+1}$ is injective for every $i \geq 1$. We see from Lemma 2.3 that $\delta^{n-1}_{R/(x)}(I^l/xI^l) \geq \delta^n_R(R/I^l) > 0$ in the same way as (2.5.1). Applying Lemma 2.4 (3), we get an isomorphism $I^l/xI^l \cong I^{l-1}/I^l \oplus I^l/xI^{l-1}$ and then we see that

$$\delta^{n-1}_{R/(x)}(I^l/xI^l) = \delta^{n-1}_{R/(x)}(I^{l-1}/I^l) + \delta^{n-1}_{R/(x)}(I^l/xI^{l-1}).$$

Since I^{l-1}/I^l is a free R/I-module, we have $\delta^{n-1}_{R/(x)}(R/I) > 0$ or $\delta^{n-1}_{R/(x)}(I^l/xI^{l-1}) > 0$. In the case that $\delta^{n-1}_{R/(x)}(R/I) > 0$, we already showed that I is a parameter ideal. So we may assume that $\delta^{n-1}_{R/(x)}(I^l/xI^{l-1}) > 0$. The equality $xI^{l-1} = I^l \cap (x)$ in Lemma 2.4 (1) shows that the image $\overline{I^l}$ of I^l in $R/(x)$ coinsides with I^l/xI^{l-1}. Thus it holds that $\delta^{n-1}_{R/(x)}(\overline{I}^l) = \delta^{n-1}_{R/(x)}(\overline{I^l}) > 0$. We also note that $\overline{I}^i/\overline{I}^{i+1}$ is free over $\overline{R}/\overline{I}$ by Lemma 2.4 (3). By the induction hypothesis, $\overline{I}$ is a parameter ideal of $R/(x)$. This implies that I is also a parameter ideal of R.

Finally, the implication (b) $\Rightarrow$ (a) follows from the proof of [**10**, Theorem 11.42]. $\square$

Next, to prove Theorem 1.5, we start by recalling the definition of regularity; see

[**11**, Definition 3].

Definition 2.6. Let A be a positively graded homogeneous ring and M be a finitely generated graded A-module. Then the (*Castelnuovo-Mumford*) *regularity* of M is defined by $\mathrm{reg}_A(M) = \sup\{i + j \mid H^i_{A_+}(M)_j \neq 0\}$.

Here we state some properties of regularity.

Remark 2.7. Let A and M be the same as in the definition above.

(1) Let $a \in A$ be a homogeneous M-regular element of degree 1. Then we have $\mathrm{reg}_{A/(a)}(M/aM) = \mathrm{reg}_A(M)$.

(2) If A is an artinian ring, then $\mathrm{reg}(M) = \max\{p \mid M_p \neq 0\}$.

Now let us state the proof of Theorem 1.5.

Proof of Theorem 1.5. Since k is infinite, there exists a regular sequence $x_1, \dots, x_d$ of R in I such that the sequence of initial forms $x_1^*, \dots, x_d^*$ makes a homogeneous system of parameters of $\mathrm{gr}_I(R)$. Then the equality $\mathrm{gr}_I(R)/(x_1^*, \dots, x_d^*) = \mathrm{gr}_{I'}(R')$ holds, where $R' = R/(x_1, \dots, x_d)$ and $I' = I/(x_1, \dots, x_d)$. It holds that

$$\begin{aligned}\mathrm{reg}(\mathrm{gr}_I(R)) &= \mathrm{reg}(\mathrm{gr}_{I'}(R')) \\ &= \max\{p \mid \mathrm{gr}_{I'}(R')_p \neq 0\} \\ &= \max\{p \mid \mathrm{gr}_I(R)_p \not\subset (x_1^*, \dots, x_d^*)\} \\ &= \max\{p \mid I^p \not\subset (x_1, \dots, x_d)\}.\end{aligned}$$

To show the inequality $\mathrm{index}(I) \geq \mathrm{reg}(\mathrm{gr}_I(R)) + 1$, it is enough to check that $I^p \subset (x_1, \dots, x_d)$ if $p = \mathrm{index}(I)$. We prove this by induction on d.

Let $\overline{R}$ be the quotient ring $R/(x_1)$ and $\overline{I}$ be the ideal $I/(x_1)$ of $\overline{R}$. Now put $p = \mathrm{index}(I)$ and we have $\delta_R(R/I^p) > 0$ by definition. Since there is a surjection from $J := I^p + (x)$ to R/I^p by Lemma 2.4 (4), $\delta_R(J)$ is greater than 0 . Lemma 2.3 yields that $\delta_{\overline{R}}(J/x_1J) \geq \delta_R(J) > 0$. Using Lemma 2.4 (5), we obtain an isomorphism $J/x_1J \cong R/J \oplus I^p/x_1I^{p-1}$, and hence $\delta_{\overline{R}}(J/x_1J) = \delta_{\overline{R}}(R/J) + \delta_{\overline{R}}(I^p/x_1I^{p-1})$. Therefore we see that $\delta_{\overline{R}}(R/J) > 0$ or $\delta_{\overline{R}}(I^p/x_1I^{p-1}) > 0$. Now assume that $d = 1$. If $\delta_{\overline{R}}(I^p/x_1I^{p-1}) > 0$, then $I^p/x_1I^{p-1} = \overline{R}$ since $I^p/x_1I^{p-1} = I^p/(x_1) \cap I^p$ is an ideal of the Artinian ring $\overline{R}$ and we apply Lemma 2.5. Therefore $I^p = R$ and this is a contradiction. So we get $\delta_{\overline{R}}(R/J) > 0$. In this case, R/J must have an $\overline{R}$-free summand. This shows that $J = (x_1)$ and $I^p \subset (x_1)$.

Next we assume that $d > 1$. By Theorem 1.2, $\delta_{\overline{R}}(I^p/x_1I^{p-1}) = 0$. So we have $\delta_{\overline{R}}(R/J) > 0$. Then $R/J = R/(I^p + (x_1)) = \overline{R}/\overline{I}^p$ hold. By the induction hypothesis, $\overline{I}^p \subset (x_1, x_2, \dots, x_d)/(x_1)$. Hence we get $I^p \subset (x_1, \dots, x_d)$.

It remains to show that $\mathrm{index}(I) = \mathrm{reg}(\mathrm{gr}_I(R)) + 1$ if $I \subset \mathrm{tr}(\omega)$. We only need to prove that $I^p \subset (x_1, \dots, x_d)$ implies $\delta(R/I^p) > 0$. This immediately follows from the inequalities $\delta(R/I^p) \geq \delta(R/(x_1, \dots, x_d))$ and $\delta(R/(x_1, \dots, x_d)) > 0$ by applying Theorem 1.2 (b)$\Rightarrow$ (a) to the ideal $(x_1, \dots, x_d)$. $\square$

3. Examples.

In this section, $(R, \mathfrak{m}, k)$, and d are the same as in the previous section. Let I be an $\mathfrak{m}$-primary ideal of R. To begin with, let us recall the definition of Ulrich ideals.

Definition 3.1. We say that I is an Ulrich ideal of R if it satisfies the following.

(1) $\mathrm{gr}_I(R)$ is a Cohen–Macaulay ring with $a(\mathrm{gr}_I(R)) \leq 1 - d$.

(2) I/I^2 is a free R/I-module.

Here we denote by $a(\mathrm{gr}_I(R))$ the a-invariant of $a(\mathrm{gr}_I(R))$. Since k is infinite, the condition (1) of Definition 3.1 is equivalent to saying that $I^2 = QI$ for some minimal reduction Q of I.

Next, we prove that Ulrich ideals satisfies the assumption of Theorem 1.2 and 1.5.

Proposition 3.2. *Let I be an Ulrich ideal of R. Then I^l/I^{l+1} is a free R/I-module for any $l \geq 1$.*

Proof. By definition, I/I^2 is free over R/I. Take a minimal reduction Q of I. Consider the canonical exact sequence

$$0 \to I^l/Q^l \to Q^{l-1}/Q^l \to Q^{l-1}/I^l \to 0$$

of R/Q-modules. Then Q^{l-1}/Q^l is a free R/Q-module and

$$Q^{l-1}/I^l = Q^{l-1}/IQ^{l-1} = R/I \otimes_{R/Q} Q^{l-1}/Q^l$$

is a free R/I-module. Therefore

$$I^l/Q^l = \Omega_{R/Q}((R/I)^{\oplus m}) = \Omega_{R/Q}(R/I)^{\oplus m} = (I/Q)^{\oplus m}$$

for some m. Since I/Q is free over R/I, I^l/Q^l is also a free R/I-module. We now look at the canonical exact sequence $0 \to Q^l/I^{l+1} \to I^l/I^{l+1} \to I^l/Q^l \to 0$ of R/I-modules. Then as we already saw, I^l/Q^l and Q^l/I^{l+1} are both free over R/I. Thus the sequence is split exact and I^l/I^{l+1} is a free R/I-module. $\square$

Now we give the proof of Corollary 1.6.

Proof of Corollary 1.6. It follows from [**4**, Theorem 1.1] that $\mathrm{index}(R)$ is finite number. Since I is not a parameter ideal, we have $\delta(R/I) = 0$ by Theorem 1.2. If $I \subset \mathfrak{m}^{\mathrm{index}(R)}$, then we have a surjective homomorphism $R/I \to R/\mathfrak{m}^{\mathrm{index}(R)}$ and thus $\delta(R/I) \geq \delta(R/\mathfrak{m}^{\mathrm{index}(R)}) > 0$. This is a contradiction. $\square$

To end this section, we give an example of an ideal showing that the condition I/I^2 is free over R/I does not imply that I^l/I^{l+1} is free over R/I for any $l \geq 1$.

Example 3.3. Let $S = k[[x, y]]$ be the formal power series ring in two variables, $\mathfrak{n}$ be the maximal ideal of S, $L = (x^4)S$, $J = (x^2, y)S$, $R = S/L$ be the quotient ring of S by L and I be the ideal J/L of R. Then I/I^2 is free over R/I but I^2/I^3 is not so.

Proof. We note that J is a parameter ideal of S and therefore J^l/J^{l+1} is free over S/J for any $l \geq 1$. Since $I^2 = (J^2+L)/L = J^2/L$, we have $I/I^2 = (J/L)/(J^2/L) \cong J/J^2$ which is free over $S/J \cong R/I$. On the other hand, we have $l_R(I^2/I^3) = l_S(J^2/(J^3+L)) = 4$, $l_R(R/I) = l_S(S/J) = 2$ and $\mu_R(I^2) = 3$, here we denote by $l_A(M)$ the length of A-module M for a commutative ring A and by $\mu_A(M)$ the number of minimal generator of M. Thus $l_R(I^2/I^3) \neq \mu_R(I^2) l_R(R/I)$. This shows that I^2/I^3 is not free over R/I. $\square$

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