J. Math. Soc. Japan
Vol. 56, No. 2, 2004

Relations between cone-parameter Lévy processes and convolution semigroups

By Jan PEDERSEN and Ken-iti SATO

(Received Nov. 25, 2002)

Abstract. Cone-parameter Lévy processes and convolution semigroups on $\boldsymbol{R}^d$ are defined. Here, cone-parameter Lévy processes have stationary independent increments along increasing sequences on the cone. This property ensures that subordination of a cone-parameter Lévy process by an independent cone-valued cone-parameter Lévy process yields a new cone-parameter Lévy process. It is shown that a cone-parameter Lévy process induces a cone-parameter convolution semigroup. The converse statement, that any convolution semigroup appears in this way, is however not true. In particular we show that there is no Brownian motion with parameter in the set of nonnegative-definite symmetric $d \times d$ matrices. The question when a given cone-parameter convolution semigroup is generated by a Lévy process is studied. It is shown that this is the case if one of the following three conditions is satisfied: $d = 1$; the convolution semigroup is purely non-Gaussian; or K is isomorphic to $\boldsymbol{R}_+^N$.

1. Introduction.

Recall that a Lévy process in law on $\boldsymbol{R}^d$ starts at 0, has stationary independent increments and is continuous in probability; a Lévy process appears by assuming in addition that the paths are cadlag. The following properties are fundamental. (i) If $\{X_t : t \geq 0\}$ is a Lévy process in law then $\{\mu_t : t \geq 0\}$ defined by $\mu_t = \mathscr{L}(X_t)$ is a convolution semigroup; (ii) conversely, if $\{\mu_t : t \geq 0\}$ is a convolution semigroup then there exists a Lévy process in law $\{X_t : t \geq 0\}$ with $\mu_t = \mathscr{L}(X_t)$ for all t; (iii) the law of $\{X_t : t \geq 0\}$ in (ii) is uniquely determined by $\{\mu_t : t \geq 0\}$. In fact, if $t_0 > 0$ then μ_{t_0} determines both $\{\mu_t : t \geq 0\}$ and the law of $\{X_t : t \geq 0\}$; (iv) we have stability under subordination, that is, if $\{T_t : t \geq 0\}$ is a subordinator, independent of a Lévy process $\{X_t : t \geq 0\}$, then $\{X_{T_t} : t \geq 0\}$ is a Lévy process.

Several papers discuss extensions of Lévy process to the case where the parameter is multidimensional. Often the stationary independent increment property is replaced by an assumption saying that a multiparameter Lévy process $\{X_t : t \in \boldsymbol{R}_+^N\}$ on $\boldsymbol{R}^d$ has stationary independent increments over half-open intervals of $\boldsymbol{R}_+^N$. (See e.g. Adler, Monrad, Scissors and Wilson [**1**] for a precise formulation of this.) Examples include the Brownian sheet (Orey and Pruitt [**13**], Talagrand [**16**], Khosnevisan and Shi [**9**]) and processes considered by Ehm [**5**], Vares [**17**] and Lagaize [**11**]. In these papers the law

2000 *Mathematics Subject Classification*. Primary 60B99; Secondary 60G10, 60G51.

Key Words and Phrases. cone-parameter Lévy processes and convolution semigroups, generative and non-generative convolution semigroups.

Jan Pedersen was supported by the Danish Natural Science Research Council.

Ken-iti Sato was partially supported by MaPhySto, Aarhus, Denmark and Centro de Investigación en Matemáticas, Guanajuato, Mexico.

of $\{X_t : t \in \boldsymbol{R}_+^N\}$ is uniquely determined by the law of X_{t_0}, where $t_0 \in \boldsymbol{R}_+^N$ has positive coordinates. Moreover, whenever μ_0 is an infinitely divisible distribution on $\boldsymbol{R}^d$ there is a multiparameter Lévy process $\{X_t : t \in \boldsymbol{R}_+^N\}$ with $\mathscr{L}(X_{t_0}) = \mu_0$. It is, however, readily seen that subordination of multiparameter Lévy processes does not always result in a new multiparameter Lévy process. That is, (i)–(iii) generalize to the multiparameter case while (iv) does not. Let us also mention Lévy's [**12**] multiparameter Brownian motion as an example of a process with parameter in $\boldsymbol{R}^N$, which has independent increments only along straight lines in $\boldsymbol{R}^N$.

In this paper processes have parameter s in a cone K. Besides the case $K = \boldsymbol{R}_+^N$ we also consider interesting examples where K is the set $\boldsymbol{S}_d^+$ of nonnegative-definite symmetric $d \times d$ matrices. We consider another natural generalization of the stationary independent increment property by assuming stationary independent increments along K-increasing sequences. This leads to what we call K-parameter Lévy processes in law, see Definition 3.1. We have the convenient property that subordination of K-parameter Lévy processes in law results in a new K-parameter Lévy process in law. Moreover, it is readily seen that a K-parameter Lévy process in law induces a K-parameter convolution semigroup $\{\mu_s : s \in K\}$ by $\mu_s = \mathscr{L}(X_s)$. Here a convolution semigroup satisfies $\mu_{s^1+s^2} = \mu_{s^1} * \mu_{s^2}$ and has a continuity property. A K-parameter convolution semigroup $\{\mu_s\}$ is said to be generative if there is a K-parameter Lévy process in law satisfying $\mu_s = \mathscr{L}(X_s)$ for all s; otherwise $\{\mu_s\}$ is non-generative. We show that if $K = \boldsymbol{S}_d^+$ with $d \geq 2$ and $\mu_s = N_d(0, s)$ then $\{\mu_s : s \in K\}$ is non-generative. This can be rephrased as the property that there is no Brownian motion with nonnegative-definite symmetric matrix parameter. In particular we see that (ii) does not generalize to the K-parameter case. A second purpose of the paper is to investigate the question when a given cone-parameter convolution semigroup is generative. The main results are that $\{\mu_s\}$ is generative if one of the following three conditions is satisfied: $d = 1$; the convolution semigroup is purely non-Gaussian; or K is isomorphic to $\boldsymbol{R}_+^N$.

Even for a generative K-parameter convolution semigroup $\{\mu_s\}$ it is generally not true that the law of an associated cone-parameter Lévy process in law is uniquely determined. That is, neither (iii) generalizes. In the case where K is isomorphic to $\boldsymbol{R}_+^N$ we give conditions on $\{\mu_s\}$ under which the law of an associated cone-parameter convolution semigroup is in fact unique.

The paper is organized as follows. In Section 3 cone-parameter Lévy processes and convolution semigroups are defined and some properties are derived. In particular we show stability under subordination as mentioned above. In Section 4 we construct non-generative $\boldsymbol{S}_d^+$-parameter convolution semigroups and finally Section 5 contains an analysis of generative convolution semigroups.

2. Preliminaries.

Throughout the paper let N, M and d be positive integers. Elements of $\boldsymbol{R}^d$ are column vectors. We denote the coordinates of $x \in \boldsymbol{R}^d$ by x_j, and use either the notation $x = (x_j)_{1 \leq j \leq d}$ or $x = (x_1, \dots, x_d)^\top$. The inner product on $\boldsymbol{R}^d$ is $\langle x, y \rangle$ and the norm is $|x|$. When $d_1, \dots, d_n$ are positive integers and $x^j \in \boldsymbol{R}^{d_j}$ for $j = 1, \dots, n$, then $(x^1, \dots, x^n)^\top$ denotes the stacked vector

$$(x^1, \dots, x^n)^\top = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \tag{2.1}$$

which is an element of $\boldsymbol{R}^{d_1+\cdots+d_n}$.

Let $ID(\boldsymbol{R}^d)$ be the class of infinitely divisible distributions on $\boldsymbol{R}^d$ equipped with the Borel σ-algebra $\mathscr{B}(\boldsymbol{R}^d)$. For $\mu \in ID(\boldsymbol{R}^d)$ and $t \geq 0$, denote $\mu^t = \mu^{t*}$. The characteristic function of μ is $\hat{\mu}(z) = \int_{\boldsymbol{R}^d} e^{i\langle z,x\rangle}\mu(dx)$, $z \in \boldsymbol{R}^d$. Let $\mathscr{L}(X)$ be the distribution (law) of a random variable X. By $X \stackrel{d}{=} Y$ we mean $\mathscr{L}(X) = \mathscr{L}(Y)$. Thus, by $\{X_s\} \stackrel{d}{=} \{Y_s\}$ we mean that the two stochastic processes $\{X_s\}$ and $\{Y_s\}$ have an identical system of finite-dimensional distributions. For probability measures μ_n $(n = 1, 2, \dots)$ and μ on $\boldsymbol{R}^d$, $\mu_n \to \mu$ means weak convergence of μ_n to μ. Let δ_c denote a distribution concentrated at a point c. Such a distribution is called trivial. For $z, x \in \boldsymbol{R}^d$ let $g(z,x)$ be the function $g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\leq 1\}}(x)$. For $\mu \in ID(\boldsymbol{R}^d)$ and $r \in \boldsymbol{R}$, we define $\hat{\mu}(z)^r$, $z \in \boldsymbol{R}^d$, as $\hat{\mu}(z)^r = e^{r \log \hat{\mu}(z)}$, where $\log \hat{\mu}(z)$ is the distinguished logarithm of $\hat{\mu}(z)$ in [**15**], p. 33. In other words,

$$\hat{\mu}(z)^r = \exp\left[r\left(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\boldsymbol{R}^d} g(z,x)\nu(dx)\right)\right],$$

where (A, ν, γ) is the triplet or the generating triplet of μ in [**15**], p. 38. The matrix A and the measure ν are respectively the Gaussian covariance matrix and the Lévy measure of μ, and $\gamma \in \boldsymbol{R}^d$.

Definition 2.1. A subset K of $\boldsymbol{R}^M$ is a *cone* if it is a non-empty closed convex set closed under multiplication by nonnegative reals ($s \in K$ and $a \geq 0$ imply $as \in K$) and containing no straight line through 0 ($s \in K$ and $-s \in K$ imply $s = 0$) and if $K \neq \{0\}$.

Throughout this paper, K is a cone in $\boldsymbol{R}^M$ unless otherwise stated. Notice that K is closed under addition. Therefore, if $s^1, \dots, s^n$ are in K, then $t_1 s^1 + \cdots + t_n s^n \in K$ for any nonnegative reals $t_1, \dots, t_n$. Let L be the linear subspace generated by K, that is, the smallest linear subspace of $\boldsymbol{R}^M$ that contains K. If $\dim L = N$, then we say that K is an *N-dimensional cone*. If $\dim L = M$, then K is said to be *nondegenerate*.

If $\{e^1, \dots, e^N\}$ is a linearly independent system in K such that $K = \{s_1 e^1 + \cdots + s_N e^N : s_1, \dots, s_N \geq 0\}$ then $\{e^1, \dots, e^N\}$ is called a *strong basis of* K. If $\{e^1, \dots, e^N\}$ is a basis of L and $e^1, \dots, e^N$ are in K then $\{e^1, \dots, e^N\}$ is called a *weak basis of* K. For example, a cone in $\boldsymbol{R}$ is either $[0, \infty)$ or $(-\infty, 0]$, and has a strong basis. Any nondegenerate cone in $\boldsymbol{R}^2$ is a closed sector with angle $< \pi$ and has a strong basis. A nondegenerate cone in $\boldsymbol{R}^3$ has a strong basis if and only if it is a triangular cone. For any M, the nonnegative orthant $\boldsymbol{R}_+^M$ is a cone with a strong basis. Any cone has a weak basis.

Write $s^1 \leq_K s^2$ if $s^2 - s^1 \in K$. A sequence $\{s^n\}_{n=1,2,\dots}$ in $\boldsymbol{R}^M$ is *K-increasing* if $s^n \leq_K s^{n+1}$ for each n; *K-decreasing* if $s^{n+1} \leq_K s^n$ for each n. A mapping f from $[0, \infty)$ into $\boldsymbol{R}^M$ is *K-increasing* if $f(t_1) \leq_K f(t_2)$ for $t_1 \leq t_2$; *K-decreasing* if $f(t_2) \leq_K f(t_1)$ for $t_1 \leq t_2$.

More generally, let K_1 and K_2 be cones in $\boldsymbol{R}^{M_1}$ and $\boldsymbol{R}^{M_2}$, respectively. A mapping f from K_1 into $\boldsymbol{R}^{M_2}$ is (K_1, K_2)-*increasing* if $s^1 \leq_{K_1} s^2$ implies $f(s^1) \leq_{K_2} f(s^2)$; (K_1, K_2)-*decreasing* if $s^1 \leq_{K_1} s^2$ implies $f(s^2) \leq_{K_2} f(s^1)$.

Let $K' = \{u \in \boldsymbol{R}^M : \langle u, s \rangle \geq 0 \text{ for all } s \in K\}$. Then K' is again a cone in $\boldsymbol{R}^M$. It is called the *dual cone* of K. We have $(K')' = K$. If $K = \boldsymbol{R}_+^M$, then $K = K'$. For two cones K_1, K_2 in $\boldsymbol{R}^M$, we have $K_1 \subseteq K_2$ if and only if $K_1' \supseteq K_2'$.

Remark 2.2. Let K be an N-dimensional cone in $\boldsymbol{R}^M$. Let L be the linear subspace generated by K and let T be a linear transformation from L to $\boldsymbol{R}^{\tilde{M}}$ such that $\dim(TL) = N$. Denote by T^{-1} the inverse of T defined on TL. Define $\tilde{K} = TK$, the image of K by T. Then, $\tilde{K}$ is an N-dimensional cone in $\boldsymbol{R}^{\tilde{M}}$. We have $u^1 \leq_{\tilde{K}} u^2$ if and only if $T^{-1}u^1 \leq_K T^{-1}u^2$. A system $\{u^1, \ldots, u^N\}$ is a strong basis (resp. a weak basis) of $\tilde{K}$ if and only if $\{T^{-1}u^1, \ldots, T^{-1}u^N\}$ is a strong basis (resp. a weak basis) of K. We say that K and $\tilde{K}$ are isomorphic cones. Any N-dimensional cone K with a strong basis is isomorphic to $\boldsymbol{R}_+^N$. The isomorphism is given by a mapping between strong bases. From this follows that a strong basis $\{e^1, \ldots, e^N\}$ of K is unique up to permutation and scaling, if it exists. Indeed, it is readily seen that up to scaling and permutation the standard basis in $\boldsymbol{R}^N$ is the only strong basis of $\boldsymbol{R}_+^N$.

Definition 2.3. Let f be a mapping from a cone K in $\boldsymbol{R}^M$ into $\boldsymbol{R}^d$.

(i) We say that f is K-right continuous at $s^0 \in K$, if, for every K-decreasing sequence $\{s^n\}_{n=1,2,\ldots}$ in K with $|s^n - s^0| \to 0$, we have $|f(s^n) - f(s^0)| \to 0$.

(ii) We say that f has K-left limits at $s^0 \in K \backslash \{0\}$, if, for every K-increasing sequence $\{s^n\}_{n=1,2,\ldots}$ in $K \backslash \{s^0\}$ satisfying $|s^n - s^0| \to 0$, $\lim_{n\to\infty} f(s^n)$ exists in $\boldsymbol{R}^d$.

(iii) We say f is K-cadlag if it is K-right continuous at each $s^0 \in K$ and has K-left limits at each $s^0 \in K \backslash \{0\}$.

When $f : K \to \boldsymbol{R}$ has K-left limits at $s^0 \in K$ then $\lim_{n\to\infty} f(s^n)$ may depend on the choice of the K-increasing sequence $\{s^n\}$. But, we now show that if K is an N-dimensional cone with a strong basis, then any mapping with K-left limits has at most $2^N - 1$ different left limits at each point. Let K be with a strong basis $\{e^1, \ldots, e^N\}$. Let $s^0 \in K$ and $\{s^n\}_{n=1,2,\ldots}$ be a sequence in K. Write s^0 and s^n as $s^0 = s_1^0 e^1 + \cdots + s_N^0 e^N$ and $s^n = s_1^n e^1 + \cdots + s_N^n e^N$. Note that $s^n \leq_K s^{n+1}$ if and only if $s_j^n \leq s_j^{n+1}$ for all $j = 1, \ldots, N$. Thus, $\{s^n\}_{n=1,2,\ldots}$ is K-increasing with $|s^n - s^0| \to 0$ if and only if $\{s_j^n\}_{n=1,2,\ldots}$ is an increasing sequence in $\boldsymbol{R}_+$ which tends to s_j^0 for each j. Let a be a nonempty subset of $\{1, \ldots, N\}$. We use the notation $s^n \uparrow_a s^0$ if $\{s^n\}_{n=1,2,\ldots}$ is K-increasing with $|s^n - s^0| \to 0$ such that $s_j^n < s_j^0$ for $j \in a$ and all n, and $s_j^n = s_j^0$ for $j \notin a$ and n sufficiently large. Let $p_{s^0} = \{j : s_j^0 > 0\}$. We have the following easy result.

Lemma 2.4. *Let K have a strong basis $\{e^1, \ldots, e^N\}$.*

(i) *Let $\{s^n\}_{n=1,2,\ldots}$ be K-increasing in $K \backslash \{s^0\}$ with $|s^n - s^0| \to 0$. Then there is a unique nonempty subset a of p_{s^0} such that $s^n \uparrow_a s^0$. This particular a is given by $a = \{j : s_j^n < s_j^0 \text{ for all } n\}$.*

(ii) *Let $f : K \to \boldsymbol{R}^d$ have K-left limits at $s^0 \in K \backslash \{0\}$. Then there is a family $\{f^a(s_0) : a \subseteq p_{s^0},\ a \text{ nonempty}\}$ in $\boldsymbol{R}^d$ such that if a is a nonempty subset of p_{s^0} and $\{s^n\}_{n=1,2,\ldots}$ is a sequence in K with $s^n \uparrow_a s^0$, then $f(s^n) \to f^a(s^0)$.*

3. Cone-parameter Lévy processes and convolution semigroups.

In this section we define cone-parameter Lévy processes and convolution semigroups. Some examples and properties will be discussed as well.

Definition 3.1. Let $\{X_s : s \in K\}$ be a collection of random variables on $\boldsymbol{R}^d$ defined on a probability space $(\Omega, \mathscr{F}, P)$. Then, $\{X_s : s \in K\}$ is a *K-parameter Lévy process* on $\boldsymbol{R}^d$ if the following five conditions are satisfied.

(i) If $n \geq 3$ and $\{s^j\}_{j=1,\dots,n}$ is K-increasing in K, then $X_{s^{j+1}} - X_{s^j}$, $j = 1, \dots, n-1$, are independent.

(ii) If $s^1, \dots, s^4 \in K$ and $s^2 - s^1 = s^4 - s^3 \in K$, then $X_{s^2} - X_{s^1} \stackrel{\mathrm{d}}{=} X_{s^4} - X_{s^3}$.

(iii) $X_0 = 0$ almost surely (a.s.).

(iv) $X_s(\omega)$ is K-cadlag in s for almost all $\omega \in \Omega$.

(v) If $s^0 \in K$ and $\{s^n\}_{n=1,2,\dots}$ is a sequence in K with $|s^n - s^0| \to 0$, then $X_{s^n} \to X_{s^0}$ in probability.

If $\{X_s : s \in K\}$ satisfies (i)–(iii) and (v), then $\{X_s : s \in K\}$ is called a *K-parameter Lévy process in law*.

Remark 3.2. (i) Note that with $K = \boldsymbol{R}_+$ the definition of an $\boldsymbol{R}_+$-parameter Lévy process reduces to the definition of a Lévy process in [**15**]. Similarly, an $\boldsymbol{R}_+$-parameter Lévy process in law is a Lévy process in law, as defined in [**15**].

(ii) Recall that $\{X_s : s \in K\}$ is called measurable if the mapping $X_s(\omega)$ from $(\omega, s) \in \Omega \times K$ into $\boldsymbol{R}^d$ is measurable with respect to $(\mathscr{F} \times \mathscr{B}(K), \mathscr{B}(\boldsymbol{R}^d))$. A K-parameter Lévy process is automatically measurable if condition (iv) of Definition 3.1 holds for all ω (not only for almost all ω), or if the underlying probability space is complete. More generally, any K-parameter Lévy process in law has a measurable modification. This follows from the fact that any process which is continuous in probability has a measurable modification; see [**3**], Theorem 2.

Let us provide some examples of K-parameter Lévy processes.

Example 3.3. Let K be a cone in $\boldsymbol{R}^M$ and K' be the dual cone of K. Let $u \in K'$. Let $\{V_t : t \geq 0\}$ be a Lévy process on $\boldsymbol{R}^d$. Then, we get a K-parameter Lévy process $\{X_s : s \in K\}$ on $\boldsymbol{R}^d$ by letting $X_s = V_{\langle u, s \rangle}$.

Example 3.4. Let K have a strong basis $\{e^1, \dots, e^N\}$. Then, in each of the following three constructions of X_s for $s = s_1 e^1 + \cdots + s_N e^N \in K$, we obtain a K-parameter Lévy process $\{X_s : s \in K\}$ on $\boldsymbol{R}^d$.

(i) Let $\{V_t : t \geq 0\}$ be a Lévy process on $\boldsymbol{R}^d$. Fix $(c_j)_{1 \leq j \leq N}$ with $c_j \geq 0$ for $1 \leq j \leq N$. Define $X_s = V_{c_1 s_1 + \cdots + c_N s_N}$.

(ii) Let $\{V_t^j : t \geq 0\}$, $j = 1, \dots, N$, be independent Lévy processes on $\boldsymbol{R}^d$. Define $X_s = V_{s_1}^1 + \cdots + V_{s_N}^N$.

(iii) For each $j = 1, \dots, N$, let $\{U_t^j : t \geq 0\}$ be a Lévy process on $\boldsymbol{R}^{d_j}$. Assume that they are independent. Let $d = d_1 + \cdots + d_N$. Define $X_s = (U_{s_1}^1, \dots, U_{s_N}^N)^\top$.

The processes in the preceding example have been studied in the literature. Indeed, Dynkin [**4**], Evans [**6**] and Fitzsimmons and Salisbury [**7**] worked on processes which generalize (iii), while Hirsch [**8**] and Khoshnevisan, Xiao and Zhong [**10**] studied (ii).

Related to Lévy processes is the notion of a convolution semigroup.

DEFINITION 3.5. A family $\{\mu_s : s \in K\}$ of probability measures on $\boldsymbol{R}^d$ is a K-*parameter convolution semigroup* if

(i) $\mu_{s^1} * \mu_{s^2} = \mu_{s^1+s^2}$ for all $s^1, s^2 \in K$,

(ii) $\mu_{ts} \to \delta_0$ for $s \in K$ as $t \downarrow 0$.

EXAMPLE 3.6. Let $d \geq 2$ and $\boldsymbol{S}_d^+$ be the set of symmetric nonnegative-definite $d \times d$ matrices. Let $s = (s_{jk})_{j,k=1}^d \in \boldsymbol{S}_d^+$. The lower triangle, $(s_{jk})_{k \leq j}$ with $d(d+1)/2$ entries, determines s. We identify $\boldsymbol{S}_d^+$ with a subset of $\boldsymbol{R}^{d(d+1)/2}$, considering $(s_{jk})_{k \leq j}$ as a column vector. Then $\boldsymbol{S}_d^+$ is a nondegenerate cone in $\boldsymbol{R}^{d(d+1)/2}$ and does not have a strong basis. For $s \in \boldsymbol{S}_d^+$ let μ_s be the Gaussian measure on $\boldsymbol{R}^d$, defined as $\mu_s = N_d(0, s)$, the d-dimensional Gaussian distribution with mean zero and covariance matrix s. Then, obviously, $\{\mu_s : s \in \boldsymbol{S}_d^+\}$ is an $\boldsymbol{S}_d^+$-parameter convolution semigroup on $\boldsymbol{R}^d$. We call it the *canonical* $\boldsymbol{S}_d^+$*-parameter convolution semigroup*.

We refer to Pedersen and Sato [**14**] for a detailed analysis of cone-parameter convolution semigroups. Here we just recall the following important result.

REMARK 3.7. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\boldsymbol{R}^d$. Then $\mu_s = (\mu_{s/n})^n$ for all $s \in K$ and $n \geq 1$, which shows that $\mu_s \in ID(\boldsymbol{R}^d)$. Let $\{e^1, \ldots, e^N\}$ be a weak basis of K. For $s = s_1 e^1 + \cdots + s_N e^N \in K$ we have

$$\hat{\mu}_s(z) = \hat{\mu}_{e^1}(z)^{s_1} \cdots \hat{\mu}_{e^N}(z)^{s_N}, \quad z \in \boldsymbol{R}^d. \tag{3.1}$$

In particular, if $\{s^n\}$ is a sequence in K with $s^n \to s$ then $\mu_{s^n} \to \mu_s$. This follows from Theorem 2.8 and Corollary 2.9 in [**14**].

LEMMA 3.8. *Let* $\{X_s : s \in K\}$ *be a family of random variables satisfying* (i)–(ii) *of* Definition 3.1 *together with the following condition* (v)$'$:

(v)$'$ *If* $s \in K$ *and if* $\{\varepsilon_n\}$ *is a sequence of real numbers strictly decreasing to* 0, *then* $X_{\varepsilon_n s} \to 0$ *in probability.*

Then $\{X_s : s \in K\}$ *is a K-parameter Lévy process in law and* $\{\mu_s : s \in K\}$ *defined by* $\mu_s = \mathscr{L}(X_s)$ *is a K-parameter convolution semigroup.*

In particular, if $\{X_s : s \in K\}$ *satisfies* (i)–(iv) *of* Definition 3.1 *then it is a K-parameter Lévy process.*

PROOF. It is readily seen that $\{\mu_s : s \in K\}$ is a convolution semigroup. Since $\mu_0 = \delta_0$ it follows that $X_0 = 0$ almost surely. We verify Definition 3.1 (v). Let $\{s^n\}_{n=1,2,\ldots} \subseteq K$ and $s^0 \in K$ with $|s^n - s^0| \to 0$. Let $\{e^1, \ldots, e^N\}$ be a weak basis of K and decompose s^n and s^0 as $s^n = s_1^n e^1 + \cdots + s_N^n e^N$ and $s^0 = s_1^0 e^1 + \cdots + s_N^0 e^N$ where $s_j^n, s_j^0 \in \boldsymbol{R}$ for all j and n. Define u^n by $u^n = u_1^n e^1 + \cdots + u_N^n e^N$, where $u_j^n = s_j^n \vee s_j^0$ for $j = 1, \ldots, N$. Since $u_j^n - s_j^n \geq 0$ for all j we have $u^n - s^n \in K$, that is $s^n \leq_K u^n$ and $u^n \in K$. Similarly, $s^0 \leq_K u^n$. Since $X_{s^n} - X_{s^0} = [X_{u^n} - X_{s^0}] - [X_{u^n} - X_{s^n}]$ it suffices to prove that the two terms on the right-hand side converge to zero in probability. As $u^n - s^n, u^n - s^0 \to 0$, the result follows from Definition 3.1 (ii) and Remark 3.7.

The last statement follows from the fact that Definition 3.1 (iii)–(iv) readily imply (v)$'$. $\square$

The preceding lemma shows that a K-parameter Lévy process in law $\{X_s : s \in K\}$ induces a K-parameter convolution semigroup by $\mu_s = \mathscr{L}(X_s)$. The converse statement, that any cone-parameter convolution semigroup appears in this way, is, however, not true as we show in Section 4.

Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup. We say that a K-parameter Lévy process in law $\{X_s : s \in K\}$ is *associated with* $\{\mu_s : s \in K\}$ if $\mu_s = \mathscr{L}(X_s)$ for all $s \in K$. We say that $\{\mu_s : s \in K\}$ is *generative* if there is a K-parameter Lévy process in law associated with it; otherwise $\{\mu_s : s \in K\}$ is *non-generative*.

If $\{\mu_s : s \in K\}$ is generative and $\{X_s : s \in K\}$ is a K-parameter Lévy process in law associated with it, then for any K-increasing sequence $\{s^1, \dots, s^n\}$ the distribution of $(X_{s^1}, \dots, X_{s^n})^\top$ is uniquely determined by $\{\mu_s : s \in K\}$. This fact is readily seen from (i)–(iii) of Definition 3.1. But, generally the distribution of the entire process $\{X_s : s \in K\}$ is not uniquely determined; see for example Remark 3.11. This was essentially also recognized by Barndorff-Nielsen, Pedersen and Sato [**2**]. We say that $\{\mu_s : s \in K\}$ is *unique-generative* if it is generative and any two K-parameter Lévy processes in law $\{X_s^1 : s \in K\}$ and $\{X_s^2 : s \in K\}$ associated with it satisfy $\{X_s^1 : s \in K\} \stackrel{\mathrm{d}}{=} \{X_s^2 : s \in K\}$. If $\{\mu_s : s \in K\}$ is generative but not unique-generative we say that it is *multiple-generative*.

Let $(\boldsymbol{R}^d)^K$ be the set of mappings $\omega = (\omega(s))_{s \in K}$ from K into $\boldsymbol{R}^d$ and let $\mathscr{B}(\boldsymbol{R}^d)^K$ be the σ-algebra generated by the coordinate mappings $\xi_s(\omega) = \omega(s)$, $s \in K$. If $\{X_s : s \in K\}$ is a K-parameter Lévy process in law, then it induces a unique probability measure Q on $((\boldsymbol{R}^d)^K, \mathscr{B}(\boldsymbol{R}^d)^K)$ such that $\{X_s : s \in K\}$ is identical in law with $\{\xi_s : s \in K\}$ under Q. We call Q the distribution (or law) of $\{X_s : s \in K\}$ and denote $Q = \mathscr{L}(\{X_s : s \in K\})$. For a K-parameter convolution semigroup $\{\mu_s : s \in K\}$ denote the set of distributions of K-parameter Lévy processes in law associated with it by $\boldsymbol{L}(\{\mu_s : s \in K\})$. Then, $\{\mu_s : s \in K\}$ is generative (resp. multiple-generative, unique-generative, non-generative) if and only if $\boldsymbol{L}(\{\mu_s : s \in K\})$ is nonempty (resp. has more than one element, is a singleton, is empty).

Let us give a method of constructing K-parameter Lévy processes in law.

Proposition 3.9. *Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\boldsymbol{R}^d$.*

(i) *Let $n \geq 2$. For each $j = 1, \dots, n$ let $\{X_s^j : s \in K\}$ be a K-parameter Lévy process* (*resp. Lévy process in law*) *associated with $\{\mu_s : s \in K\}$. Let U_j be non-negative random variables such that $1 = U_1 + \cdots + U_n$ almost surely. Suppose that $\{X_s^1 : s \in K\}, \dots, \{X_s^n : s \in K\}$ and $(U_1, \dots, U_n)^\top$ are independent. Define $\{X_s : s \in K\}$ by $X_s = X_{U_1 s}^1 + \cdots + X_{U_n s}^n$ for $s \in K$. Then $\{X_s : s \in K\}$ is a K-parameter Lévy process* (*resp. Lévy process in law*) *associated with $\{\mu_s : s \in K\}$.*

(ii) *Let $\{\mu_s : s \in K\}$ be a multiple-generative. Then $\boldsymbol{L}(\{\mu_s : s \in K\})$ is a convex set of probability measures.*

Proof. (i) First assume that $U_1, \dots, U_n$ are nonrandom. Then $\{X_s\}$ is a K-parameter Lévy process in law. Moreover, for $s \in K$ we have

$$\mathscr{L}(X_s) = \mathscr{L}(X_s^1)^{U_1} * \cdots * \mathscr{L}(X_s^n)^{U_n} = \mu_s^{U_1} * \cdots * \mu_s^{U_n} = \mu_s.$$

That is, $\{X_s\}$ is associated with $\{\mu_s\}$.

If $U_1, \dots, U_n$ are random we hence have that $\{X_s\}$ is a K-parameter Lévy process in law associated with $\{\mu_s\}$ conditional on $(U_1, \dots, U_n)$. It is easily seen that the same holds in the unconditional distribution.

If the paths of $\{X_s^j\}$ are K-cadlag almost surely, then the same holds for $\{X_s\}$. Thus, the property of being a K-parameter Lévy process is inherited from $\{X_s^j\}$ to $\{X_s\}$.

(ii) Let $Q^0, Q^1 \in \boldsymbol{L}(\{\mu_s : s \in K\})$ and $p \in [0,1]$. Let $\{X_s^0 : s \in K\}$ and $\{X_s^1 : s \in K\}$ be K-parameter Lévy processes in law with $Q^j = \mathscr{L}(\{X_s^j : s \in K\})$ for $j = 0, 1$, and U be a random variable such that $\{X_s^0 : s \in K\}$, $\{X_s^1 : s \in K\}$ and U are independent and $p = P(U = 1) = 1 - P(U = 0)$. Define $X_s = X_{Us}^0 + X_{(1-U)s}^1$ for $s \in K$. Then from (i) it follows that $\{X_s : s \in K\}$ is a K-parameter Lévy process in law associated with $\{\mu_s : s \in K\}$. Let $Q = \mathscr{L}(\{X_s : s \in K\})$. For $n \geq 1$, $s^1, \dots, s^n \in K$ and $B_1, \dots, B_n \in \mathscr{B}(\boldsymbol{R}^d)$, we have

$$\begin{aligned} Q(\xi_{s^1} \in B_1, \dots, \xi_{s^n} \in B_n) &= P(X_{s^1} \in B_1, \dots, X_{s^n} \in B_n) \\ &= pP(X_{s^1}^0 \in B_1, \dots, X_{s^n}^0 \in B_n) + (1-p)P(X_{s^1}^1 \in B_1, \dots, X_{s^n}^1 \in B_n), \end{aligned}$$

that is, $pQ^0 + (1-p)Q^1 = Q \in \boldsymbol{L}(\{\mu_s : s \in K\})$, as desired. □

Remark 3.10. It is an interesting problem to characterize the extremal points of the convex set $\boldsymbol{L}(\{\mu_s : s \in K\})$. At present we do not have any results in this direction.

Remark 3.11. In general the finite-dimensional marginals of a K-parameter Lévy process in law are not infinitely divisible. To illustrate, let $\{\mu_s : s \in \boldsymbol{R}_+^2\}$ be the convolution semigroup on $\boldsymbol{R}$ given by $\mu_s = N(0, s_1 + s_2)$ for $s = (s_1, s_2)^\top \in \boldsymbol{R}_+^2$. For $j = 1, 2, 3$, let $\{V_t^j : t \geq 0\}$ be independent standard Wiener processes on $\boldsymbol{R}$. Define $\{X_s^0 : s \in \boldsymbol{R}_+^2\}$ by $X_s^0 = V_{s_1}^1 + V_{s_2}^2$, and $\{X_s^1 : s \in \boldsymbol{R}_+^2\}$ by $X_s^1 = V_{s_1+s_2}^3$. It is readily seen that $\{X_s^0 : s \in \boldsymbol{R}_+^2\}$ and $\{X_s^1 : s \in \boldsymbol{R}_+^2\}$ are associated with $\{\mu_s : s \in \boldsymbol{R}_+^2\}$. Since these two $\boldsymbol{R}_+^2$-parameter Lévy processes are not identical in law $\{\mu_s : s \in \boldsymbol{R}_+^2\}$ is multiple-generative. Let $X_s = X_{Us}^0 + X_{(1-U)s}^1$ where U, independent of $(\{X_s^0 : s \in \boldsymbol{R}_+^2\}, \{X_s^1 : s \in \boldsymbol{R}_+^2\})$, is a random variable with $0 < p = P(U = 1) = 1 - P(U = 0) < 1$. By Proposition 3.9 (i) $\{X_s : s \in K\}$ is an $\boldsymbol{R}_+^2$-parameter Lévy process associated with $\{\mu_s : s \in \boldsymbol{R}_+^2\}$. The distribution μ of $(X_{e^1}, X_{e^2})^\top$ is not infinitely divisible, where $e^1 = (1,0)^\top$ and $e^2 = (0,1)^\top$.

The proof is as follows. For any $B \in \mathscr{B}(\boldsymbol{R}^2)$, $\mu(B) = pN_2(0, \mathrm{diag}(1,1))(B) + (1-p)\rho(B)$, where ρ is a degenerate Gaussian concentrated on the line $L_1 = \{(x_1, x_2)^\top \in \boldsymbol{R}^2 : x_1 = x_2\}$. Suppose that μ is infinitely divisible. Then the projection σ of μ onto the line $L_2 = \{(x_1, x_2)^\top \in \boldsymbol{R}^2 : x_1 = -x_2\}$ has to be infinitely divisible by Proposition 11.10 of [**15**]. But σ is a mixture of a Gaussian distribution and a point mass at the origin, which is not infinitely divisible by Remark 26.3 of [**15**].

Another way of constructing cone-parameter Lévy processes is by subordination, as we discuss in the following. Let K_1 be a cone in $\boldsymbol{R}^{M_1}$ and K_2 be a cone in $\boldsymbol{R}^{M_2}$.

If $\{Z_s : s \in K_1\}$ is a K_1-parameter Lévy process (resp. Lévy process in law) on $\boldsymbol{R}^{M_2}$ such that $Z_s \in K_2$ almost surely for each $s \in K_1$, then we call it a *K_2-valued K_1-parameter Lévy process* (resp. *Lévy process in law*). If $\{Z_s : s \in K_1\}$ is a K_1-parameter Lévy process on $\boldsymbol{R}^{M_2}$ then it is a K_2-valued K_1-parameter Lévy process if and only if, almost surely, Z_s is (K_1, K_2)-increasing as a function of s. There is no analogous characterization of the sample paths of a K_2-valued K_1-parameter Lévy process in law.

In order to define subordination we have to impose the regularity condition that the processes involved (the subordinator and the subordinand) are measurable processes. But this is essentially no restriction since any K-parameter Lévy process in law has a measurable modification by Remark 3.2 (ii). Thus, we introduce *subordination of a measurable K_2-parameter Lévy process in law by a measurable K_2-valued K_1-parameter Lévy process in law*. This is an extension of the multivariate subordination introduced in [**2**].

Theorem 3.12. *Let $\{Z_s : s \in K_1\}$ be a measurable K_2-valued K_1-parameter Lévy process in law and $\{X_u : u \in K_2\}$ a measurable K_2-parameter Lévy process in law on $\boldsymbol{R}^d$. Suppose that they are independent. Define $Y_s = X_{Z'_s}$, where $Z'_s = Z_s 1_{K_2}(Z_s)$. Then $\{Y_s : s \in K_1\}$ is a measurable K_1-parameter Lévy process in law on $\boldsymbol{R}^d$.*

If in addition $\{Z_s : s \in K_1\}$ is a measurable K_2-valued K_1-parameter Lévy process on K_2 and $\{X_u : u \in K_2\}$ a measurable K_2-parameter Lévy process on $\boldsymbol{R}^d$, then $\{Y_s : s \in K_1\}$ is a measurable K_1-parameter Lévy process on $\boldsymbol{R}^d$.

The processes $\{X_u : u \in K_2\}$, $\{Z_s : s \in K_1\}$ and $\{Y_s : s \in K_1\}$ are subordinand, subordinator and subordinated, respectively.

Proof of the theorem. Since $\{Y_s : s \in K_1\}$ appears by composition of two measurable mappings, it is itself measurable. The other properties defining a cone-parameter Lévy process in law are essentially verified as in the first part of the proof of Theorem 3.3 of [**2**].

Assume that $\{Z_s : s \in K_1\}$ is a K_2-valued K_1-parameter Lévy process and $\{X_u : u \in K_2\}$ a K_2-parameter Lévy process on $\boldsymbol{R}^d$. Then, by (K_1, K_2)-increasingness of $\{Z_s : s \in K_1\}$, $\{Y_s : s \in K_1\}$ is K_1-cadlag almost surely and is hence a measurable K_1-parameter Lévy process on $\boldsymbol{R}^d$. □

Related to subordination of cone-parameter Lévy processes in law is the notion of subordination of cone-parameter convolution semigroups. The latter is treated in [**14**]. More precisely, since cone-parameter Lévy processes in law are associated with cone-parameter convolution semigroups, the distribution $\mathscr{L}(Y_s)$, $s \in K_1$, of the subordinated process $\{Y_s\}$ can be considered as a special case of subordination of cone-parameter convolution semigroups. But, since there are non-generative convolution semigroups, subordination of some convolution semigroups does not appear in this way.

4. Non-generativeness of the canonical $\boldsymbol{S}_d^+$-parameter convolution semigroup.

We say that a K-parameter convolution semigroup $\{\mu_s : s \in K\}$ is trivial if μ_s is trivial for all $s \in K$. Our main result in this section reads as follows.

Theorem 4.1. *Let $K = \boldsymbol{S}_d^+$ with $d \geq 2$. Let $\{\mu_s : s \in K\}$ be a nontrivial K-parameter convolution semigroup on $\boldsymbol{R}^d$ such that $\int |x|^2 \mu_s(dx) < \infty$ and the covariance matrix v_s of μ_s satisfies $v_s \leq_K s$ for all $s \in K$. Then $\{\mu_s\}$ is non-generative. In particular, the canonical $\boldsymbol{S}_d^+$-parameter convolution semigroup is non-generative.*

For the proof of the main Theorem we need a result of independent interest. Recall that a subset L of $\boldsymbol{R}^d$ is an additive subgroup if $x - y \in L$ whenever x and y are in L. For instance, a linear subspace is an additive subgroup. As another example note that $\boldsymbol{Q}$ is an additive subgroup of $\boldsymbol{R}$; in particular we see that additive subgroups need not be closed. The following result shows that when K has a strong basis any convolution semigroup is generative, and it gives a characterization of the unique-generative convolution semigroups.

THEOREM 4.2. *Let K have a strong basis $\{e^1, \dots, e^N\}$ and let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\boldsymbol{R}^d$. Let $Y_s = V^1_{s_1} + \cdots + V^N_{s_N}$ for $s = s_1 e^1 + \cdots + s_N e^N \in K$, where $\{V^j_t : t \geq 0\}$, $j = 1, \dots, N$, are independent Lévy processes satisfying $\mathscr{L}(V^j_1) = \mu_{e^j}$ for $j = 1, \dots, N$.*

(i) *The semigroup $\{\mu_s\}$ is generative. In particular, $\{Y_s : s \in K\}$ is a K-parameter Lévy process associated with $\{\mu_s\}$.*

(ii) *The following three statements* (a)–(c) *are equivalent*:

(a) *$\{\mu_s\}$ is unique-generative.*

(b) *Any K-parameter Lévy process in law $\{X_s : s \in K\}$ associated with $\{\mu_s : s \in K\}$ satisfies $\{X_s : s \in K\} \overset{\mathrm{d}}{=} \{Y_s : s \in K\}$.*

(c) *For any K-parameter Lévy process in law $\{X_s : s \in K\}$ associated with $\{\mu_s : s \in K\}$ and any $s = s_1 e^1 + \cdots + s_N e^N \in K$ we have $X_s = X_{s_1 e^1} + \cdots + X_{s_N e^N}$ almost surely.*

(iii) *For $j = 1, \dots, N$ let L_j be an additive subgroup of $\boldsymbol{R}^d$ such that $L_j \in \mathscr{B}(\boldsymbol{R}^d)$. Assume that for all $i \neq j$ we have $L_i \cap L_j = \{0\}$. Let $\mu_{te^j}(L_j) = 1$ for $t \geq 0$ and $j = 1, \dots, N$. Then $\{\mu_s\}$ is unique-generative.*

REMARK 4.3. From Theorem 4.2 (ii) it follows that if $\{\mu_s : s \in K\}$ is unique-generative and $\{X_s : s \in K\}$ is a K-parameter Lévy process in law associated with $\{\mu_s\}$, then the N processes $\{X_{te^1} : t \geq 0\}, \dots, \{X_{te^N} : t \geq 0\}$ are independent.

PROOF OF THEOREM 4.2. (i) It is easily verified that $\{Y_s\}$ is a K-parameter Lévy process in law. To see that it is associated with $\{\mu_s\}$, note that for $s = s_1 e^1 + \cdots + s_N e^N$ we have $\mathscr{L}(Y_s) = \mathscr{L}(V^1_{s_1}) * \cdots * \mathscr{L}(V^N_{s_N}) = \mu^{s_1}_{e^1} * \cdots * \mu^{s_N}_{e^N} = \mu_s$ by Remark 3.7.

(ii) It follows directly from (i) that (a) and (b) are equivalent. Assume that $\{\mu_s\}$ is unique-generative. Let $\{X_s\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$ and let $s = s_1 e^1 + \cdots + s_N e^N \in K$. Then, from (b),

$$P(X_{s_1 e^1 + \cdots + s_N e^N} = X_{s_1 e^1} + \cdots + X_{s_N e^N}) = P(Y_{s_1 e^1 + \cdots + s_N e^N} = Y_{s_1 e^1} + \cdots + Y_{s_N e^N})$$

and since this probability is trivially 1, we get (c).

Conversely, assume that (c) holds. Let $\{X_s\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$. Let $n \geq 1$ and $0 = s_0 \leq s_1 \leq \cdots \leq s_n$. Define random vectors $Z_{i,j}$ for $i = 1, \dots, N$, $j = 0, \dots, n$ by

$$Z_{i,j} = X_{s_n e^1 + \cdots + s_n e^{i-1} + s_j e^i}.$$

Thus, $Z_{i,0} = Z_{i-1,n}$ for $i \geq 2$ and $Z_{1,0} = 0$. It follows from (i) of Definition 3.1 that $Z_{i,j} - Z_{i,j-1}$ with $i = 1, \dots, N$ and $j = 1, \dots, n$ are independent. Since

$$Z_{i,j} = X_{s_n e^1} + \cdots + X_{s_n e^{i-1}} + X_{s_j e^i} \quad \text{almost surely}$$

by (c), we see that $X_{s_j e^i} - X_{s_{j-1} e^i}$ with $i = 1, \ldots, N$ and $j = 1, \ldots, n$ are independent. Since this holds for arbitrary $n \geq 1$ and $0 \leq s_1 \leq \cdots \leq s_n$, $\{X_{te^1} : t \geq 0\}, \ldots, \{X_{te^N} : t \geq 0\}$ are independent Lévy processes in law with $\mathscr{L}(X_{e^j}) = \mu_{e^j}$ for all j. Choosing their modifications which are Lévy processes we now see that (b) holds.

(iii) We use induction in N. In the case $N = 1$ the theorem is trivially true. Assume that the theorem holds for $N - 1$ in place of N. Let $\{X_s : s \in K\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$. By (ii) it is enough to verify condition (c). Consider the $(N-1)$-dimensional cones K_1 and K_2 generated by $\{e^2, \ldots, e^N\}$ and by $\{e^1, e^3, \ldots, e^N\}$, respectively. Then, by the induction hypothesis, both $\{\mu_s : s \in K_1\}$ and $\{\mu_s : s \in K_2\}$ are unique-generative. The restrictions $\{X_s : s \in K_1\}$ and $\{X_s : s \in K_2\}$ are associated with $\{\mu_s : s \in K_1\}$ and $\{\mu_s : s \in K_2\}$, respectively. Let $s = s_1 e^1 + \cdots + s_N e^N \in K$ and define $s^1 = s - s_1 e^1 \in K_1$ and $s^2 = s - s_2 e^2 \in K_2$. Using condition (c) for the two restrictions, we decompose X_s as

$$X_s = X_{s^1} + (X_s - X_{s^1}) \overset{\text{a.s.}}{=} X_{s_2 e^2} + \cdots + X_{s_N e^N} + (X_s - X_{s^1}), \tag{4.1}$$

$$X_s = X_{s^2} + (X_s - X_{s^2}) \overset{\text{a.s.}}{=} X_{s_1 e^1} + X_{s_3 e^3} + \cdots + X_{s_N e^N} + (X_s - X_{s^2}). \tag{4.2}$$

By equating (4.1) and (4.2) it follows that $(X_s - X_{s^1}) - X_{s_1 e^1} \overset{\text{a.s.}}{=} (X_s - X_{s^2}) - X_{s_2 e^2}$. The left-hand side is concentrated on L_1 and the right-hand side on L_2. Hence they are zero almost surely. Therefore, $X_s - X_{s^1} = X_{s_1 e^1}$ almost surely. Inserting this in (4.1) we get the almost surely identity in (c) for $\{X_s : s \in K\}$. □

Example 4.4. In the case $N = 2$ the additive subgroups $L_1 = \boldsymbol{Q}^d$ and $L_2 = (c\boldsymbol{Q})^d$ with $c \in \boldsymbol{R} \backslash \boldsymbol{Q}$ satisfy the condition $L_1 \cap L_2 = \{0\}$. We can make examples of (iii) with these L_1 and L_2, using compound Poisson convolution semigroups with Lévy measures restricted to $\boldsymbol{Q}^d$ or $(c\boldsymbol{Q})^d$.

We can now prove the main Theorem.

Proof of Theorem 4.1. We may and do assume that μ_s has zero mean for all s. The covariance matrix satisfies $v_{s^1+s^2} = v_{s^1} + v_{s^2}$ and $v_{ts} = t v_s$.

Step 1. Proof in the case $d = 2$. Suppose there exists a K-parameter Lévy process in law $\{X_s : s \in K\}$ on $\boldsymbol{R}^2$ associated with $\{\mu_s\}$. Let

$$e^1 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}.$$

Let $K_0 = \{s_1 e^1 + s_2 e^2 : s_1, s_2 \geq 0\}$ be the least cone containing $\{e^1, e^2\}$. Since e^1 and e^2 have rank one, there are $t_1, t_2 \in [0, 1]$ such that $v_{e^1} = t_1 e^1$ and $v_{e^2} = t_2 e^2$. This is easily seen using diagonalization by orthogonal matrices. It follows that for any $t \geq 0$, μ_{te^1} and μ_{te^2} are concentrated on L_1 and L_2, respectively, where $L_1 = \{(a, \sqrt{2}a) : a \in \boldsymbol{R}\}$ and $L_2 = \{(\sqrt{2}a, a) : a \in \boldsymbol{R}\}$. Hence, by Theorem 4.2 (iii), the restriction $\{\mu_s : s \in K_0\}$ is unique-generative. Since $\{X_s : s \in K_0\}$ is a K_0-parameter Lévy process in law associated with $\{\mu_s : s \in K_0\}$, it follows from Remark 4.3 that X_{e^1} and X_{e^2} are independent. Let $(X_s)_j$ denote the jth coordinate of X_s. Since $v_{e^3 - e^1} \leq_K e^3 - e^1 = \operatorname{diag}(1, 0)$ and since $X_{e^3} - X_{e^1} \stackrel{\mathrm{d}}{=} X_{e^3-e^1}$, we have $(X_{e^3} - X_{e^1})_2 = 0$ almost surely. Similarly, $(X_{e^3} - X_{e^2})_1 = 0$ almost surely. Now, using $X_{e^3} = X_{e^j} + (X_{e^3} - X_{e^j})$ for $j = 1, 2$, we get $(X_{e^3})_1 = (X_{e^2})_1$ and $(X_{e^3})_2 = (X_{e^1})_2$ almost surely. Hence $(X_{e^3})_2$ and $(X_{e^3})_1$ are independent. It follows that v_{e^3} is diagonal, say, $v_{e^3} = \mathrm{diag}(a_1, a_2)$ with $a_1, a_2 \geq 0$. We have $v_{e^3-e^1} = \mathrm{diag}(t, 0)$ with $t \geq 0$ since $v_{e^3-e^1} \leq_K e^3 - e^1$. Now, looking at nondiagonal entries of $v_{e^1} = v_{e^3} - v_{e^3-e^1}$ and $v_{e^1} = t_1 e^1$, we conclude that $t_1 = 0$. Thus $v_{e^1} = 0$. Hence $v_{e^3} = v_{e^3-e^1} \leq_K e^3 - e^1$, which shows that $a_2 = 0$. The same kind of argument gives $a_1 = 0$ and $v_{e^2} = v_{e^3} = 0$. It follows that $\mu_{e^1} = \mu_{e^2} = \mu_{e^3} = \delta_0$. Since the system $\{e^1, e^2, e^3\}$ is linearly independent, it is a weak basis of K. Hence, by Remark 3.7 $\mu_s = \delta_0$ for all $s \in K$, contradicting the assumption of nontriviality. Therefore, the associated Lévy process in law does not exist.

STEP 2. Proof in the case $d \geq 2$. Suppose that we can find a K-parameter Lévy process in law $\{X_s : s \in K\}$ on $\boldsymbol{R}^d$ associated with $\{\mu_s : s \in K\}$. Since $\{\mu_s\}$ is nontrivial, there is $s^0 \in K$ such that $v_{s^0} \neq 0$. Let $p = \mathrm{rank}(s^0)$. Then $p \geq 1$. Using diagonalization, we can decompose s^0 as $s^0 = s^1 + \cdots + s^p$, where, for each $j \geq 1$, $s^j \in K$ and $\mathrm{rank}(s^j) = 1$. Since $v_{s^0} = v_{s^1} + \cdots + v_{s^p}$, we have $v_{s^j} \neq 0$ for some $j \geq 1$. Thus we may and do assume that $\mathrm{rank}(s^0) = 1$ and $v_{s^0} \neq 0$. There is a $d \times d$ orthogonal matrix r such that $rs^0r' = \mathrm{diag}(a, 0, \ldots, 0)$ with $a > 0$, where r' is the transpose of r. Define

$$K_0 = \{s = (s_{jk})_{j,k=1}^d \in K : s_{jk} = 0 \text{ except for } j,k \in \{1,2\}\},$$

$$K_1 = \{r'sr : s \in K_0\}.$$

Then K_1 is a cone and $s^0 \in K_1$.

Notice that $\mathrm{cov}(rX_s) = rv_sr'$ for $s \in K$, since $\mathrm{cov}(X_s) = v_s$. If $s \in K_1$, then $rv_sr' \leq_K rsr' \in K_0$ and hence $rv_sr' \in K_0$. Therefore, if $s \in K_1$, then $(rX_s)_j = 0$ almost surely for $j \neq 1, 2$.

For $u \in \boldsymbol{S}_2^+$ let $T_0u \in K_0$ be the natural extension of u and let $Tu = r'(T_0u)r$. Then T is an isomorphism from $\boldsymbol{S}_2^+$ to K_1. Define $X_u^0 = ((rX_{Tu})_1, (rX_{Tu})_2)^\top$ for $u \in \boldsymbol{S}_2^+$. Then $\{X_{Tu} : u \in \boldsymbol{S}_2^+\}$ is an $\boldsymbol{S}_2^+$-parameter Lévy process in law on $\boldsymbol{R}^d$, and such is $\{rX_{Tu} : u \in \boldsymbol{S}_2^+\}$. It follows that $\{X_u^0 : u \in \boldsymbol{S}_2^+\}$ is an $\boldsymbol{S}_2^+$-parameter Lévy process in law on $\boldsymbol{R}^2$. Let $\mu_u^0 = \mathscr{L}(X_u^0)$. Then $\{\mu_u^0 : u \in \boldsymbol{S}_2^+\}$ is an $\boldsymbol{S}_2^+$-parameter convolution semigroup on $\boldsymbol{R}^2$ and $\mathrm{cov}(\mu_u^0)$ equals the restriction of $rv_{Tu}r'$ to the first 2×2 block. Since $rv_{Tu}r' \leq_K r(Tu)r' = T_0u \in K_0$, we see that $\mathrm{cov}(\mu_u^0) \leq_{\boldsymbol{S}_2^+} u$. We have $\mathrm{cov}(\mu_{u^0}^0) \neq 0$, where u^0 is chosen so that $Tu^0 = s^0$. But this is impossible in view of Step 1. Hence, $\{X_s : s \in K\}$ does not exist. □

The main Theorem 4.1 shows that there does not exist a Brownian motion with parameter in $\boldsymbol{S}_d^+$. We can refine this by showing that there is no Brownian motion with parameter in the set $\boldsymbol{S}_d^{++}$ of positive-definite symmetric $d \times d$ matrices.

PROPOSITION 4.5. *Let $d \geq 2$. There is no family $\{X_s : s \in \boldsymbol{S}_d^{++}\}$ of random variables on $\boldsymbol{R}^d$ satisfying the following two conditions* (i)–(ii).

(i) *If $s^1, \ldots, s^n \in \boldsymbol{S}_d^{++}$ with $n \geq 2$ and $s^j - s^{j-1} \in \boldsymbol{S}_d^{++}$ for $j = 2, \ldots, n$, then $X_{s^2} - X_{s^1}, \ldots, X_{s^n} - X_{s^{n-1}}$ are independent.*

(ii) *If $s^2, s^1 \in \boldsymbol{S}_d^{++}$ with $s^2 - s^1 \in \boldsymbol{S}_d^{++}$, then $\mathscr{L}(X_{s^2} - X_{s^1}) = N_d(0, s^2 - s^1)$.*

Proof. Assume that $\{X_s : s \in \boldsymbol{S}_d^{++}\}$ satisfies (i)–(ii). We show that there exists an $\boldsymbol{S}_d^{+}$-parameter Lévy process in law $\{\tilde{X}_s : s \in \boldsymbol{S}_d^{+}\}$ associated with the canonical $\boldsymbol{S}_d^{+}$-parameter convolution semigroup, whereby we get a contradiction by Theorem 4.1.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers strictly decreasing to 0 and I denote the $d \times d$ identity matrix. Let $s \in \boldsymbol{S}_d^{+}$. By (ii), $\mathscr{L}(X_{s+\varepsilon_k I} - X_{s+\varepsilon_l I}) = N_d(0, (\varepsilon_k - \varepsilon_l)I)$ for $k < l$. Hence $\tilde{X}_s := \lim_{k\to\infty} X_{s+\varepsilon_k I}$ exists in probability.

Let $s^1, s^2 \in \boldsymbol{S}_d^{+}$ with $s^1 \leq_{\boldsymbol{S}_d^{+}} s^2$. Then $\mathscr{L}(X_{s^2+\varepsilon_k I} - X_{s^1+\varepsilon_{k+1} I}) \to \mathscr{L}(\tilde{X}_{s^2} - \tilde{X}_{s^1})$ as $k \to \infty$ by construction of $\tilde{X}_{s^2}$ and $\tilde{X}_{s^1}$, and $\mathscr{L}(X_{s^2+\varepsilon_k I} - X_{s^1+\varepsilon_{k+1} I}) = N_d(0, s^2 - s^1 + (\varepsilon_k - \varepsilon_{k+1})I) \to N_d(0, s^2 - s^1)$ by (ii). That is, $\mathscr{L}(\tilde{X}_{s^2} - \tilde{X}_{s^1}) = N_d(0, s^2 - s^1)$. In particular $\mathscr{L}(\tilde{X}_s) = N_d(0, s)$ and the family $\{\tilde{X}_s : s \in \boldsymbol{S}_d^{+}\}$ satisfies Definition 3.1 (ii) and Lemma 3.8 (v)$'$. Let $\{s^j\}_{j=1,\dots,n}$ be $\boldsymbol{S}_d^{+}$-increasing. Since $X_{s^{j+1}+\varepsilon_{k+j} I} - X_{s^j+\varepsilon_{k+j+1} I}$, $j = 1, \dots, n-1$, are independent by (i) it follows by letting $k \to \infty$ that $\tilde{X}_{s^{j+1}} - \tilde{X}_{s^j}$, $j = 1, \dots, n-1$, are independent. Thus $\{\tilde{X}_s : s \in \boldsymbol{S}_d^{+}\}$ satisfies Definition 3.1 (i). By Lemma 3.8 it follows that $\{\tilde{X}_s : s \in \boldsymbol{S}_d^{+}\}$ is an $\boldsymbol{S}_d^{+}$-parameter Lévy process in law associated with the canonical $\boldsymbol{S}_d^{+}$-parameter convolution semigroup. □

5. Generative convolution semigroups.

In this section the main results are concerned with the problem whether a given cone-parameter is generative. We have already seen that if K has a strong basis then any K-parameter Lévy process is generative. Next we consider the case where μ_s is purely non-Gaussian.

Lemma 5.1. *Let $\{e^1, \dots, e^N\}$ be a weak basis of K and $\{\mu_s : s \in K\}$ be a convolution semigroup such that μ_s has triplet $(0, \nu_s, 0)$ for $s \in K$. Let $\nu = \nu_{e^1} + \dots + \nu_{e^N}$. Then, for each $s \in K$, ν_s is absolutely continuous with respect to ν. Moreover, the family $\{\phi_s : s \in K\}$ of densities ϕ_s of ν_s with respect to ν can be chosen such that*

(i) $\phi_{e^1}(x) + \dots + \phi_{e^N}(x) \leq 1$ *for* $x \in \boldsymbol{R}^d$,

(ii) $\phi_s(x) = s_1\phi_{e^1}(x) + \dots + s_N\phi_{e^N}(x)$ *for* $s = s_1e^1 + \dots + s_Ne^N \in K$ *and* $x \in \boldsymbol{R}^d$,

(iii) $s^n \to s$ *implies* $\phi_{s^n}(x) \to \phi_s(x)$ *for* $x \in \boldsymbol{R}^d$,

(iv) $\phi_s(x) \geq 0$ *for* $s \in K$ *and* $x \in \boldsymbol{R}^d$.

Proof. Let $s = s_1e^1 + \dots + s_Ne^N$ and let $K_0 = \{s \in K\colon s_1, \dots, s_N \in \boldsymbol{Q}\}$. Note that $e^1, \dots, e^N \in K_0$. Since $\nu_s = s_1\nu_{e^1} + \dots + s_N\nu_{e^N}$ by Remark 3.7 it follows that ν_s is absolutely continuous with respect to ν. Fix a density ϕ_s^0 of ν_s with respect to ν for $s \in K_0$. Then

$$\phi_{e^1}^0(x) + \dots + \phi_{e^N}^0(x) = 1, \quad \phi_s^0(x) = s_1\phi_{e^1}^0(x) + \dots + s_N\phi_{e^N}^0(x), \quad \phi_s^0(x) \geq 0, \tag{5.1}$$

each holding for ν-almost every x. Let $B = \{x \in \boldsymbol{R}^d\colon \text{(5.1) holds for all } s \in K_0\}$. Then $\nu(\boldsymbol{R}^d \backslash B) = 0$. Define

$$\phi_s(x) = \phi_s^0(x) \quad \text{for } s \in K_0 \text{ and } x \in B,$$

$$\phi_s(x) = s_1\phi_{e^1}^0(x) + \dots + s_N\phi_{e^N}^0(x) \quad \text{for } s \in K \backslash K_0 \text{ and } x \in B,$$

$$\phi_s(x) = 0 \quad \text{for } s \in K \text{ and } x \in \boldsymbol{R}^d \backslash B.$$

Then, ϕ_s is a desired density of ν_s with respect to ν; (i) and (ii) are from the definition of ϕ_s; (iii) is from (ii) since $s^n \to s$ if and only if $s_j^n \to s$ for $j = 1, \dots, N$; (iv) is from the definition for $s \in K_0$ and by approximation using (iii) for $s \in K \backslash K_0$. $\square$

Consider the family $\{\phi_s : s \in K\}$ of densities of Lemma 5.1 and define, for $s \in K$,

$$D_s = \{(t, x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^d : 0 \le t \le \phi_s(x)\}. \tag{5.2}$$

THEOREM 5.2. *Let K be an arbitrary cone with a weak basis $\{e^1, \dots, e^N\}$. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\boldsymbol{R}^d$ such that μ_s is purely non-Gaussian for all s, that is, μ_s has triplet $(0, \nu_s, \gamma_s)$. Then $\{\mu_s\}$ is generative.*

To construct an associated K-parameter Lévy process in law, let $\{J(A) : A \in \mathscr{B}(\boldsymbol{R}_+ \times \boldsymbol{R}^d)\}$, defined on a probability space $(\Omega, \mathscr{F}, P)$, be a Poisson random measure with intensity measure $\lambda(d(t, x)) = dt\nu(dx)$, where $\nu = \nu_{e^1} + \dots + \nu_{e^N}$. For $s \in K$ define

$$X_s = \int_{D_s} x 1_{\{|x| \le 1\}}(x)(J(d(t, x)) - \lambda(d(t, x))) + \int_{D_s} x 1_{\{|x| > 1\}}(x) J(d(t, x)) + \gamma_s. \tag{5.3}$$

Then $\{X_s : s \in K\}$ is a K-parameter Lévy process in law associated with $\{\mu_s\}$.

If, in addition, $\int_{\boldsymbol{R}^d}(1 \wedge |x|)\nu_s(dx) < \infty$ for all $s \in K$, then $\{X_s : s \in K\}$ is a K-parameter Lévy process.

The first integral on the right-hand side of (5.3) is a stochastic integral only determined up to null sets. Hence, we may change $X_s(\omega)$ on a null set of ω's while (5.3) remains true. Thus, the last statement says that it is possible to choose $X_s(\omega)$ for $\omega \in \Omega$ and $s \in K$ such that all paths are K-cadlag.

PROOF OF THE THEOREM. We may and do assume $\gamma_s = 0$ for all s. Let $D_s^1 = D_s \cap \{(t, x) : |x| \le 1\}$, $D_s^2 = D_s \cap \{(t, x) : |x| > 1\}$, $f_s^1(t, x) = x 1_{D_s^1}(t, x)$ and $f_s^2(t, x) = x 1_{D_s^2}(t, x)$. Let $U_s^1 = \int f_s^1(t, x)(J(d(t, x)) - \lambda(d(t, x)))$ and $U_s^2 = \int f_s^2(t, x) J(d(t, x))$. That is, U_s^j is the jth term on the right-hand side of (5.3) for $j = 1, 2$. Using $d\nu_s = \phi_s \, d\nu$ it follows that

$$\lambda(D_s^2) = \nu_s(\{x : |x| > 1\}) < \infty,$$

$$\int |f_s^1|^2(t, x)\lambda(d(t, x)) = \int_{|x| \le 1} |x|^2 \nu_s(dx) < \infty.$$

Hence, U_s^2 exists as Lebesgue-Stieltjes integral with respect to $J(d(t, x))$ while U_s^1 exists as stochastic integral with respect to the compensated measure $J(d(t, x)) - \lambda(d(t, x))$. Moreover, it is well-known that for $s^1, s^2 \in K$ and $z \in \boldsymbol{R}^d$ we have

$$Ee^{\mathrm{i}\langle z, U_{s^2}^1 - U_{s^1}^1 \rangle} = \exp \int (e^{\mathrm{i}\langle z, (f_{s^2}^1 - f_{s^1}^1)(t, x) \rangle} - 1 - \mathrm{i}\langle z, (f_{s^2}^1 - f_{s^1}^1)(t, x) \rangle)\lambda(d(t, x)), \tag{5.4}$$

$$Ee^{\mathrm{i}\langle z, U_{s^2}^2 - U_{s^1}^2 \rangle} = \exp \int (e^{\mathrm{i}\langle z, (f_{s^2}^2 - f_{s^1}^2)(t, x) \rangle} - 1)\lambda(d(t, x)). \tag{5.5}$$

STEP 1. Let $s^1, s^2 \in K$ with $s^1 \le_K s^2$. Then, $D_{s^1} \subseteq D_{s^2}$ and

$$(5.6) \qquad (f^1_{s^2} - f^1_{s^1})(t,x) = x1_{D^1_{s^2}\backslash D^1_{s^1}}(t,x) = \begin{cases} x1_{\{|x|\le 1\}}(x) & \text{if } \phi_{s^1}(x) < t \le \phi_{s^2}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, using $\phi_{s^2} - \phi_{s^1} = \phi_{s^2-s^1}$ and $\nu_{s^2-s^1}(dx) = \phi_{s^2-s^1}(x)\nu(dx)$, we find that

$$\int (e^{\mathrm{i}\langle z,(f^1_{s^2}-f^1_{s^1})(t,x)\rangle} - 1 - \mathrm{i}\langle z,(f^1_{s^2}-f^1_{s^1})(t,x)\rangle)\lambda(d(t,x))$$

$$= \int_{\boldsymbol{R}^d} x1_{\{|x|\le 1\}}(x)(e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle)\nu_{s^2-s^1}(dx).$$

Inserting this in (5.4) we find that $\mathscr{L}(U^1_{s^2} - U^1_{s^1})$ has triplet $(0, 1_{\{|x|\le 1\}}(x)\nu_{s^2-s^1}(dx), 0)$. Similar arguments show that $\mathscr{L}(U^2_{s^2} - U^2_{s^1})$ has triplet $(0, 1_{\{|x|>1\}}(x)\nu_{s^2-s^1}(dx), 0)$.

Step 2. Let $n \ge 2$ and $\{s^j\}_{j=1,\dots,n}$ be K-increasing. Then $D^j_{s^{k-1}} \subseteq D^j_{s^k}$ and $(f^j_{s^k} - f^j_{s^{k-1}})(t,x) = x1_{D^j_{s^k}\backslash D^j_{s^{k-1}}}(t,x)$ for $j = 1,2$ and $k = 2,\dots,n$. Hence, since the sets $D^1_{s^2}\backslash D^1_{s^1},\dots,D^1_{s^n}\backslash D^1_{s^{n-1}}, D^2_{s^2}\backslash D^2_{s^1},\dots,D^2_{s^n}\backslash D^2_{s^{n-1}}$ are disjoint, $U^j_{s^k} - U^j_{s^{k-1}}$, $j = 1,2$, $k = 2,\dots,n$, are independent; consequently also $X_{s^k} - X_{s^{k-1}} = (U^1_{s^k} - U^1_{s^{k-1}}) + (U^2_{s^k} - U^2_{s^{k-1}})$, $k = 2,\dots,n$, are independent. Moreover, by Step 1, $\mathscr{L}(X_{s^k} - X_{s^{k-1}}) = \mu_{s^k - s^{k-1}}$.

Step 3. Let $s^n, s \in K$ with $s^n \to s$. By Lemma 5.1 (iii) we have $\phi_{s^n}(x) \to \phi_s(x)$ for all $x \in \boldsymbol{R}^d$. Hence, $1_{D_{s^n}}(t,x) \to 1_{D_s}(t,x)$ for λ-a.e. (t,x). Moreover, by Lemma 5.1 (i), (ii), (iv) it follows that

$$(5.7) \qquad 0 \le \phi_r(x) \le |r_1| + \cdots + |r_N| \quad \text{for } r = r_1e^1 + \cdots + r_Ne^N \in K.$$

Decompose s^n and s as $s^n = s^n_1e^1 + \cdots + s^n_Ne^N$ and $s = s_1e^1 + \cdots + s_Ne^N$. Since $s^n_j \to s_j$ for all $j = 1,\dots,N$, (5.7) shows that there exists a constant $c > 0$ such that $1_{D_s}(t,x), 1_{D_{s^n}}(t,x) \le 1_{[0,c]}(t)$. Since

$$|e^{\mathrm{i}\langle z,(f^1_{s^n}-f^1_s)(t,x)\rangle} - 1 - \mathrm{i}\langle z,(f^1_{s^n}-f^1_s)(t,x)\rangle| \le \frac{1}{2}|\langle z,(f^1_{s^n}-f^1_s)(t,x)\rangle|^2$$

$$\le \frac{1}{2}|z|^2|(f^1_{s^n}-f^1_s)(t,x)|^2 \le \frac{1}{2}|z|^2|x|^2 1_{\{|x|\le 1\}}(x)1_{[0,c]}(t),$$

$$|e^{\mathrm{i}\langle z,(f_{s^n}-f_s)(t,x)\rangle} - 1| \le 2\ 1_{\{|x|>1\}}(x)1_{[0,c]}(t),$$

it follows from (5.4)–(5.5) that $\mathscr{L}(U^j_{s^n} - U^j_s) \to \delta_0$ for $j = 1,2$.

Step 4. Note that by Step 2 $\{X_s : s \in K\}$ satisfies (i)–(ii) of Definition 3.1. It is immediate that $X_0 = 0$ almost surely. Since $X_s = U^1_s + U^2_s$ it follows from Step 3 that $\{X_s : s \in K\}$ is continuous in probability. Thus, we have shown that $\{X_s : s \in K\}$ is a K-parameter Lévy process in law. Moreover, it is associated with $\{\mu_s : s \in K\}$ since we have $\mathscr{L}(X_s) = \mu_s$ for $s \in K$ by Step 2.

Step 5. Now assume in addition that $\int_{\boldsymbol{R}^d}(1 \wedge |x|)\nu_s(dx) < \infty$ for all s. Let $(T_1, Y_1), (T_2, Y_2), \dots,$ be a random sequence such that $J(d(t,x)) = \sum_m \delta_{(T_m, Y_m)}(d(t,x))$ almost surely. Then, using (5.3) we have that

$$(5.8) \qquad X_s = \sum_{m: T_m \le \phi_s(Y_m)} Y_m - \int_{|x|\le 1} x\phi_s(x)\nu(dx) \quad \text{almost surely.}$$

where $\sum_{m:T_m \le \phi_s(Y_m)} |Y_m| < \infty$ almost surely. We stress that X_s is only determined up to null sets by (5.8). Let us define $X_s(\omega)$ such that all paths are K-cadlag. Let $p \in \boldsymbol{N}$ and define $u^p = p(e^1 + \cdots + e^N) \in K$. Choose a null set $N \in \mathscr{F}$ such that $\sum_{m:T_m(\omega) \le \phi_{u^p}(Y_m(\omega))} Y_m(\omega)$ is absolutely convergent for all $p \in \boldsymbol{N}$ and $\omega \in N^c$. Note that if $s \in K$ then there is some $p \in \boldsymbol{N}$ such that $s \le_K u^p$. Hence, since $\phi_s(x) \le \phi_{u^p}(x)$ by Lemma 5.1 (ii) and (iv), the series $\sum_{m:T_m(\omega) \le \phi_s(T_m(\omega))} Y_m(\omega)$ is absolutely convergent for all $s \in K$ and all $\omega \in N^c$. For $s \in K$ let

$$X_s(\omega) = \begin{cases} \sum_{m:T_m(\omega) \le \phi_s(Y_m(\omega))} Y_m(\omega) - \int_{|x| \le 1} x\phi_s(x)\nu(dx) & \text{if } \omega \in N^c \\ 0 & \text{if } \omega \in N. \end{cases}$$

Note that $s^1 \le_K s^2$ implies $\phi_{s^1} \le \phi_{s^2}$. Using this it follows that all paths of $\{X_s : s \in K\}$ are K-cadlag. In fact, the K-left limits can be calculated as follows. Let $\{s^n\}$ in $K \backslash \{s\}$ be K-increasing with $s^n \to s$. Then

$$X_{s^n} \to \sum_{m:T_m \le \phi_{s^n}(Y_m) \text{ for some } n} Y_m - \int_{|x| \le 1} x\phi_s(x)\nu(dx)$$

pointwise on N^c. Thus, $\{X_s : s \in K\}$ is a K-parameter Lévy process. $\square$

In the next result we specialize to the case $d = 1$.

Theorem 5.3. *Let K be an arbitrary cone. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\boldsymbol{R}$. Then $\{\mu_s\}$ is generative.*

Proof. Let (A_s, ν_s, γ_s) be the triplet of μ_s. Here A_s is a nonnegative number. By the previous theorem there exists a K-parameter Lévy process in law $\{X_s^1\}$ associated with the convolution semigroup $\{\tilde{\mu}_s\}$, where $\tilde{\mu}_s$ is the distribution with triplet $(0, \nu_s, \gamma_s)$. Let $\{V_t : t \ge 0\}$ be a standard Wiener process, independent of $\{X_s^1 : s \in K\}$. If $s^1 \le_K s^2$, then $A_{s^1} \le A_{s^2}$. Hence, $\{X_s^2 : s \in K\}$ defined by $X_s^2 = V_{A_s}$ is a K-parameter Lévy process in law such that $\mathscr{L}(X_s^2)$ has triplet $(A_s, 0, 0)$. Hence, $\{X_s\}$ defined by $X_s = X_s^1 + X_s^2$ is a K-parameter Lévy process in law associated with $\{\mu_s\}$. $\square$

The following fact on $\boldsymbol{S}_d^+$-parameter convolution semigroups is a consequence of Theorem 5.2 combined with Theorem 4.1.

Proposition 5.4. *Let $K = \boldsymbol{S}_d^+$ with $d \ge 2$. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\boldsymbol{R}^d$ such that $\int |x|^2 \mu_s(dx) < \infty$ and $v_s \le_K s$ for all $s \in K$, where v_s is the covariance matrix of μ_s. Then μ_s is Gaussian, that is, the Lévy measure ν_s of μ_s is zero.*

Proof. Let (A_s, ν_s, γ_s) be the triplet of μ_s. Decompose μ_s as $\mu_s = \mu_s' * \mu_s''$, where μ_s' and μ_s'' are infinitely divisible with triplets $(0, \nu_s, \gamma_s)$ and $(A_s, 0, 0)$, respectively. Then μ_s' and μ_s'' have finite second moments and the covariance matrices v_s' and v_s'' of μ_s', μ_s'' satisfy $v_s = v_s' + v_s''$. Hence, $v_s', v_s'' \le_K s$. Since $\{\mu_s'\}$ is a K-parameter convolution semigroup there is a K-parameter Lévy process associated with it by Theorem 5.2. But Theorem 4.1 says that this is impossible if $\{\mu_s'\}$ is nontrivial. It follows that $\nu_s = 0$. That is, μ_s is Gaussian. $\square$

Remark 5.5. Let $d \geq 1$ and consider the problem of constructing a family of probability measures $\{\mu_s : s \in \boldsymbol{S}_d^+\}$ on $\boldsymbol{R}^d$ which is closed under convolution and satisfies that s is the covariance matrix of μ_s. In the case $d = 1$ let $\boldsymbol{S}_d^+ = \boldsymbol{R}_+$. Then the latter condition is that $s \in \boldsymbol{R}_+$ is the variance of μ_s. In this case there are many such families. In fact, any infinitely divisible distribution on $\boldsymbol{R}$ with unit variance corresponds to a family with the desired properties.

Let $d \geq 2$. It is remarkable that, up to a change of drift, the canonical $\boldsymbol{S}_d^+$-parameter convolution semigroup is the only family with the desired properties. Precisely, if $\{\mu_s : s \in \boldsymbol{S}_d^+\}$ satisfies the conditions stated above, then $\mu_s = \mu_s^\# * \delta_{m_s}$, where m_s is the mean of μ_s and $\{\mu_s^\# : s \in \boldsymbol{S}_d^+\}$ is the canonical $\boldsymbol{S}_d^+$-parameter convolution semigroup. This follows since $\{\mu_s * \delta_{-m_s} : s \in \boldsymbol{S}_d^+\}$ is a convolution semigroup on $\boldsymbol{R}^d$ satisfying the assumptions of the preceding proposition.

Let K have a strong basis. As stated in Theorem 4.2, any convolution semigroup is generative. We consider the following question: When is a K-parameter convolution semigroup unique-generative?

Theorem 5.6. *Let $N \geq 2$ and K have a strong basis $\{e^1, \dots, e^N\}$. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\boldsymbol{R}^d$ with triplet (A_s, ν_s, γ_s).*

(i) *Assume that for some i and k with $i \neq k$ we have either* (a) *or* (b), *where*

(a) $A_{e^i}(\boldsymbol{R}^d) \cap A_{e^k}(\boldsymbol{R}^d) \neq \{0\}$;

(b) *ν_{e^i} and ν_{e^k} are not mutually singular.*

Then $\{\mu_s : s \in K\}$ is multiple-generative.

(ii) *Assume that $\{\mu_s\}$ is Gaussian, that is, $\nu_s = 0$ for all $s \in K$. Then $\{\mu_s\}$ is unique-generative if and only if $A_{e^i}(\boldsymbol{R}^d) \cap A_{e^k}(\boldsymbol{R}^d) = \{0\}$ for all $i \neq j$.*

(iii) *If $\{\mu_s\}$ is unique-generative, then any K-parameter Lévy process in law $\{X_s : s \in K\}$ associated with $\{\mu_s\}$ has a K-parameter Lévy process modification.*

Remark 5.7. We do not know whether every K-parameter Lévy process in law has a K-parameter Lévy process modification.

Proof of Theorem 5.6. (i) Let us for simplicity assume that either (a) or (b) holds with $i = 1$ and $k = 2$. Then there are three generating triplets (A^j, ν^j, γ^j), $j = 0, 1, 2$, such that A^0 or ν^0 is non-zero and such that $(A_{e^j}, \nu_{e^j}, \gamma_{e^j}) = (A^0 + A^j, \nu^0 + \nu^j, \gamma^0 + \gamma^j)$ for $j = 1, 2$. Let $\{V_t^j : t \geq 0\}$, $j = 0, \dots, N$, be independent Lévy processes on $\boldsymbol{R}^d$ such that $\mathscr{L}(V_1^j)$ has triplet (A^j, ν^j, γ^j) for $j = 0, 1, 2$ and $\mathscr{L}(V_1^j)$ has triplet $(A_{e^j}, \nu_{e^j}, \gamma_{e^j})$ for $j = 3, \dots, N$. Define $\{X_s : s \in K\}$ by $X_s = V_{s_1+s_2}^0 + V_{s_1}^1 + \cdots + V_{s_N}^N$ for $s = s_1 e^1 + \cdots + s_N e^N \in K$. Then $\{X_s\}$ is a K-parameter Lévy process and it is associated with $\{\mu_s\}$. Since $\{V_t^0\}$ is a non-trivial Lévy process, $\{X_{te^1}\}$ and $\{X_{te^2}\}$ are not independent. Thus, by Theorem 4.2 (ii) $\{\mu_s : s \in K\}$ is multiple-generative.

(ii) If for some $i \neq j$ we have $A_{e^i}(\boldsymbol{R}^d) \cap A_{e^j}(\boldsymbol{R}^d) \neq \{0\}$ then by (i) $\{\mu_s\}$ is multiple-generative. Conversely assume that $A_{e^i}(\boldsymbol{R}^d) \cap A_{e^j}(\boldsymbol{R}^d) = \{0\}$ for all $i \neq j$. Let $L_j = A_{e^j}(\boldsymbol{R}^d)$ for $j = 1, \dots, N$. Let $\mu_s^\# = \mu_s * \delta_{-\gamma_s}$. Then $\mu_{te^j}^\#(L_j) = 1$ for every $t \geq 0$ and j. By Theorem 4.2 the convolution semigroup $\{\mu_s^\#\}$ is unique-generative.

(iii) Let $\{\mu_s\}$ be unique-generative. Let $\{X_s\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$. Since $\{X_{te^j} : t \geq 0\}$ is a Lévy process in law it has a Lévy process modification $\{U_t^j : t \geq 0\}$. For simplicity let $\{U_t^j : t \geq 0\}$ be chosen such that all paths are cadlag. For $s = s_1 e^1 + \cdots + s_N e^N \in K$ define X'_s as $X'_s = U^1_{s_1} + \cdots + U^N_{s_N}$. Then $\{X'_s : s \in K\}$ is a modification of $\{X_s : s \in K\}$ by (c) of Theorem 4.2. We claim that all paths of $\{X'_s : s \in K\}$ are K-cadlag. Indeed, K-right continuity follows from right continuity of U^j_t. If $s^n = s^n_1 e^1 + \cdots + s^n_N e^N$ is K-increasing, $s^n \in K \backslash \{s^0\}$ and $s^n \to s^0 = s^0_1 e^1 + \cdots + s^0_N e^N$, then, by Lemma 2.4, there exists a unique nonempty subset a of $\{1, \dots, N\}$ such that $s^n \uparrow_a s^0$. Therefore, $\lim_{n\to\infty} X'_{s^n} = \sum_{j \notin a} U^j_{s^0_j} + \sum_{j \in a} \lim_{n\to\infty} U^j_{s^n_j}$ exists. □

EXAMPLE 5.8. Let $K = \boldsymbol{S}_2^+$ and $\mu_s = N_2(0, s)$. Note that $\boldsymbol{S}_2^+$ has a weak basis $\{e^1, e^2, e^3\}$, where

$$e^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $K_0 = \{s_1 e^1 + s_2 e^2 + s_3 e^3 : s_1, s_2, s_3 \geq 0\}$ be the least cone containing $\{e^1, e^2, e^3\}$. Then, from Theorem 5.6 it follows that $\{\mu_s : s \in K_0\}$ is a unique-generative K_0-parameter convolution semigroup. Note also that, by Theorem 4.2 (ii), any K_0-parameter Lévy process in law associated with $\{\mu_s : s \in K_0\}$ is identical in law with

$$\{(V^1_{s_1}, 0)^\top + (0, V^2_{s_2})^\top + (V^3_{s_3}, V^3_{s_3})^\top : s = s_1 e^1 + s_2 e^2 + s_3 e^3 \in K_0\},$$

where $\{V^1_t : t \geq 0\}$, $\{V^2_t : t \geq 0\}$ and $\{V^3_t : t \geq 0\}$ are independent standard Wiener processes on $\boldsymbol{R}$. In particular, it follows that any K_0-parameter Lévy process in law associated with $\{\mu_s : s \in K_0\}$ has a continuous modification.

References

[1] R. J. Adler, D. Monrad, R. H. Scissors and R. Wilson, Representations, decompositions and sample function continuity of random fields with independent increments, Stochastic Process. Appl., **15** (1983), 3–30.

[2] O. E. Barndorff-Nielsen, J. Pedersen and K. Sato, Multivariate subordination, selfdecomposability and stability, Adv. in Appl. Probab., **33** (2001), 160–187.

[3] D. L. Cohn, Measurable choice of limit points and the existence of separable and measurable processes, Z. Wahrsch. Verw. Gebiete, **22** (1972), 161–165.

[4] E. B. Dynkin, Additive functionals of several time-reversible Markov processes, J. Funct. Anal., **42** (1981), 64–101.

[5] W. Ehm, Sample function properties of multi-parameter stable processes, Z. Wahrsch. Verw. Gebiete, **56** (1981), 195–228.

[6] S. N. Evans, Potential theory for a family of several Markov processes, Ann. Inst. H. Poincaré Probab. Statist., **23** (1987), 499–530.

[7] P. J. Fitzsimmons and T. S. Salisbury, Capacity and energy for multiparameter Markov processes, Ann. Inst. H. Poincaré Probab. Statist., **25** (1989), 325–350.

[8] F. Hirsch, Potential theory related to some multiparameter processes, Potential Anal., **4** (1995), 245–267.

[9] D. Khoshnevisan and Z. Shi, Brownian sheet and capacity, Ann. Probab., **27** (1999), 1135–1159.

[10] D. Khoshnevisan, Y. Xiao and Y. Zhong, Local times of additive Lévy processes, Stochastic Process. Appl., **104** (2003), 193–216.

[11] S. Lagaize, Hölder exponent for a two-parameter Lévy process, J. Multivariate Anal., **77** (2001), 270–285.

[12] P. Lévy, Processus Stochastiques et Mouvement Brownien, Gauthier-Villars, Paris, 1948 (2e éd. 1965).

[13] S. Orey and W. E. Pruitt, Sample functions of the N-parameter Wiener process, Ann. Probab., **1** (1973), 138–163.

[14] J. Pedersen and K. Sato, Cone-parameter convolution semigroups and their subordination, Tokyo J. Math., **26** (2003), 503–525.

[15] K. Sato, Lévy Processes and Infinitely Divisible Distributions, (ed. W. Fulton et al.), Cambridge Stud. Adv. Math., **68**, Cambridge Univ. Press, Cambridge, 1999.
[16] M. Talagrand, The small ball problem for the Brownian sheet, Ann. Probab., **22** (1994), 1331–1354.
[17] M. E. Vares, Local times for two-parameter Lévy processes, Stochastic Process. Appl., **15** (1983), 59–82.

Jan Pedersen
Department of Mathematical Sciences
University of Aarhus
Ny Munkegade
DK-8000 Aarhus C
Denmark
E-mail: jan@imf.au.dk

Ken-iti Sato
Hachiman-yama 1101-5-103
Tenpaku-ku
Nagoya, 468-0074
Japan
E-mail: ken-iti.sato@nifty.ne.jp