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Covering for category and combinatorics on $P_\kappa(\lambda)$

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Abstract. We study combinatorics on $P_\kappa(\lambda)$ under the assumption that $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$.

0. Introduction.

Galvin (see [**3**]) established that if Martin's axiom holds and λ is an uncountable cardinal $< 2^{\aleph_0}$, then $I^+_{\omega,\lambda} \to (I^+_{\omega,\lambda})^2$. Jech and Shelah [**12**] observed that the conclusion can be strengthened to "$I^+_{\omega,\lambda} \to (I^+_{\omega,\lambda})^n$ whenever $0 < n < \omega$". Moreover, they proved that $I^+_{\omega,\omega_1} \to (I^+_{\omega,\omega_1})^n$ for all n with $0 < n < \omega$ in the Cohen model for $2^{\aleph_0} = \aleph_2$.

Johnson [**14**] asked the following question: if κ is mildly λ-ineffable, where κ is a regular uncountable cardinal and λ a cardinal $> \kappa$, is it the case that $I_{\kappa,\lambda}$ is $(\lambda,2)$-distributive? Abe [**2**] answered the question in the negative by showing that if κ is an uncountable strongly compact cardinal and λ a strong limit cardinal $> \kappa$ of cofinality $< \kappa$, then (a) $I_{\kappa,\lambda}$ is not $(\lambda,2)$-distributive, and (b) $I^+_{\kappa,\lambda} \not\to (I^+_{\kappa,\lambda})^2$. This led him to ask whether the following are theorems in ZFC: (1) $I_{\kappa,\lambda}$ is not $(\lambda,2)$-distributive for any regular uncountable cardinal κ and cardinal $\lambda > \kappa$. (2) $I^+_{\kappa,\lambda} \not\to (I^+_{\kappa,\lambda})^2$ for any κ and λ as in (1). Shioya [**32**] provided a negative answer to Abe's questions by establishing the consistency, relative to a supercompact cardinal, of "there is a regular uncountable cardinal κ such that $I^+_{\kappa,\kappa^+} \to (I^+_{\kappa,\kappa^+})^n$ for all n with $0 < n < \omega$ (and therefore I_{κ,κ^+} is $(\kappa^+,2)$-distributive)".

Concerning another combinatorial aspect of $P_\kappa(\lambda)$, let $S_\kappa(\lambda)$ assert the following: For every function $g : P_\kappa(\lambda) \to P_\kappa(\lambda)$, there is $A \in I^+_{\kappa,\lambda}$ such that for every $B \subseteq A$ with B in $I_{\kappa,\lambda}$, $g``B$ is in $I_{\kappa,\lambda}$. Strengthening a result of Galvin (see [**38**]), Fleissner (see [**38**]) established that if $D(\omega)^\lambda$ is not the union of λ nowhere dense sets, where λ is an uncountable cardinal and $D(\omega)$ denotes the discrete topological space of cardinality $\aleph_0$, then $S_\omega(\lambda)$ holds. Zwicker [**38**] wondered whether these results of Galvin and Fleissner could be extended to show $S_\kappa(\lambda)$ consistent for regular uncountable κ.

The results of Galvin and Jech-Shelah were revisited in [**25**] where it was shown that if λ is an uncountable cardinal such that $\lambda < \mathbf{cov}(\mathbf{M}_{\omega,\lambda})$, then $\mathfrak{p}_{\omega,\lambda} \geq \lambda^+$ and $I_{\omega,\lambda}$ is weakly selective (and hence $I^+_{\omega,\lambda} \to (I^+_{\omega,\lambda})^n$ whenever $0 < n < \omega$). The present paper is a continuation of [**25**]. Its purpose is to generalize results of [**25**] to the case of an uncountable κ. Specifically we prove the following (focusing for simplicity on the case $\lambda = \kappa^+$):

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Proposition 1. *Suppose that κ is a regular infinite cardinal and* $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa^+}) > \kappa^+$. *Then*:

(i) I_{κ,κ^+} *is weakly selective.*
(ii) $\mathfrak{p}_{\kappa,\kappa^+} > \kappa^+$.
(iii) *For every infinite cardinal* $\theta \le \kappa$, $\kappa \to (\kappa,\theta)^2$ *if and only if* $I^+_{\kappa,\kappa^+} \to (I^+_{\kappa,\kappa^+},\theta)^2$.
(iv) κ *is* κ^+*-mildly ineffable if and only if* $I^+_{\kappa,\kappa^+} \to (I^+_{\kappa,\kappa^+})^n$ *for all* n *with* $0 < n < \omega$.
(v) *If* $\tau^{\aleph_0} < \kappa$ *for every cardinal* τ *with* $2 \le \tau < \kappa$, *then*

$$(NS_{\kappa,\kappa^+}|A)^+ \to \big(I^+_{\kappa,\kappa^+},\omega \oplus 1\big)^2,$$

where

$$A = \big\{a \in P_\kappa(\kappa^+) : cf\big(\cup(a \cap \kappa)\big) = cf(\cup a) = \omega\big\}.$$

(vi) *Any two cofinal subsets of* $P_\kappa(\kappa^+)$ *have isomorphic cofinal subsets.*
(vii) $S_\kappa(\kappa^+)$ *holds. In fact, given* $g : P_\kappa(\kappa^+) \to P_\kappa(\kappa^+)$ *and* $A \in I^+_{\kappa,\kappa^+}$, *there is* $D \in I^+_{\kappa,\kappa^+} \cap P(A)$ *such that* $g``B \in I_{\kappa,\lambda}$ *for all* $B \in I_{\kappa,\lambda} \cap P(D)$.

Proposition 2. *It is consistent, relative to a supercompact cardinal, that for the least uncountable measurable cardinal* κ, $I^+_{\kappa,\kappa^+} \to (I^+_{\kappa,\kappa^+})^n$ *for all* n *with* $0 < n < \omega$.

We also prove the following:

Proposition 3. *Suppose that κ is a regular uncountable cardinal and* $\mathfrak{d}_\kappa = \kappa^+$. *Then*:

(i) $\{A : \mathfrak{p}_{I_{\kappa,\lambda}|A} \ge \kappa^+\}$ *is not dense in* $(I^+_{\kappa,\kappa^+},\subseteq)$.
(ii) $I^+_{\kappa,\kappa^+} \not\to (I^+_{\kappa,\kappa^+},\omega_1)^2$.
(iii) $(I_{\kappa,\kappa^+}|C)^+ \to (I^+_{\kappa,\kappa^+},\omega+1)^2$ *for some* $C \in I^+_{\kappa,\kappa^+}$.

We mention that Galvin (see [**38**]) showed that if κ is a regular infinite cardinal such that $\mathfrak{d}_\kappa = \kappa^+$, then $S_\kappa(\kappa^+)$ fails.

The structure of this paper is as follows. Section 1 contains standard definitions concerning ideals on $P_\kappa(\lambda)$. Section 2 is devoted to the notion of a cofinal Kurepa family on $P_\kappa(\lambda)$ (the existence of such a family is hypothesized in many results of the paper). Section 3 reviews a number of facts concerning the covering number $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda})$ and the uniformity number $\mathbf{non}(\mathbf{M}_{\kappa,\lambda})$. In Section 4 we give sufficient conditions for an ideal H on $P_\kappa(\lambda)$ to be weakly selective and verify $\mathfrak{p}_H > \lambda^{<\kappa}$. Section 5 deals with the partition property $H^+ \to (H^+,\theta)^2$, Section 6 with a $P_\kappa(\lambda)$ version of the topological partition relation $\kappa \to (\kappa, \mathrm{top}\ \omega+1)^2$, and Section 7 with $H^+ \to (H^+)^n$. Section 8 investigates the assertion $S_\kappa(\lambda)$ and the existence of isomorphisms between cofinal sets.

We do not know whether the two assumptions (namely that $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$ and there exists a cofinal Kurepa family on $P_\kappa(\lambda)$) made in Sections 4–7 to derive properties of $I_{\kappa,\lambda}$ are necessary. Some negative results are presented in the final section of the paper.

1. Ideals.

Throughout the paper κ and λ will denote, respectively, a regular infinite cardinal and a cardinal $> \kappa$.

In this section we review various definitions concerning ideals on $P_\kappa(\lambda)$.

For a set A and a cardinal $\mu, P_\mu(A) = \{a \subseteq A : |a| < \mu\}$.

An *ideal* on a set S is a collection H of subsets of S such that: (i) $S \notin H$, (ii) $P(A) \subseteq H$ for all $A \in H$, (iii) $A \cup B \in H$ for all $A \in H$, and (iv) $\{s\} \in H$ for every $s \in S$.

$\mathrm{cof}(H)$ is the least cardinality of any $X \subseteq H$ such that $H = \bigcup_{A \in X} P(A)$.

Let $H^+ = P(S) - H$ and $H^* = \{A \subseteq S : S - A \in H\}$.

For $A \in H^+, H|A = \{B \subseteq S : B \cap A \in H\}$. It is readily seen that $H|A$ is an ideal on S and $\mathrm{cof}(H|A) \leq \mathrm{cof}(H)$.

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H is *κ-complete* if $\cup Y \in H$ for every $Y \in P_\kappa(H)$.

If H is κ-complete, $\overline{\mathrm{cof}}(H)$ is the least cardinality of any $X \subseteq H$ such that for every $A \in H$, there is $x \in P_\kappa(X)$ with $A \subseteq \cup x$.

NS_κ (respectively, $NS_{\kappa,\lambda}$) denotes the nonstationary ideal on κ (respectively, $P_\kappa(\lambda)$). That is, NS_κ (respectively, $NS_{\kappa,\lambda}$) is the collection of all subsets B of κ (respectively, $P_\kappa(\lambda)$) such that $\{\gamma \in B : \forall \alpha < \gamma\ (f(\alpha) < \gamma)\} \subseteq \{0\}$ (respectively, $\{a \in B : \forall \alpha, \beta \in a (f(\alpha, \beta) \subseteq a)\} \subseteq \{\phi\}$) for some $f : \kappa \to \kappa$ (respectively, $f : \lambda \times \lambda \to P_\kappa(\lambda)$).

Given two cardinals ρ and μ with $\omega \leq \rho \leq \mu$, $I_{\rho,\mu}$ is the set of all $A \subseteq P_\rho(\mu)$ such that $\{b \in A : a \subseteq b\} = \phi$ for some $a \in P_\rho(\mu)$. It is simple to see that $I_{\rho,\mu}$ is an ideal on $P_\rho(\mu)$. $u(\rho, \mu)$ denotes the least cardinality of any $A \in I^+_{\rho,\mu}$.

The following lists some elementary properties (see e.g. [**10**] and [**23**]):

Proposition 1.1.

- (i) $u(\kappa, \lambda) \geq \lambda$.
- (ii) $cf(u(\kappa, \lambda)) \geq \kappa$.
- (iii) $\lambda^{<\kappa} = \max\{2^{<\kappa}, u(\kappa, \lambda)\} = u(\kappa, \lambda^{<\kappa})$.
- (iv) $u(\kappa, \kappa^{+n}) = \kappa^{+n}$ *whenever* $0 < n < \omega$.
- (v) $u(\kappa, \lambda) \leq u(\kappa^+, \lambda)$.
- (vi) $u(\kappa, \lambda) = \mathrm{cof}(I_{\kappa,\lambda}|A)$ *for every* $A \in I^+_{\kappa,\lambda}$.

An ideal H on $P_\kappa(\lambda)$ is fine if $I_{\kappa,\lambda} \subseteq H$.

We adopt the convention that the phrase "ideal on $P_\kappa(\lambda)$" means "κ-complete fine ideal on $P_\kappa(\lambda)$".

For $A \subseteq P_\kappa(\lambda)$ and an ordinal $\delta \leq \kappa, [A]^\delta$ denotes the set of all $B \subseteq A$ such that $(B, \subsetneq)$ has ordertype δ. Abusing notation, we write

$$[A]^\delta = \{(a_\alpha : \alpha < \delta) : a_0, a_1, \ldots \in A \text{ and } a_0 \subsetneq a_1 \subsetneq \ldots\}.$$

2. Cofinal Kurepa families.

This section is concerned with existence of cofinal Kurepa families. For more on the subject, see [**34**], [**35**] and [**36**].

Definition. $\mathfrak{K}_{\kappa,\lambda}$ is the set of all $A \in I^+_{\kappa,\lambda}$ such that $|A \cap P(a)| < \kappa$ for every $a \in A$.

First let us establish two easy facts concerning members of $\mathfrak{K}_{\kappa,\lambda}$.

Lemma 2.1. *Suppose $A \in \mathfrak{K}_{\kappa,\lambda}$. Then $|A \cap P(b)| < \kappa$ for every $b \in P_\kappa(\lambda)$.*

Proof. Given $b \in P_\kappa(\lambda)$, pick $a \in A$ with $b \subseteq a$. Then $A \cap P(b) \subseteq A \cap P(a)$. □

Proposition 2.2. *Suppose $A \in \mathfrak{K}_{\kappa,\lambda}$. Then $|A| = u(\kappa,\lambda)$.*

Proof. Pick $B \in I^+_{\kappa,\lambda}$ with $|B| = u(\kappa,\lambda)$. Then $A = \bigcup_{b\in B}(A \cap P(b))$ and therefore by Lemma 2.1 $|A| \le \kappa \cdot |B| = u(\kappa,\lambda)$. □

If $\mathfrak{K}_{\kappa,\lambda}$ is not empty, then it is a large (i.e. dense) subset of $I^+_{\kappa,\lambda}$.

Proposition 2.3. *Suppose $\mathfrak{K}_{\kappa,\lambda} \neq \phi$, and let J be an ideal on $P_\kappa(\lambda)$ with* $\mathrm{cof}(J) =$ *$u(\kappa,\lambda)$. Then $J^+ \cap \mathfrak{K}_{\kappa,\lambda}$ is dense in $(J^+, \subseteq)$.*

Proof. Fix $A \in \mathfrak{K}_{\kappa,\lambda}$ and $B \in J^+$. Select $C_a \in J$ for $a \in A$ so that $J = \bigcup_{a\in A} P(C_a)$. Define $f : A \to B$ so that $a \subseteq f(a)$ and $f(a) \notin C_a$ for every $a \in A$. Obviously, $\mathrm{ran}(f) \in J^+ \cap P(B)$. For $a \in A$,

$$\{c \in A : f(c) \subseteq f(a)\} \subseteq A \cap P\big(f(a)\big)$$

and consequently by Lemma 2.1, $|\mathrm{ran}(f) \cap P(f(a))| < \kappa$. □

Corollary 2.4.

i) *Suppose $\mathfrak{K}_{\kappa,\lambda} \neq \phi$. Then $\mathfrak{K}_{\kappa,\lambda}$ is dense in $(I^+_{\kappa,\lambda}, \subseteq)$.*
ii) *Suppose $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and λ is a strong limit cardinal of cofinality less than κ. Then $NS^+_{\kappa,\lambda} \cap \mathfrak{K}_{\kappa,\lambda}$ is dense in $(NS^+_{\kappa,\lambda}, \subseteq)$.*

Let us now turn to the question whether $\mathfrak{K}_{\kappa,\lambda} \neq \phi$. The following observation is immediate.

Proposition 2.5. *If κ is inaccessible, then $\mathfrak{K}_{\kappa,\lambda} = I^+_{\kappa,\lambda}$.*

Proposition 2.6 ([**24**]). *The following are equivalent*:

(i) $\mathfrak{K}_{\kappa,\lambda} \neq \phi$.
(ii) *There is $D \in I^+_{\kappa,u(\kappa,\lambda)}$ such that $\overline{\mathrm{cof}}(I_{\kappa,u(\kappa,\lambda)}|D) \le \lambda$.*
(iii) *There is $A \subseteq P_\kappa(\lambda)$ such that $|A| = u(\kappa,\lambda)$ and $|A \cap P(b)| < \kappa$ for every $b \in P_\kappa(\lambda)$.*

Corollary 2.7. *If $u(\kappa,\lambda) = \lambda$, then $\mathfrak{K}_{\kappa,\lambda} \neq \phi$.*

It follows that $\mathfrak{K}_{\kappa,\lambda^{<\kappa}} \neq \phi$, since $(\lambda^{<\kappa})^{<\kappa} = \lambda^{<\kappa}$.

COROLLARY 2.8. *Suppose $\mathfrak{K}_{\kappa,\lambda} \neq \phi$. Then $\mathfrak{K}_{\kappa,\nu} \neq \phi$ for every cardinal ν with $\lambda \leq \nu \leq u(\kappa,\lambda)$.*

PROOF. If $\mathfrak{K}_{\kappa,\lambda} \neq \phi$, then $u(\kappa, u(\kappa,\lambda)) = u(\kappa,\lambda)$ by Proposition 2.6, and so

$$u(\kappa,\lambda) \leq u(\kappa,\nu) \leq u(\kappa, u(\kappa,\lambda)) \leq u(\kappa,\lambda)$$

for every cardinal ν with $\lambda \leq \nu \leq u(\kappa,\lambda)$. □

PROPOSITION 2.9 ([**24**]). *Suppose that $\kappa < \lambda$, $u(\kappa,\lambda) = u(\kappa^+,\lambda)$ and $\mathfrak{K}_{\kappa,\lambda} \neq \phi$. Then $\mathfrak{K}_{\kappa^+,\lambda} \neq \phi$.*

PROPOSITION 2.10 ([**24**]). *Suppose that $cf(\lambda) < \kappa$, $\tau^{cf(\lambda)} < \kappa$ for every infinite cardinal $\tau < \kappa$, and $u(\kappa,\lambda) \leq \lambda^{cf(\lambda)}$. Then $\mathfrak{K}_{\kappa,\lambda} \neq \phi$.*

Note that if $cf(\lambda) < \kappa$ and $u(\kappa,\mu) < \lambda$ for every cardinal μ with $\kappa < \mu < \lambda$, then $u(\kappa,\lambda) \leq \lambda^{cf(\lambda)}$.

PROPOSITION 2.11 ([**24**]). *If μ is a cardinal with $cf(\mu) < \kappa < \mu$, and $\kappa \to [\kappa]^2_{cf(\mu),<cf(\mu)}$, then $\mathfrak{K}_{\kappa,2^{<\mu}} \neq \phi$.*

Hence, by a result of Solovay [**33**], if κ bears an ω_1-saturated κ-complete ideal, then $\mathfrak{K}_{\kappa,2^{<\mu}} \neq \phi$ for every cardinal μ with $cf(\mu) < \kappa < \mu$.

DEFINITION. $\mathrm{cov}(\lambda,\lambda,\kappa,2)$ is the least cardinality of any $X \subseteq P_\lambda(\lambda)$ such that for every $a \in P_\kappa(\lambda)$, there is $b \in X$ with $a \subseteq b$.

It is readily checked that

$$\lambda^{<\kappa} = \max\left\{2^{<\kappa}, \mathrm{cov}(\lambda,\lambda,\kappa,2), \bigcup_{\kappa \leq \nu < \lambda} u(\kappa,\nu)\right\}.$$

The following is due to Shelah (see [**24**]).

PROPOSITION 2.12. *Suppose that $cf(\lambda) < \kappa$ and $u(\lambda^+, u(\kappa,\lambda)) < \mathrm{cov}(\lambda,\lambda,\kappa,2)$. Then $\mathfrak{K}_{\kappa,\lambda} \neq \phi$.*

The following is due to Todorcevic [**37**] and Cummings, Foreman and Magidor [**8**].

PROPOSITION 2.13. *Suppose $cf(\lambda) < \kappa$, $u(\kappa,\lambda) = \lambda^+$ and either $\Box^*_\lambda$ holds or there is a very good scale on λ. Then $\mathfrak{K}_{\kappa,\lambda} \neq \phi$.*

Todorcevic (see [**24**]) established that $\omega_{\omega+1} \to [\omega_1]^2_{\omega_\omega,<\omega_1}$ implies that $\mathfrak{K}_{\omega_1,\omega_{\omega+1}} = \phi$. Now we prove two generalizations of this result. The following key lemma is due to Todorcevic (see [**24**]).

LEMMA 2.14. *Suppose ρ and χ are two cardinals such that $\kappa \leq \rho \leq \chi$ and $\chi \to [\kappa]^2_{\rho,<\kappa}$, and let $B \subseteq P(\rho)$ with $|B| = \chi$. Then there is $z \in P_\kappa(\rho)$ such that $|\{b \cap z : b \in B\}| \geq \kappa$.*

Lemma 2.15. *Suppose $A \in \mathfrak{K}_{\kappa,\lambda}, T \subseteq P_\kappa(\lambda)$, and τ is an infinite cardinal with $\tau < \kappa$. Suppose further there is $\varphi : A \to \{x \subseteq T : |x| = \tau\}$ such that $a \subseteq \cup y$ for each $a \in A$ and each $y \subseteq \varphi(a)$ with $|y| = \tau$. Then there is $B \subseteq P_{\tau^+}(T)$ such that $|B| = u(\kappa, \lambda)$ and $|\{b \in B : |b \cap z| = \tau\}| < \kappa$ for every $z \in P_\kappa(T)$.*

Proof. Set $B = \mathrm{ran}(\varphi)$. Then $u(\kappa, \lambda) \leq |B|$ since $\{\cup x : x \in B\} \in I^+_{\kappa,\lambda}$. Conversely, $|B| \leq u(\kappa, \lambda)$ by Proposition 2.2. It remains to observe that for every $z \in P_\kappa(T)$,

$$\{a \in A : |\varphi(a) \cap z| = \tau\} \subseteq \{a \in A : a \subseteq \cup z\}. \qquad \square$$

Lemma 2.16. *Suppose $cf(\lambda) < \kappa$ and $\mathfrak{K}_{\kappa,\lambda} \neq \phi$. Then setting $\rho = \bigcup_{\kappa<\sigma<\lambda} u(\kappa, \sigma)$ and $\tau = cf(\lambda)$, there is $B \subseteq P_{\tau^+}(\rho)$ such that $|B| = u(\kappa, \lambda)$ and $|\{b \in B : |b \cap z| = \tau\}| < \kappa$ for every $z \in P_\kappa(\rho)$.*

Proof. Select a strictly increasing sequence $< \lambda_\xi : \xi < \tau >$ of cardinals greater than κ so that $\lambda = \bigcup_{\xi<\tau} \lambda_\xi$. For $\xi < \tau$, choose $T_\xi \in I^+_{\kappa,\lambda_\xi}$ with $|T_\xi| = u(\kappa, \lambda_\xi)$. Set $T = \bigcup_{\xi<\tau} T_\xi$. Note that $|T| = \rho$. Fix $A \in \mathfrak{K}_{\kappa,\lambda}$ with $A \subseteq \{a \in P_\kappa(\lambda) : \cup a = \lambda\}$. For $a \in A$ and $\xi < cf(\lambda)$, pick $a_\xi \in T_\xi$ so that $a \cap \lambda_\xi \subseteq a_\xi$. Define $\varphi : A \to P(T)$ by $\varphi(a) = \{a_\xi : \xi < \tau\}$. Now apply Lemma 2.15. $\square$

Proposition 2.17. *Suppose $cf(\lambda) < \kappa$, $u(\kappa, \lambda) \to [\kappa]^2_{\rho,<\kappa}$, where $\rho = \bigcup_{\kappa<\sigma<\lambda} u(\kappa, \sigma)$, and $\mu^{<cf(\lambda)} < \kappa$ for every cardinal $\mu < \kappa$. Then $\mathfrak{K}_{\kappa,\lambda} = \phi$.*

Proof. By Lemmas 2.14 and 2.16. $\square$

In particular, if $\lambda = \kappa^{+\omega}$ and $u(\kappa, \lambda) \to [\kappa]^2_{\lambda,<\kappa}$, then $\mathfrak{K}_{\kappa,\lambda} = \phi$.

Lemma 2.18. *Suppose $\kappa = \nu^+$ and $\mathfrak{K}_{\kappa,\lambda} \neq \phi$. Then setting $\rho = u(\nu, \lambda)$ and $\tau = cf(\nu)$, there is $B \subseteq P_{\tau^+}(\rho)$ such that $|B| = u(\kappa, \lambda)$ and $|\{b \in B : |b \cap z| = \tau\}| < \kappa$ for every $z \in P_\kappa(\rho)$.*

Proof. Fix $T \in I^+_{\nu,\lambda}$ with $|T| = \rho$, and $A \in \mathfrak{K}_{\kappa,\lambda}$ with $A \subseteq \{a \in P_\kappa(\lambda) : |a| = \nu\}$. Pick a strictly increasing sequence $< \nu_\xi : \xi < \tau >$ of nonzero ordinals so that $\nu = \bigcup_{\xi<\tau} \nu_\xi$. For $a \in A$, choose a bijection $i_a : \nu \to a$ and select $a_\xi \in T$ for $\xi < \tau$ so that $i``_a\, \nu_\xi \subseteq a_\xi$. Define $\varphi : A \to P(T)$ by $\varphi(a) = \{a_\xi : \xi < \tau\}$. Now apply Lemma 2.15. $\square$

Proposition 2.19. *Suppose $\kappa = \nu^+$, $u(\kappa, \lambda) \to [\kappa]^2_{u(\nu,\lambda),<\kappa}$ and $\nu^{<cf(\nu)} = \nu$. Then $\mathfrak{K}_{\kappa,\lambda} = \phi$.*

Proof. By Lemmas 2.14 and 2.18. $\square$

In particular, if $\kappa = \omega_1$ and $u(\kappa, \lambda) \to [\kappa]^2_{\lambda,<\kappa}$, then $\mathfrak{K}_{\kappa,\lambda} = \phi$.

To conclude this section, we establish two consequences of "$\mathfrak{K}_{\kappa,\lambda} \neq \phi$". The proof uses ideas from [**36**].

Proposition 2.20. *Suppose κ is a successor cardinal and $\mathfrak{K}_{\kappa,\lambda} \neq \phi$, and let H be any ideal on $P_\kappa(\lambda)$. Then every $D \in H^+$ can be partitioned into $u(\kappa, \lambda)$ disjoint members of H^+.*

Proof. Set $\kappa = \nu^+$ and fix $A \in \mathfrak{K}_{\kappa,\lambda}$ and $D \in H^+$. For $b \in P_\kappa(\lambda)$ pick a one-to-one

$f_b : A \cap P(b) \to \nu$. Put

$$D_a^\alpha = \{b \in D : a \subseteq b \text{ and } f_b(a) = \alpha\}$$

for $a \in A$ and $\alpha \in \nu$. Now define $g \in {}^A\nu$ so that $D_a^{g(a)} \in H^+$ for each $a \in A$. Select $B \in I^+_{\kappa,\lambda} \cap P(A)$ and $\beta \in \nu$ so that g takes the constant value β on B. Then $< D_a^\beta : a \in B >$ is a sequence of disjoint members of $H^+ \cap P(D)$. $\square$

DEFINITION. A partially ordered set $(Q, <)$ is κ-directed if for every $x \in P_\kappa(Q)$, there is $r \in Q$ such that $q \leq r$ for all $q \in x$.

PROPOSITION 2.21. *Suppose that $(Q, <)$ is a κ-directed partially ordered set such that $|Q| = \lambda$ and $|\{r \in Q : r < q\}| < \kappa$ for all $q \in Q$. Then there is $A \in \mathfrak{K}_{\kappa,\lambda}$ such that $(Q, <)$ and $(A, \subsetneq)$ are isomorphic.*

PROOF. Set $Q = \{q_\alpha : \alpha < \lambda\}$ and define a one-to-one $g : Q \to P_\kappa(\lambda)$ by $g(q_\alpha) = \{\beta \in \lambda : q_\beta \leq q_\alpha\}$. It is simple to check that $\mathrm{ran}(g) \in I^+_{\kappa,\lambda}$ and

$$q_\alpha < q_\gamma \leftrightarrow g(q_\alpha) \subsetneq g(q_\gamma)$$

for all $\alpha, \gamma < \lambda$. $\square$

Thus, if $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $B \in I^+_{\kappa,\lambda}$, then there are $C \in I^+_{\kappa,\lambda} \cap P(B)$ and $A \in \mathfrak{K}_{\kappa,u(\kappa,\lambda)}$ such that $(C, \subsetneq)$ and $(A, \subsetneq)$ are isomorphic.

3. Covering for category.

DEFINITION. For a set A,

$$Fn(A, 2, \kappa) = \cup\{{}^a2 : a \in P_\kappa(A)\}.$$

$Fn(A, 2, \kappa)$ is ordered by : $p \leq q$ if and only if $q \subseteq p$.

DEFINITION. Suppose ρ is a cardinal $\geq \kappa$.

${}^\rho2$ is endowed with the topology obtained by taking as basic open sets ϕ and O_s^ρ for $s \in Fn(\rho, 2, \kappa)$, where $O_s^\rho = \{f \in {}^\rho2 : s \subsetneq f\}$.

$\mathbf{M}_{\kappa,\rho}$ is the set of all $W \subseteq {}^\rho2$ such that $W \cap (\cap X) = \phi$ for some collection X of dense open subsets of ${}^\rho2$ with $0 < |X| \leq \kappa$.

$\mathbf{cov}(\mathbf{M}_{\kappa,\rho})$ is the least cardinality of any $Y \subseteq \mathbf{M}_{\kappa,\rho}$ with ${}^\rho2 = \cup Y$.

$\mathbf{non}(\mathbf{M}_{\kappa,\rho})$ is the least cardinality of any $W \subseteq {}^\rho2$ with $W \notin \mathbf{M}_{\kappa,\rho}$.

In this section we review some well-known facts concerning the cardinal coefficients $\mathbf{cov}(\mathbf{M}_{\kappa,\rho})$ and $\mathbf{non}(\mathbf{M}_{\kappa,\rho})$.

DEFINITION. For a set $A, \mathscr{A}_\kappa^A$ is the collection of all maximal antichains in $Fn(A, 2, \kappa)$.

PROPOSITION 3.1 ([**17**], [**28**]). *Let ρ be a cardinal $\geq \kappa$. Then:*

(i) $\mathbf{cov}(\mathbf{M}_{\kappa,\rho})$ *is the least cardinality of any nonempty family of dense open subsets of* ${}^{\rho}2$ *with empty intersection.*
(ii) $\mathbf{cov}(\mathbf{M}_{\kappa,\rho})$ *is the least cardinality of any collection* Z *of dense subsets of* $Fn(\rho, 2, \kappa)$ *(or of members of* $\mathscr{A}^{\rho}_{\kappa}$*) such that for every filter* $G \subseteq Fn(\rho, 2, \kappa)$*, there is* $D \in Z$ *with* $D \cap G = \phi$*.*

Proposition 3.2 ([**17**], [**28**]). *Suppose that* ρ *and* μ *are two cardinals such that* $\kappa \leq \mu \leq \rho$*. Then* $\mathbf{cov}(\mathbf{M}_{\kappa,\mu}) \geq \mathbf{cov}(\mathbf{M}_{\kappa,\rho})$ *and* $\mathbf{non}(\mathbf{M}_{\kappa,\mu}) \leq \mathbf{non}(\mathbf{M}_{\kappa,\rho})$*.*

Definition. $\mathfrak{b}_\kappa$ (respectively, $\mathfrak{d}_\kappa$) is the least cardinality of any $F \subseteq {}^{\kappa}\kappa$ such that for every $g \in {}^{\kappa}\kappa$, there is $f \in F$ with $|\{\alpha \in \kappa : f(\alpha) \geq g(\alpha)\}| = \kappa$ (respectively, $|\{\alpha \in \kappa : g(\alpha) > f(\alpha)\}| < \kappa$).

Proposition 3.3 ([**30**]). $\mathbf{non}(\mathbf{M}_{\kappa,\kappa}) \geq \mathfrak{b}_\kappa$.

Proof. Fix $W \subseteq {}^{\kappa}2$ with $W \notin \mathbf{M}_{\kappa,\kappa}$. For $t \in \bigcup_{\delta \leq \kappa} {}^{\delta}2$, define a partial function $\widetilde{t}$ from κ to κ by: $\widetilde{t}(\alpha) = \gamma$ if and only if γ is the least $\xi \in \mathrm{dom}(t)$ such that $(t \restriction \xi)^{-1}(\{1\})$ has ordertype α.

Let $g \in {}^{\kappa}\kappa$. For $\beta < \kappa$, stipulate that S_β is the set of all $s \in \bigcup_{\gamma \in \kappa} {}^{\gamma}2$ such that there is $\alpha \geq \beta$ with $\alpha \in \mathrm{dom}(\widetilde{s})$ and $\widetilde{s}(\alpha) \geq g(\alpha)$. It is simple to check that $U_\beta = \bigcup_{s \in S_\beta} O^{\kappa}_s$ is dense. Hence there is $f \in W$ such that $f \in \bigcup_{\beta < \kappa} U_\beta$. Obviously, $|\{\alpha \in \kappa : \widetilde{f}(\alpha) \geq g(\alpha)\}| = \kappa$. □

It is straightforward to check that $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \mathfrak{d}_\kappa$.

Proposition 3.4 ([**17**], [**28**]). $\mathbf{cov}(\mathbf{M}_{\kappa,\rho}) \geq \kappa^+$ *for every cardinal* $\rho \geq \kappa$*.*

Proposition 3.5 ([**17**]). *Suppose* $2^{<\kappa} > \kappa$*. Then* $\mathbf{cov}(\mathbf{M}_{\kappa,\rho}) = \kappa^+$ *for every* $\rho \geq \kappa$*.*

Proof. By Propositions 3.2 and 3.4 it suffices to show that $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \kappa^+$. Fix a cardinal $\tau < \kappa$ with $2^\tau > \kappa$. Let $\varphi : \kappa \times \tau \to \kappa$ be a bijection. For $y \in {}^{\tau}2$, define an open subset W_y of ${}^{\kappa}2$ by: $f \in W_y$ if and only if there is $\gamma < \kappa$ such that $f\big(\varphi(\gamma, \xi)\big) = y(\xi)$ for all $\xi < \tau$. Then $\bigcap_{y \in Y} W_y = \phi$ for any $Y \subseteq {}^{\tau}2$ with $|Y| > \kappa$. □

Corollary 3.6. *Suppose that* $\mathbf{cov}(\mathbf{M}_{\kappa,\rho}) > \kappa^+$ *for some cardinal* $\rho \geq \kappa$*. Then* $u(\kappa, \mu) = \mu^{<\kappa}$ *for every cardinal* $\mu \geq \kappa$*.*

Proof. By Propositions 1.1 and 3.5. □

Lemma 3.7. *Suppose* ρ *is a cardinal* $\geq \kappa$*. Then*:

(i) $|A| \leq 2^{<\kappa}$ *for every* $A \in \mathscr{A}^{\rho}_{\kappa}$*.*
(ii) *If* $X \subseteq \mathscr{A}^{\rho}_{\kappa}$ *is such that* $2^{<\kappa} \leq |X| < \rho$*, then* $X \subseteq \mathscr{A}^{B}_{\kappa}$ *for some* $B \subseteq \rho$ *with* $|B| = |X|$*.*

Proof.

(i) See Lemma VII.6.10 in [**15**].
(ii) Use (i). □

Proposition 3.8 ([**17**]). *$\mathbf{cov}(\mathbf{M}_{\kappa,\rho}) = \mathbf{cov}(\mathbf{M}_{\kappa,2^\kappa})$ for every cardinal $\rho \geq \mathbf{cov}(\mathbf{M}_{\kappa,2^\kappa})$.*

Proof. Argue as for Proposition 6.4 of [**25**]. □

Note that by Proposition 3.8, $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$ if and only if $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,2^\kappa})$.

Definition. Given a cardinal $\rho \geq \kappa$, $\theta_{\kappa,\rho}$ is the least cardinality of any $X \subseteq \{A \subseteq \rho : |A| = \kappa\}$ such that for every $B \subseteq \rho$ with $|B| = \rho$, there is $A \in X$ with $A \subseteq B$.

Proposition 3.9 ([**28**]). *$\mathbf{cov}(\mathbf{M}_{\kappa,\rho}) \leq \theta_{\kappa,\rho}$ for every cardinal $\rho \geq \kappa$.*

Proposition 3.10. *Suppose that ρ is a cardinal $> \kappa$ and $V \models 2^{<\kappa} = \kappa$. Then setting $P = Fn(\rho, 2, \kappa)$:*

(i) *$(2^\mu)^{V^P} \leq (\rho^\mu)^V$ for every cardinal $\mu \geq \kappa$.*
(ii) *$V^P \models \mathbf{non}(\mathbf{M}_{\kappa,\rho}) = \kappa^+$.*
(iii) ([**17**], [**28**]) *$V^P \models \mathbf{cov}(\mathbf{M}_{\kappa,\rho}) \geq \rho$.*
(iv) ([**17**], [**28**]) *If $cf(\rho) = \kappa$ and $V \models$ "$\nu^{\kappa^+} < \rho$ for every cardinal ν with $2 \leq \nu < \rho$", then $V^P \models \theta_{\kappa,\kappa^+} = \rho$.*
(v) ([**17**]) *If $cf(\rho) < \kappa$ and $V \models$ GCH, then $V^P \models \mathbf{cov}(\mathbf{M}_{\kappa,\rho}) = \rho^+$.*

Proof.

(i) See Theorem 3.15 in [**5**].
(ii) In V, select a bijection $j : \kappa^+ \times \rho \to \rho$. Now let G be $Fn(\rho, 2, \kappa)$-generic over V. For $\alpha < \kappa^+$, define $g_\alpha \in {}^\rho 2$ by $g_\alpha(\beta) = (\cup G)(j(\alpha, \beta))$. It is readily seen that $\{g_\alpha : \alpha < \kappa^+\} \notin \mathbf{M}_{\kappa,\rho}$. □

The following two results provide models for $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa^+}) > \kappa^+$ at a large cardinal. The first one is due to Silver (see Exercise VIII.I.10 in [**15**]).

Proposition 3.11. *Suppose that $V \models$ "GCH $+ \kappa$ is weakly compact". Then there is a partially ordered set P in V such that $V^P \models$ "κ is weakly compact and $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa^+}) > \kappa^+$".*

Proposition 2 will follow from Corollary 7.5 and the following result.

Proposition 3.12 ([**4**]). *Suppose that $V \models$ "κ is supercompact". Then there is a generic extension W of V such that for every cardinal $\rho \geq \kappa$ in W,*

$$W^P \models \text{“}\kappa \textit{ is both strongly compact and the least uncountable measurable cardinal”},$$

where $P = Fn(\rho, 2, \kappa)$.

4. Selectivity and pseudointersections.

Definition. An ideal H on $P_\kappa(\lambda)$ is weakly selective if given $A \in H^+$ and $B_a \in H$ for $a \in A$, there is $C \in H^+ \cap P(A)$ such that $b \notin B_a$ for every $(a, b) \in [C]^2$.

Definition. For an ideal H on $P_\kappa(\lambda)$, $\mathscr{Z}_H$ is the set of all $\mathscr{F} \subseteq H^+$ such that (a) $\cap X \in H^+$ for every $X \subseteq \mathscr{F}$ with $0 < |X| < \kappa$, and (b) for every $C \in H^+$, there is $A \in \mathscr{F}$ with $C - A \in H^+$.

$\mathfrak{p}_H$ is defined by: $\mathfrak{p}_H =$ the least cardinality of any member of $\mathscr{Z}_H$ if $\mathscr{Z}_H \neq \phi$, and $\mathfrak{p}_H = (2^{\lambda^{<\kappa}})^+$ otherwise.

For $H = I_{\kappa,\lambda}$, we set $\mathfrak{p}_H = \mathfrak{p}_{\kappa,\lambda}$.

Note that $\mathfrak{p}_{H|E} \geq \mathfrak{p}_H$ for every $E \in H^+$.

Lemma 4.1. *Suppose H is an ideal on $P_\kappa(\lambda)$ such that $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$. Then the following are equivalent*:

(i) *H is weakly selective and $\mathfrak{p}_H > u(\kappa, \lambda)$.*
(ii) *Given $A \in H^+$ and $S_a \subseteq A$ for $a \in A$ such that $\bigcap_{a \in x} S_a \in H^+$ for every $x \in P_\kappa(A) - \{\phi\}$, there is $C \in H^+ \cap P(A)$ such that $b \in S_a$ for every $(a, b) \in [C]^2$.*

Proof. (i) $\to$ (ii): Suppose (i) holds and $A \in H^+$ and $S_a \subsetneq a$ for $a \in A$ are such that $\bigcap_{a \in x} S_a \in H^+$ for every $x \in P_\kappa(A) - \{\phi\}$. Fix $B \in H^* \cap \mathfrak{K}_{\kappa,\lambda}$. By Proposition 2.2 $\mathfrak{p}_H > |A \cap B|$, so there is $C \in H^+$ such that $C - S_a \in H$ for every $a \in A \cap B$. Select $D \in H^+ \cap (A \cap B \cap C)$ so that $b \notin C - S_a$ for each $(a, b) \in [D]^2$. Then $b \in S_a$ whenever $(a, b) \in [D]^2$.

(ii) $\to$ (i): Suppose (ii) holds. Then given $A \in H^+$ and $B_a \in H$ for $a \in A$, there is $C \in H^+ \cap P(A)$ such that $b \in A - B_a$ for every $(a, b) \in [C]^2$. Hence H is weakly selective. To show that $\mathfrak{p}_H > u(\kappa, \lambda)$, let $\mathscr{F} \subseteq H^+$ be such that $|\mathscr{F}| \leq u(\kappa, \lambda)$ and $\cap X \in H^+$ for every $X \subseteq \mathscr{F}$ with $0 < |X| < \kappa$. Fix $D \in H^* \cap \mathfrak{K}_{\kappa,\lambda}$. By Proposition 2.2, there is an onto $j : D \to \mathscr{F}$. Set $S_a = D \cap (\bigcap_{c \in D \cap P(a)} j(c))$ for $a \in D$. Select $T \in H^+ \cap P(D)$ so that $b \in S_a$ for every $(a, b) \in [T]^2$. It is easy to see that $T - W \in H$ for every $W \in \mathscr{F}$. $\square$

For $H = I_{\kappa,\lambda}$, (ii) of Lemma 4.1 can be reformulated as follows:

Proposition 4.2. *The following are equivalent*:

(i) *Given $A \in I^+_{\kappa,\lambda}$ and $S_a \subseteq A$ for $a \in A$ such that $\bigcap_{a \in x} S_a \in I^+_{\kappa,\lambda}$ for every $x \in P_\kappa(A) - \{\phi\}$, there is $C \in I^+_{\kappa,\lambda} \cap P(A)$ such that $b \in S_a$ for every $(a, b) \in [C]^2$.*
(ii) *Given a cardinal $\mu \geq \lambda$ and $B \in I^+_{\kappa,\mu}$, there is $C \in I^+_{\kappa,\lambda}$ such that $[C]^2 \subseteq \{(c \cap \lambda, d \cap \lambda) : (c, d) \in [B]^2\}$.*

Proof. (i) $\to$ (ii): Suppose that (i) holds and let $B \in I^+_{\kappa,\mu}$, where μ is a cardinal with $\mu \geq \lambda$. Set $A = \{d \cap \lambda : d \in B\}$. Note that $A \in I^+_{\kappa,\lambda}$. Pick $g : A \to B$ so that $a = \lambda \cap g(a)$ for every $a \in A$. For $a \in A$, let S_a be the set of all $b \in A$ with the property that there is $d \in B$ such that $g(a) \subsetneq d$ and $b = d \cap \lambda$. It is simple to see that $\bigcap_{a \in x} S_a \in I^+_{\kappa,\lambda}$ for every $x \in P_\kappa(A) - \{\phi\}$. By our assumption there is $C \in I^+_{\kappa,\lambda} \cap P(A)$ such that $b \in S_a$ for every $(a, b) \in [C]^2$. Then for every $(a, b) \in [C]^2$, there is d such that $(g(a), d) \in [B]^2$ and $b = d \cap \lambda$.

(ii) $\to$ (i): Suppose that (ii) holds, and fix $A \in I^+_{\kappa,\lambda}$ and $S_a \subseteq A$ for $a \in A$ with $\{\bigcap_{a \in x} S_a : x \in P_\kappa(A) - \{\phi\}\} \subseteq I^+_{\kappa,\lambda}$. Now fix a cardinal μ with $\mu > \lambda$ and $\mu \geq |A|$. Select a one-to-one $j : A \to \mu - \lambda$. Let B be the set of all $d \in P_\kappa(\mu)$ such that (a) $d \cap \lambda \in A$, (b) $j(d \cap \lambda) \in d$, and (c) $d \cap \lambda \in S_a$ for all $a \in A$ such that $a \subsetneq d \cap \lambda$ and $j(a) \in d$.

Let us show that $B \in I^+_{\kappa,\mu}$. Fix $e \in P_\kappa(\mu)$. Pick $c \in A$ so that $e \cap \lambda \subsetneq c$ and $c \in S_a$ for all $a \in A$ such that $j(a) \in e$. Then setting $d = c \cup e \cup \{j(c)\}$, we have $d \in B$. By our assumption there is $C \in I^+_{\kappa,\lambda}$ with the property that for every $(a,b) \in [C]^2$, one can find $(c,d) \in [B]^2$ such that $a = c \cap \lambda$ and $b = d \cap \lambda$. Clearly, $C \subseteq A$. Moreover, $b \in S_a$ whenever $(a,b) \in [C]^2$. $\square$

Proposition 4.3. *Suppose H is an ideal on $P_\kappa(\lambda)$ such that $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$ and* $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. *Then H is weakly selective and $\mathfrak{p}_H > \lambda^{<\kappa}$.*

Proof. By Corollary 3.6, $\lambda^{<\kappa} = u(\kappa,\lambda)$ since $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \mathrm{cof}(H) > \kappa$. So it suffices to show that (ii) of Lemma 4.1 holds. Thus let $A \in H^+$ and $S_a \subseteq A$ for $a \in A$ with the property that $\bigcap_{a\in x} S_a \in H^+$ for every $x \in P_\kappa(A) - \{\phi\}$. Fix $Z \in H^* \cap \mathfrak{K}_{\kappa,\lambda}$ and $Y \subseteq H$ such that $|Y| = \mathrm{cof}(H)$ and $H = \bigcup_{B\in Y} P(B)$. For $B \in Y$, let $\mathscr{D}_B$ be the set of all $p \in Fn(A \cap Z, 2, \kappa)$ such that there is $b \in \mathrm{dom}(p)$ with the following properties:

(0) $b \notin B$.
(1) $\mathrm{dom}(p) = (A \cap Z) \cap P(b)$.
(2) $p(b) = 1$.
(3) $b \in S_a$ for every $a \in \mathrm{dom}(p)$ such that $a \neq b$ and $p(a) = 1$.

Let us show that $\mathscr{D}_B$ is dense. Thus fix $q \in Fn(A\cap Z, 2, \kappa)$. Pick $b \in (A\cap Z) - B$ so that (i) $a \subsetneq b$ for all $a \in \mathrm{dom}(q)$, and (ii) $b \in S_a$ for each $a \in \mathrm{dom}(q)$ with $q(a) = 1$. Define $p : (A \cap Z) \cap P(b) \to 2$ by:

(α) $p(b) = 1$.
(β) $p \upharpoonright \mathrm{dom}(q) = q$.
(γ) $p(c) = 0$ for every $c \in (A \cap Z) \cap P(b)$ such that $c \neq b$ and $c \notin \mathrm{dom}(q)$.

Obviously, $q \subseteq p$ and $p \in \mathscr{D}_B$.

Let $G \subseteq Fn(A \cap Z, 2, \kappa)$ be a filter such that $G \cap \mathscr{D}_B \neq \phi$ for every $B \in Y$. Pick $\varphi \in \prod_{B\in Y}(G \cap \mathscr{D}_B)$. For $B \in Y$, let $b_B \in A \cap Z$ be such that $\mathrm{dom}\big(\varphi(B)\big) = P(b_B)$. Set $C = \{b_B : B \in Y\}$. Then clearly $C \in H^+ \cap P(A)$. Now suppose $B_0, B_1 \in Y$ are such that $b_{B_0} \subsetneq b_{B_1}$. There is $r \in G$ such that $\varphi(B_0) \cup \varphi(B_1) \subseteq r$. Then

$$(\varphi(B_1))(b_{B_0}) = r(b_{B_0}) = (\varphi(B_0))(b_{B_0}) = 1$$

and consequently $b_{B_1} \in S_{b_{B_0}}$. Thus $d \in S_a$ whenever $(a,d) \in [C]^2$. $\square$

Corollary 4.4. *Suppose $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$. Then $I_{\kappa,\lambda}$ is weakly selective and $\{C : \mathfrak{p}_{I_{\kappa,\lambda}|C} > \lambda^{<\kappa}\}$ is dense in $(I^+_{\kappa,\lambda}, \subseteq)$.*

Proof. Use the following observation: Let $A \in I^+_{\kappa,\lambda}$. Then by Proposition 2.3 there is $C \in \mathfrak{K}_{\kappa,\lambda} \cap P(A)$. Now setting $H = I_{\kappa,\lambda}|C$, $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. $\square$

It is immediate from Proposition 4.3 that if κ is inaccessible and $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$, then $\mathfrak{p}_{\kappa,\lambda} > \lambda^{<\kappa}$. More generally, Corollary 8.5 and Proposition 8.6 below yield that if $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$, then $\mathfrak{p}_{\kappa,\lambda} > \lambda^{<\kappa}$.

5. $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda}, \theta)^2$.

Definition. Given $X, Y \subseteq P\big(P_\kappa(\lambda)\big)$ and an ordinal $\eta \leq \kappa$, $X \to (Y, \eta)^2$ means that for all $A \in X$ and $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$, there is $B \subseteq A$ such that either $B \in Y$ and F is constantly 0 on $[B]^2$, or $B \in [P_\kappa(\lambda)]^\eta$ and F is constantly 1 on $[B]^2$.

The negation of this and other partition relations is indicated by crossing the arrow.

In this section we investigate the problem of getting $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda}, \theta)^2$ for a given infinite cardinal $\theta \leq \kappa$. First, a simple observation:

Proposition 5.1. *Suppose η is an ordinal $\leq \kappa$ such that $\{P_\kappa(\lambda)\} \to (I^+_{\kappa,\lambda}, \eta)^2$. Then $\kappa \to (\kappa, \eta)^2$.*

Proof. Given $f : \kappa \times \kappa \to 2$, consider $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$ defined by: $F(a, b) = 1$ if and only if $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ and $f\big(\cup(a \cap \kappa), \cup(b \cap \kappa)\big) = 1$. □

Lemma 5.2. *Suppose $\kappa \to (\kappa, \theta)^2$, where θ is an infinite cardinal $< \kappa$, and μ, τ are two cardinals such that $\theta \leq \mu < \kappa$ and $\omega \leq \tau < \theta$. Then $\mu^\tau < \kappa$.*

Proof. By Corollary 19.7 in [**11**], $\mu^\tau \not\to (\mu^+, \tau^+)^2$. □

Definition. Given an ideal H on $P_\kappa(\lambda)$, $A \in H^+$ and $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2, (H, A, F)$ is 0-nice if there is $C \in H^+ \cap P(A)$ such that

$$\{b \in C : \forall a \in x\ (F(a, b) = 0)\} \in H^+$$

for every $x \in P_\kappa(C) - \{\phi\}$.

Lemma 5.3. *Suppose $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$, $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$ and (H, A, F) is 0-nice, where H is an ideal on $P_\kappa(\lambda), A \in H^+$ and $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$. Then there is $D \in H^+ \cap P(A)$ such that F is constantly 0 on $[D]^2$.*

Proof. By Proposition 4.3. □

Definition. For an ideal H on $P_\kappa(\lambda)$ and $C \in H^+, M^{\mathrm{d}}_{H,C}$ is the set of all $Q \subseteq H^+ \cap P(C)$ such that (a) any two distinct members of Q are disjoint, and (b) for every $A \in H^+ \cap P(C)$, there is $B \in Q$ with $A \cap B \in H^+$.

Lemma 5.4. *Suppose $\kappa \to (\kappa, \theta)^2$, where θ is an infinite cardinal $< \kappa$, and (H, A, F) is not 0-nice, where H is an ideal on $P_\kappa(\lambda), A \in H^+$ and $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$. Then there is $D \in [A]^{\theta+1}$ such that F is constantly 1 on $[D]^2$.*

Proof. First, we define two functions φ and ψ so that for each $C \in H^+ \cap P(A)$,

(a) $\varphi(C) \in M^{\mathrm{d}}_{H,C}$ and $|\varphi(C)| < \kappa$,
(b) $\psi(C)$ is a one-to-one function from $\varphi(C)$ to C,
(c) If $b \in B \in \varphi(C)$, then $(\psi(C))(B) \subsetneq b$ and $F((\psi(C))(B), b) = 1$.

Given $C \in H^+ \cap P(A)$, pick $x \in P_\kappa(C) - \{\phi\}$ so that

$$\{b \in C : \forall a \in x\ (F(a, b) = 0)\} \in H.$$

Select a bijection $j : |x| \to x$. For $\delta < |x|$, let B_δ be the set of all $b \in C$ such that $\cup x \subsetneq b$ and $\delta =$ the least $\gamma < |x|$ such that $F(j(\gamma), b) = 1$. Now set $\varphi(C) = H^+ \cap \{B_\delta : \delta < |x|\}$ and $(\psi(C))(B_\delta) = j(\delta)$ for every $\delta < |x|$ such that $B_\delta \in \varphi(C)$.

Recalling Lemma 5.2, define $R_\beta, Q_\beta \in \{W \in M^{\mathrm{d}}_{H,A} : |W| < \kappa\}$ and $\psi_\beta : Q_\beta \to A$ for $\beta < \theta$ by:

(0) $R_0 = \{A\}$.
(1) $Q_\beta = \bigcup_{C \in R_\beta} \varphi(C)$.
(2) $R_{\beta+1} = Q_\beta$.
(3) $R_\beta = H^+ \cap \{\bigcap_{\alpha<\beta} h(\alpha) : h \in \prod_{\alpha<\beta} Q_\alpha\}$ if β is a limit ordinal > 0.
(4) $\psi_\beta = \bigcup_{C \in R_\beta} \psi(C)$.

Finally, select $b \in \bigcap_{\beta<\theta}(\cup Q_\beta)$. There is $k \in \prod_{\beta<\theta} Q_\beta$ such that $b \in \bigcap_{\beta<\theta} k(\beta)$. Then

$$D = \{\psi_\beta(k(\beta)) : \beta < \theta\} \cup \{b\}$$

is as desired. □

Proposition 5.5. *Suppose $\kappa \to (\kappa, \theta)^2$, where θ is an infinite cardinal $< \kappa$, and H is an ideal on $P_\kappa(\lambda)$ such that $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. Then $H^+ \to (H^+, \theta + 1)^2$.*

Proof. By Lemmas 5.3 and 5.4. □

Corollary 5.6. *Suppose $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$. Then for every infinite cardinal $\theta < \kappa$, the following are equivalent*:

(i) $\kappa \to (\kappa, \theta)^2$.
(ii) $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda}, \theta + 1)^2$.
(iii) $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda}, \theta)^2$.

Proof.
(i) $\to$ (ii) follows from Propositions 2.3 and 5.5.
(ii) $\to$ (iii) is immediate.
(iii) $\to$ (i) is immediate by Proposition 5.1. □

It remains to handle the case $\theta = \kappa$.

Lemma 5.7. *Suppose κ is weakly compact and (H, A, F) is not 0-nice, where H is an ideal on $P_\kappa(\lambda), A \in H^+$ and $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$. Then there is $D \in [A]^\kappa$ such that F is constantly 1 on $[D]^2$.*

Proof. Proceed as in the proof of Lemma 5.4, but this time define R_β, Q_β and ψ_β for every $\beta < \kappa$. Since κ has the tree property, there is $k \in \prod_{\beta<\kappa} Q_\beta$ such that $\bigcap_{\beta\leq\gamma} k(\beta) \neq \phi$ for every $\gamma < \kappa$. Then $D = \{\psi_\beta(k(\beta)) : \beta < \kappa\}$ is as desired. □

Proposition 5.8. *Suppose κ is weakly compact and H is an ideal on $P_\kappa(\lambda)$ such that $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. Then $H^+ \to (H^+, \kappa)^2$.*

Proof. By Proposition 2.5 and Lemmas 5.3 and 5.7. □

Corollary 5.9. *Suppose* $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$. *Then the following are equivalent*:

(i) κ *is weakly compact.*
(ii) $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda}, \kappa)^2$.

Proof. By Propositions 5.1 and 5.8. $\square$

6. $H^+ \to (I^+_{\kappa,\lambda}, \theta \oplus 1)^2$.

Throughout this section κ is assumed to be uncountable.

Definition. Given $X, Y \subseteq P(P_\kappa(\lambda))$ and an infinite cardinal $\theta < \kappa$, $X \to (Y, \theta \oplus 1)^2$ means that for all $A \in X$ and $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$, there is either $B \in Y \cap P(A)$ such that F is constantly 0 on $[B]^2$, or $(a_0, a_1, \dots, a_\theta) \in [A]^{\theta+1}$ such that $a_\theta = \bigcup_{\alpha<\theta} a_\alpha$ and F is constantly 1 on $[\{a_\beta : \beta \leq \theta\}]^2$.

If θ is an infinite cardinal $< \kappa$, then $I^+_{\kappa,\lambda} \not\to (I^+_{\kappa,\lambda}, \theta \oplus 1)^2$. (Set

$$A = \{a \in P_\kappa(\lambda) : \exists \alpha < \kappa\ (a \cap \kappa = \alpha + 1)\}$$

and consider $F : [A]^2 \to 2$ defined by: $F(a, b) = 0$ if and only if $a \cap \kappa = b \cap \kappa$.) Our goal is to produce an ideal H on $P_\kappa(\lambda)$ such that $H^+ \to (I^+_{\kappa,\lambda}, \theta \oplus 1)^2$. We start by reviewing a few facts.

Definition. Suppose H is an ideal on $P_\kappa(\lambda)$ and δ is an ordinal with $\kappa \leq \delta \leq \lambda$. Then H is δ-normal if given $A \in H^+$ and $f : A \to \delta$ such that $f(a) \in a$ for all $a \in A$, there is $B \in H^+ \cap P(A)$ such that f is constant on B.
$NS^\delta_{\kappa,\lambda}$ denotes the smallest δ-normal ideal on $P_\kappa(\lambda)$.

Note that being λ-normal is the same as being normal, so that $NS^\lambda_{\kappa,\lambda} = NS_{\kappa,\lambda}$.

Definition. Suppose H is an ideal on $P_\kappa(\lambda)$, ν is a cardinal with $\kappa \leq \nu \leq \lambda$ and θ is an infinite cardinal $\leq \kappa$. Then H is $[\nu]^{<\theta}$-normal if given $A \in H^+$ and $f : A \to P(\nu)$ such that $f(a) \in P_{|a\cap\theta|}(a)$ for all $a \in A$, there is $B \in H^+ \cap P(A)$ such that f is constant on B.
The following is easy.

Lemma 6.1. *Suppose H is an ideal on $P_\kappa(\lambda)$ and ν is a cardinal with $\kappa \leq \nu \leq \lambda$. Then H is $[\nu]^{<\omega}$-normal if and only if it is ν-normal.*

Lemma 6.2 ([**7**], [**23**]).

(i) *Suppose ν is a cardinal with $\kappa \leq \nu \leq \lambda$ and θ is an uncountable cardinal $< \kappa$. Then there exists a $[\nu]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if $\tau^{<\theta} < \kappa$ for every cardinal τ with $\theta \leq \tau < \kappa$.*
(ii) *Suppose κ is a limit cardinal and ν is a cardinal with $\kappa \leq \nu \leq \lambda$. Then there exists a $[\nu]^{<\kappa}$-normal ideal on $P_\kappa(\lambda)$ if and only if κ is Mahlo.*

Definition. Suppose there exists a $[\nu]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, where ν, θ are two cardinals such that $\kappa \leq \nu \leq \lambda$ and $\omega \leq \theta \leq \kappa$. Then $NS_{\kappa,\lambda}^{[\nu]^{<\theta}}$ denotes the smallest such ideal.

Note that by Lemma 6.1 $NS_{\kappa,\lambda}^{[\lambda]^{<\omega}} = NS_{\kappa,\lambda}$.

Lemma 6.3 ([**23**]). *Suppose θ is an uncountable cardinal $< \kappa$ and ν is a cardinal with $\kappa \leq \nu \leq \lambda$. Then the set of all $a \in P_\kappa(\lambda)$ such that $cf(\cup(a \cap \tau)) < \theta$ for some regular cardinal τ with $\kappa \leq \tau \leq \nu$ lies in $NS_{\kappa,\lambda}^{[\nu]^{<\theta}}$.*

As will now be shown, $[\nu]^{<\theta}$-normality can be seen as the combination of ν-normality with a distributivity property.

Definition. For an ideal H on $P_\kappa(\lambda)$ and $A \in H^+, M_{H,A}$ is the set of all $Q \subseteq H^+ \cap P(A)$ such that (a) the intersection of any two distinct members of Q lies in H, and (b) for every $C \in H^+ \cap P(A)$, there is $B \in Q$ with $B \cap C \in H^+$.

Definition. Suppose H is an ideal on $P_\kappa(\lambda)$ and μ, ρ are two cardinals ≥ 1. Then H is (μ,ρ)-distributive (respectively, disjointly (μ,ρ)-distributive) if given $A \in H^+$ and $Q_\alpha \in M_{H,A}$ (respectively, $Q_\alpha \in M^{\rm d}_{H,A}$) for $\alpha < \mu$ with $|Q_\alpha| \leq \rho$, there are $C \in H^+ \cap P(A)$ and $h \in \prod_{\alpha<\mu} Q_\alpha$ such that $C - h(\alpha) \in H$ for all $\alpha < \mu$.

The following generalizes a result of Johnson [**14**].

Lemma 6.4. *Suppose ν is a cardinal with $\kappa \leq \nu \leq \lambda$, H is a ν-normal ideal on $P_\kappa(\lambda)$, and θ is a regular uncountable cardinal which is $\leq \kappa$ if κ is a limit cardinal, and $< \kappa$ otherwise. Then the following are equivalent*:

(i) *H is $[\nu]^{<\theta}$-normal.*
(ii) *H is $(\mu, \nu^{<\theta})$-distributive for every infinite cardinal $\mu < \theta$.*
(iii) *H is disjointly (μ,ν)-distributive for every infinite cardinal $\mu < \theta$.*

Proof. (i) $\to$ (ii): Assume (i) holds, and let μ be an infinite cardinal $< \theta$. Fix $A \in H^+$ and $Q_\alpha \in M_{H,A}$ for $\alpha < \mu$ with $|Q_\alpha| \leq \nu^{<\theta}$. Select a one-to-one $j : \mu \times \nu \to \nu$. Given $\alpha < \mu$, pick a one-to-one $f_\alpha : Q_\alpha \to P_\theta(j``(\{\alpha\} \times \nu))$ and define $k_\alpha : Q_\alpha \to P_\theta(\nu)$ by: $k_\alpha(B) = f_\alpha(B) \cup \{|f_\alpha(B)|\}$ if $\theta = \kappa$, and $k_\alpha(B) = f_\alpha(B)$ otherwise. Next, define $\ell_\alpha : Q_\alpha \to P(P_\kappa(\lambda))$ by $\ell_\alpha(B) = \{a \in B : k_\alpha(b) \subseteq a\}$, and put $R_\alpha = \mathrm{ran}(\ell_\alpha)$ and $W_\alpha = A - (\cup R_\alpha)$. Clearly $R_\alpha \in M_{H,A}$, consequently $W_\alpha \in H$.

Define C as follows: If $\theta < \kappa$, C is the set of all $a \in A - (\bigcup_{\alpha<\mu} W_\alpha)$ such that $\theta \subseteq a$. If $\theta = \kappa$, C is the set of all $a \in A - (\bigcup_{\alpha<\mu} W_\alpha)$ such that $a \cap \kappa$ is an infinite cardinal of cofinality $> \mu$. Then $C \in H^+$ by Lemma 6.3. For $a \in C$, pick $t_a \in \prod_{\alpha<\mu} Q_\alpha$ so that $a \in t_a(\alpha)$ and $k_\alpha(t_a(\alpha)) \subseteq a$ for all $\alpha < \mu$. Now define $g : C \to P(\nu)$ by $g(a) = \bigcup_{\alpha<\mu} f_\alpha(t_a(\alpha))$. Since $g(a) \in P_{|a\cap\theta|}(a)$ for all $a \in C$, there is $D \in H^+ \cap P(C)$ such that g is constant on D. Pick $a \in D$. Then $D \subseteq \bigcap_{\alpha<\mu} t_a(\alpha)$.

(ii) $\to$ (iii) is immediate.

(iii) $\to$ (i): Suppose (iii) holds and fix $A \in H^+$ and $f : A \to P(\nu)$ such that $f(a) \in P_{|a\cap\theta|}(a)$ for all $a \in A$. Define B by: $B = \{a \in A : \theta \subseteq a\}$ if $\theta < \kappa$, else

$$B = \{a \in A : a \cap \kappa \text{ is an infinite ordinal}\}.$$

Then $B \in H^+$, so by ν-normality there are $C \in H^+ \cap P(B)$ and $\mu < \theta$ such that $|f(a)| = \mu$ for all $a \in C$. If μ is finite, then f is constant on some $D \in H^+ \cap P(C)$ by Lemma 6.1. Now suppose μ is infinite. Select a bijection $j_a : \mu \to f(a)$ for each $a \in C$. Now for $\alpha < \mu$, set

$$Q_\alpha = H^+ \cap \{\{a \in C : j_a(\alpha) = \beta\} : \beta < \nu\}.$$

It is simple to check that $Q_\alpha \in M^{\mathrm{d}}_{H,C}$. Hence there is $h \in \prod_{\alpha<\mu} Q_\alpha$ such that $\bigcap_{\alpha<\mu} h(\alpha) \in H^+$. Obviously, f is constant on $\bigcap_{\alpha<\mu} h(\alpha)$. □

DEFINITION. Given $h : \lambda \to P_\kappa(\lambda)$ and a regular infinite cardinal $\theta < \kappa, U^\theta_h$ is the set of all $a \in P_\kappa(\lambda)$ such that $a = \bigcup_{\alpha\in e} h(\alpha)$ for some $e \subseteq a$ with $|e| = \theta$.

We are looking for pairs (h, θ) such that $U^\theta_h \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$.

DEFINITION. Given a regular infinite cardinal $\theta < \kappa$, $\mathscr{H}^\theta_{\kappa,\lambda}$ is the set all $h : \lambda \to P_\kappa(\lambda)$ such that for each $a \in P_\kappa(\lambda)$, there is $e \in P_{\theta^+}(\lambda)$ with $a \subseteq \bigcup_{\alpha\in e} h(\alpha)$.

The easy proof of the following is left to the reader.

LEMMA 6.5. *Suppose θ is a regular infinite cardinal $< \kappa$. Then*

$$\mathscr{H}^\theta_{\kappa,\lambda} = \{h : \lambda \to P_\kappa(\lambda) : U^\theta_h \in I^+_{\kappa,\lambda}\}.$$

LEMMA 6.6. *Suppose $h, k \in \mathscr{H}^\theta_{\kappa,\lambda}$, where θ is a regular infinite cardinal $< \kappa$. Then $U^\theta_h \Delta U^\theta_k \in NS_{\kappa,\lambda}$.*

PROOF. Define $f : \lambda \to P_{\theta^+}(\lambda)$ and $g : \lambda \to P_{\theta^+}(\lambda)$ so that for every $\alpha \in \lambda$, $h(\alpha) \subseteq \bigcup_{\beta\in f(\alpha)} k(\beta)$ and $k(\alpha) \subseteq \bigcup_{\beta\in g(\alpha)} h(\beta)$. Let D be the set of all $a \in P_\kappa(\lambda)$ such that $\theta \subseteq a$ and

$$h(\alpha) \cup k(\alpha) \cup f(\alpha) \cup g(\alpha) \subseteq a$$

for all $\alpha \in a$. Then $D \in NS^*_{\kappa,\lambda}$ and $U^\theta_h \cap D = U^\theta_k \cap D$. □

LEMMA 6.7. *Suppose $\kappa \to (\kappa, \theta)^2$ and $h \in \mathscr{H}^\theta_{\kappa,\lambda}$, where θ is a regular infinite cardinal $< \kappa$. Then $U^\theta_h \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$.*

PROOF. The existence of a $[\lambda]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ follows from Lemmas 5.2 and 6.2 (i). To establish that $U^\theta_h \notin NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}$, it suffices to show that for every $f : P_\theta(\lambda) \to P_\kappa(\lambda)$, there is $a \in U^\theta_h$ such that $\theta \subseteq a$ and $f(z) \subseteq a$ for every $z \in P_\theta(a)$. Given f, define $a_\beta, b_\beta \in P_\kappa(\lambda)$ and $e_\beta \in P_{\theta^+}(\lambda)$ for $\beta < \theta$ by:

(0) $a_0 = \theta \cup (\bigcup_{\alpha\in\theta} h(\alpha))$.
(1) $a_\beta \subseteq b_\beta$.
(2) $b_\beta = \bigcup_{\alpha\in e_\beta} h(\alpha)$.
(3) $a_{\beta+1} = b_\beta \cup (\cup\{f(z) : z \in P_\theta(b_\beta)\})$.

(4) $a_\beta = \bigcup_{\gamma<\beta} a_\gamma$ if β is a limit ordinal > 0.
(5) $\psi_\alpha = \bigcup_{C\in T_\alpha} \psi(C)$.

Now set $a = \bigcup_{\beta<\theta} b_\beta$. Obviously, $\theta \subseteq a$. Moreover, $a \in U_h^\theta$ since

$$a = \bigcup \Big\{ h(\alpha) : \alpha \in \theta \bigcup \Big(\bigcup_{\beta<\theta} e_\beta \Big) \Big\}.$$

Finally, given $z \in P_\theta(a)$, there is $\beta < \theta$ such that $z \subseteq b_\beta$, and then $f(z) \subseteq a_{\beta+1} \subseteq a$. □

It remains to discuss whether $\mathscr{H}^\theta_{\kappa,\lambda} \neq \phi$.

Lemma 6.8. *Suppose that θ is a regular infinite cardinal $< \kappa$ and either $cf(\lambda) \geq \kappa$ and $u(\kappa,\lambda) = \lambda$, or $cf(\lambda) \leq \theta$ and $u(\kappa,\mu) \leq \lambda$ for every cardinal μ with $\kappa < \mu < \lambda$. Then $\mathscr{H}^\theta_{\kappa,\lambda} \neq \phi$.*

Proof. Select $h : \lambda \to \bigcup_{\kappa\leq\xi<\lambda} P_\kappa(\xi)$ so that

$$\bigcup_{\kappa\leq\xi<\lambda} P_\kappa(\xi) = \bigcup_{\alpha<\lambda} P\big(h(\alpha)\big).$$

Then it is simple to see that $h \in \mathscr{H}^\theta_{\kappa,\lambda}$. □

Definition. Given a regular infinite cardinal $\theta < \kappa$, $T^\theta_{\kappa,\lambda}$ (respectively, $T^{\leq\theta}_{\kappa,\lambda}$) is the set of all $a \in P_\kappa(\lambda)$ such that $\cup(a \cap \tau)$ is a limit ordinal of cofinality θ (respectively, $\leq \theta$) for every regular cardinal τ with $\kappa \leq \tau \leq \lambda$.

Lemma 6.9. *Suppose $h : \lambda \to P_\kappa(\lambda)$ and θ is a regular infinite cardinal $< \kappa$. Then $U_h^\theta - T^{\leq\theta}_{\kappa,\lambda} \in NS_{\kappa,\lambda}$.*

Proof. Set $\widetilde{\beta} = \max\{\kappa, |\beta|^+\}$ for every $\beta < \lambda$. Let A be the set of all $a \in P_\kappa(\lambda)$ such that

$$\cup\big(h(\alpha) \cap \widetilde{\beta}\big) < \cup(a \cap \widetilde{\beta})$$

for all $\alpha, \beta \in a$, and $\gamma + 1 \in a$ for all $\gamma \in a$. Then $A \in NS^*_{\kappa,\lambda}$ and $A \cap U_h^\theta \subseteq T^{\leq\theta}_{\kappa,\lambda}$. □

Lemma 6.10. *Suppose $h : \lambda \to P_\kappa(\lambda)$ and θ is a regular infinite cardinal $< \kappa$. Then $U_h^\theta - T^\theta_{\kappa,\lambda} \in NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}$.*

Proof. By Lemmas 6.3 and 6.9. □

Lemma 6.11. *Suppose that θ is a regular infinite cardinal $< \kappa$ and either $\kappa = \theta^+$ or $\lambda < \kappa^{+\theta^+}$. Then $T^{\leq\theta}_{\kappa,\lambda} - U_h^\theta \in NS_{\kappa,\lambda}$ for some $h : \lambda \to P_\kappa(\lambda)$.*

Proof. If $\kappa = \theta^+$ and $h : \lambda \to P_\kappa(\lambda)$ is defined by $h(\alpha) = \{\alpha\}$, then

$$\{a \in P_\kappa(\lambda) : \theta \subseteq a\} \subseteq U_h^\theta.$$

For the other case, see Proposition 5.6 in [**10**]. □

We are now in a position to prove the main result of this section.

Proposition 6.12. *Suppose that* $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$, θ *is a regular infinite cardinal* $< \kappa$ *such that* $\kappa \to (\kappa, \theta)^2$, $h : \lambda \to P_\kappa(\lambda)$ *and* $Z \in \mathfrak{K}_{\kappa,\lambda} \cap P(U^\theta_h) \cap (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$. *Then* $(NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}|Z)^+ \to (I^+_{\kappa,\lambda}, \theta \oplus 1)^2$.

Proof. Set $H = NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}|Z$. Fix $B \in H^+$ and $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$. Set $A = B \cap Z$.

If (H, A, F) is 0-nice, then clearly so is $(I_{\kappa,\lambda}|A, A, F)$ and therefore by Lemma 5.3 F is constantly 0 on $[D]^2$ for some $D \in I^+_{\kappa,\lambda} \cap P(A)$.

Next, suppose (H, A, F) is not 0-nice. Pick $g : \theta \times A \to \lambda$ so that $\{g(\alpha, b) : \alpha < \theta\} \subseteq b$ and $b = \bigcup_{\alpha<\theta} h(g(\alpha, b))$ for every $b \in A$. For $\alpha < \theta$, define a function χ_α on $H^+ \cap P(A)$ by

$$\chi_\alpha(C) = H^+ \cap \{\{b \in C : g(\alpha, b) = \xi\} : \xi < \lambda\}.$$

It is simple to check that $\chi_\alpha(C) \in M^{\mathrm{d}}_{H,C}$. Let φ, ψ be as in the proof of Lemma 5.4. Now appealing to Lemma 6.4, define

$$R_\alpha, T_\alpha, Q_\alpha \in \{W \in M^{\mathrm{d}}_{H,A} : |W| \le \lambda^{|1+\alpha|}\}$$

and $\psi_\alpha : Q_\alpha \to A$ for $\alpha < \theta$ by:

(0) $R_0 = \{A\}$.
(1) $T_\alpha = \bigcup_{C \in R_\alpha} \chi_\alpha(C)$.
(2) $Q_\alpha = \bigcup_{C \in T_\alpha} \varphi(C)$.
(3) $R_{\alpha+1} = Q_\alpha$.
(4) $R_\alpha = H^+ \cap \{\bigcap_{\beta<\alpha} q(\beta) : q \in \prod_{\beta<\alpha} Q_\beta\}$ if α is a limit ordinal > 0.
(5) $\psi_\alpha = \bigcup_{C \in T_\alpha} \psi(C)$.

Finally, select $b \in \bigcap_{\alpha<\theta}(\cup Q_\alpha)$. Let $k \in \prod_{\alpha<\theta} Q_\alpha$ be such that $b \in \bigcap_{\alpha<\theta} k(\alpha)$. Stipulate that $a_\alpha = \psi_\alpha\big(k(\alpha)\big)$ for $\alpha < \theta$, and $a_\theta = b$. Then clearly $(a_0, a_1, \dots, a_\theta) \in [A]^{\theta+1}$ and F takes the constant value 1 on $[\{a_\delta : \delta \le \theta\}]^2$. Moreover, $g(\alpha, a_\alpha) = g(\alpha, a_\delta)$ whenever $\alpha < \delta \le \theta$. It follows that

$$a_\theta = \bigcup_{\alpha<\theta} h(g(\alpha, a_\theta)) \subseteq \bigcup_{\alpha<\theta} a_\alpha.$$ □

Corollary 6.13. *Suppose that* (a) $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$, (b) θ *is a regular infinite cardinal* $< \kappa$ *such that* $\kappa \to (\kappa, \theta)^2$, (c) $\tau^\theta < \kappa$ *for every infinite cardinal* $\tau < \kappa$, *and* (d) *either* $cf(\lambda) \ge \kappa$ *and* $\lambda^{<\kappa} = \lambda$, *or* $cf(\lambda) \le \theta$ *and* $\mu^{<\kappa} \le \lambda$ *for every cardinal* μ *with* $\kappa < \mu < \lambda$. *Then* $(NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}|Z)^+ \to (I^+_{\kappa,\lambda}, \theta \oplus 1)^2$ *for some* $Z \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$.

Proof. By Lemma 6.8, there is $h \in \mathscr{H}^\theta_{\kappa,\lambda}$. Set $Z = U^\theta_h$. Then $Z \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$ by Lemma 6.7. That $Z \in \mathfrak{K}_{\kappa,\lambda}$ follows from (c). □

Corollary 6.14. *Suppose that* (a) $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$, (b) θ *is a regular infinite cardinal* $< \kappa$ *such that* $\kappa \to (\kappa, \theta)^2$, (c) $\tau^\theta < \kappa$ *for every infinite cardinal* $\tau < \kappa$, *and* (d) *either* $\kappa = \theta^+$ *or* $\lambda < \kappa^{+\theta^+}$. *Then* $(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} | T_{\kappa,\lambda}^\theta)^+ \to (I_{\kappa,\lambda}^+, \theta \oplus 1)^2$.

Proof. By Lemma 6.11, there is $h : \lambda \to P_\kappa(\lambda)$ such that $T_{\kappa,\lambda}^{\le\theta} - U_h^\theta \in NS_{\kappa,\lambda}$. It can be checked that $T_{\kappa,\lambda}^{\le\theta} \in NS_{\kappa,\lambda}^+$, so $h \in \mathscr{H}_{\kappa,\lambda}^\theta$. Set $Z = U_h^\theta \cap T_{\kappa,\lambda}^\theta$. Then $Z \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^+$ by Lemmas 6.7 and 6.10, and $Z \in \mathfrak{K}_{\kappa,\lambda}$ because of (c). Moreover,

$$NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} | Z = NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} | T_{\kappa,\lambda}^\theta$$

by Lemma 6.10. □

Thus for example if CH holds and $\mathbf{cov}(\mathbf{M}_{\omega_2,\omega_3}) > \omega_3$, then

$$\big(NS_{\omega_3,\omega_3} | T_{\omega_2,\omega_3}^\omega\big)^+ \to \big(I_{\omega_2,\omega_3}^+, \omega \oplus 1\big)^2.$$

Left unanswered is whether it is possible that $NS_{\omega_1,\omega_2}^+ \to (I_{\omega_1,\omega_2}^+, \omega \oplus 1)^2$. Note that by the results above if $NS_{\omega_1,\omega_2}^+ \cap \mathfrak{K}_{\omega_1,\omega_2} \neq \phi$ and $\mathbf{cov}(\mathbf{M}_{\omega_1,\omega_2}) > \omega_2$, then

$$\big(NS_{\omega_1,\omega_2} | Z\big)^+ \to \big(I_{\omega_1,\omega_2}^+, \omega \oplus 1\big)^2$$

for some $Z \in NS_{\omega_1,\omega_2}^+$. We do not know whether it is consistent that $NS_{\omega_1,\omega_2}^+ \cap \mathfrak{K}_{\omega_1,\omega_2} \neq \phi$.

7. $I_{\kappa,\lambda}^+ \to (I_{\kappa,\lambda}^+)^3$.

Definition. Given $X, Y \subseteq P\big(P_\kappa(\lambda)\big)$ and $n < \omega$, $X \to (Y)^n$ means that for all $A \in X$ and $F : [P_\kappa(\lambda)]^n \to 2$, there is $B \in Y \cap P(A)$ such that F is constant on $[B]^n$.

The main purpose of this section is to discuss the partition relation $I_{\kappa,\lambda}^+ \to (I_{\kappa,\lambda}^+)^n$. To start, let us show that $I_{\kappa,\lambda}^+ \to (I_{\kappa,\lambda}^+)^3$ implies that $I_{\kappa,\lambda}^+ \to (I_{\kappa,\lambda}^+)^n$ for all $n \geq 1$.

Lemma 7.1 ([**18**]). *Suppose H is $(\nu, 2)$-distributive, where H is an ideal on $P_\kappa(\lambda)$ and ν an infinite cardinal $< \kappa$. Then $2^\nu < \kappa$.*

The following is essentially due to Johnson (see Theorem 6.2 in [**14**]).

Lemma 7.2. *Given an ideal H on $P_\kappa(\lambda)$, the following are equivalent*:

(i) *H is $(\lambda^{<\kappa}, 2)$-distributive and weakly selective.*
(ii) *$H^+ \to (H^+)^n$ for all n with $0 < n < \omega$.*
(iii) *$H^+ \to (H^+)^3$.*

Proof. (i) → (ii): Assume (i) holds. To prove (ii), we proceed by induction on n. It is immediate that $H^+ \to (H^+)^1$. Now suppose $H^+ \to (H^+)^n$ for some n with $0 < n < \omega$. Fix $A \in H^+$ and $F : [A]^{n+1} \to 2$. Since H is $(\lambda^{<\kappa}, 2)$-distributive, there are $B \in H^+ \cap P(A)$ and $f : [A]^n \to 2$ such that

$$\{b \in B : a_n \subsetneq b \text{ and } F(a_1, \ldots, a_n, b) \neq f(a_1, \ldots, a_n)\} \in H$$

for every $(a_1, \ldots, a_n) \in [A]^n$. Because κ is inaccessible by Lemma 7.1 and H is weakly selective, there is $C \in H^+ \cap P(B)$ such that $F(a_1, \ldots, a_n, b) = f(a_1, \ldots, a_n)$ whenever $(a_1, \ldots, a_n, b) \in [C]^{n+1}$. Finally, by inductive hypothesis there is $D \in H^+ \cap P(C)$ such that f is constant on $[D]^n$. Clearly, F is constant on $[D]^{n+1}$.

(ii) $\to$ (iii) is trivial.

(iii) $\to$ (i): Assume (iii) holds. To show that H is weakly selective, let $A \in H^+$ and $B_a \in H$ for $a \in A$. Define $F : [A]^2 \to 2$ by $F(a, b) = 0$ if and only if $b \in B_a$. Then F is constant on $[C]^2$ for some $C \in H^+ \cap P(A)$. It is simple to check that $b \notin B_a$ for all $(a, b) \in [C]^2$. Next, let us establish that H is $(\lambda^{<\kappa}, 2)$-distributive. Thus let $A \in H^+$ and $B_a \subseteq P_\kappa(\lambda)$ for $a \in P_\kappa(\lambda)$. Define $F : [A]^3 \to 2$ by:

$$F(a, b, c) = 0 \text{ iff } \forall e \subseteq a(b \in B_e \leftrightarrow c \in B_e).$$

Pick $C \in H^+ \cap P(A)$ so that F is constant on $[C]^3$. Since κ is clearly weakly compact and hence inaccessible, F must be identically 0 on $[C]^3$. Now define $h \in \prod_{e \in P_\kappa(\lambda)} \{B_e, P_\kappa(\lambda) - B_e\}$ as follows. Given $e \in P_\kappa(\lambda)$, pick $(a, b) \in [C]^2$ with $e \subseteq a$, and let $h(e) = B_e$ if and only if $b \in B_e$. If $d \in C$ and $a \subsetneq d$, then $d \subsetneq c$ for some $c \in C$ with $b \subsetneq c$, so that

$$d \in B_e \leftrightarrow c \in B_e \leftrightarrow b \in B_e$$

and consequently $d \in B_e$. Thus $C - B_e \in I_{\kappa,\lambda}$. □

Definition. For a cardinal $\mu \geq \kappa$, κ is mildly μ-ineffable if given $t_a \in {}^a 2$ for $a \in P_\kappa(\mu)$, there is $g \in {}^\mu 2$ such that for every $a \in P_\kappa(\mu)$,

$$\{b \in P_\kappa(\mu) : a \subseteq b \text{ and } t_b \restriction a = g \restriction a\} \in I^+_{\kappa,\mu}.$$

It is simple to see that if κ is μ-compact, then κ is mildly μ-ineffable. An immediate consequence is the result of Rado [**29**] that ω is mildly μ-ineffable for every infinite cardinal μ.

The following refines a result of Di Prisco and Zwicker [**9**].

Lemma 7.3. *Suppose κ is mildly $\lambda^{<\kappa}$-ineffable, H is an ideal on $P_\kappa(\lambda)$ and $B_\delta \subseteq P_\kappa(\lambda)$ for $\delta \in \lambda^{<\kappa}$. Then there is $h \in \prod_{\delta \in \lambda^{<\kappa}} \{B_\delta, P_\kappa(\lambda) - B_\delta\}$ such that $\bigcap_{\delta \in c} h(\delta) \in H^+$ for every $c \in P_\kappa(\lambda^{<\kappa}) - \{\phi\}$.*

Proof. For $\delta \in \lambda^{<\kappa}$, set $B^0_\delta = B_\delta$ and $B^1_\delta = P_\kappa(\lambda) - B_\delta$. Now for each $d \in P_\kappa(\lambda^{<\kappa}) - \{\phi\}$ pick $t_d \in {}^d 2$ so that $\bigcap_{\delta \in d} B^{t_d(\delta)}_\delta \in H^+$. Select $g : \lambda^{<\kappa} \to 2$ so that for every $c \in P_\kappa(\lambda^{<\kappa}) - \{\phi\}$, there is $d \in P_\kappa(\lambda^{<\kappa})$ with $c \subseteq d$ and $t_d \restriction c = g \restriction c$. Then obviously $\bigcap_{\delta \in c} B^{g(\delta)}_\delta \in H^+$ for any $c \in P_\kappa(\lambda^{<\kappa}) - \{\phi\}$. □

Proposition 7.4. *Suppose κ is mildly $\lambda^{<\kappa}$-ineffable and H is an ideal on $P_\kappa(\lambda)$ such that $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. Then $H^+ \to (H^+)^n$ for all n with $0 < n < \omega$.*

Proof. By Lemma 7.2 it suffices to establish that H is weakly selective and $(\lambda^{<\kappa}, 2)$-distributive. Weak selectivity is direct from Proposition 4.3. To show $(\lambda^{<\kappa}, 2)$-distributivity, let $A \in H^+$ and $B_\delta \subseteq A$ for $\delta \in \lambda^{<\kappa}$. By Lemma 7.3 one can find $h \in \prod_{\delta \in \lambda^{<\kappa}} \{B_\delta, A - B_\delta\}$ so that $\bigcap_{\delta \in c} h(\delta) \in H^+$ for every $c \in P_\kappa(\lambda^{<\kappa}) - \{\phi\}$. By Proposition 4.3 $\mathfrak{p}_H > \lambda^{<\kappa}$, so there must be $C \in H^+$ such that $C - h(\delta) \in H$ for all $\delta < \lambda^{<\kappa}$. □

Corollary 7.5. *Suppose κ is mildly $\lambda^{<\kappa}$-ineffable and $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$. Then $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^n$ for all n with $0 < n < \omega$.*

The following generalization is immediate from Proposition 2.21:

Corollary 7.6. *Suppose that κ is mildly $\lambda^{<\kappa}$-ineffable, $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$ and $(Q, <)$ is a κ-directed partially ordered set such that $\lambda \leq |Q| \leq \lambda^{<\kappa}$ and $|\{r \in Q : r < q\}| < \kappa$ for all $q \in Q$. Then given $f : Q^n \to 2$, where $0 < n < \omega$, there is a cofinal subset T of Q such that f is constant on*

$$\{(q_1, \ldots, q_n) \in T^n : q_1 < \cdots < q_n\}.$$

Note that Corollary 5.6 and Corollary 5.9 can be extended in the same way.

8. Isomorphisms.

We will now prove that if $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$ and $\mathfrak{K}_{\kappa,\lambda} \neq \phi$, then any two cofinal subsets of $P_\kappa(\lambda)$ have isomorphic cofinal subsets.

Lemma 8.1. *Suppose that $(Q, <)$ is a κ-directed partially ordered set with no maximal element and H is an ideal on $P_\kappa(\lambda)$ with $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. Suppose further that $A \in H^+$ and $h : A \to Q$ are such that $h``(A \cap R)$ is cofinal in Q for every $R \in H^*$. Then:*

(i) *If $|\{a \in A : h(a) < h(b)\}| < \kappa$ for every $b \in A$, then there is $C \in H^+ \cap P(A)$ such that for all $a, b \in C$,*

$$h(a) < h(b) \to a \subsetneq b.$$

(ii) *If $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$, then there is $D \in H^+ \cap P(A)$ such that for all $a, b \in D$,*

$$a \subsetneq b \to h(a) < h(b).$$

Proof.

(i) Assume that $|\{a \in A : h(a) < h(b)\}| < \kappa$ for all $b \in A$. Pick $X \subseteq H$ so that $|X| = \mathrm{cof}(H)$ and $H = \bigcup_{B \in X} P(B)$. For $B \in X$, let $\mathscr{D}_B$ be the set of all $p \in Fn(A, 2, \kappa)$ such that there is $b \in \mathrm{dom}(p)$ with the following properties:

(0) $b \notin B$.
(1) $\mathrm{dom}(p) = \{b\} \cup \{a \in A : h(a) < h(b)\}$.

(2) $p(b) = 1$.
(3) $a \subsetneq b$ for every $a \in \mathrm{dom}(p)$ such that $a \neq b$ and $p(a) = 1$.

Let us establish that $\mathscr{D}_B$ is dense. Thus fix $s \in Fn(A, 2, \kappa)$. Pick $q \in Q$ so that $h(a) < q$ for every $a \in \mathrm{dom}(s)$. There is

$$b \in \{c \in A - B : \forall a \in s^{-1}(\{1\})(a \subsetneq c)\}$$

such that $q \leq h(b)$. Define

$$p : \{b\} \cup \{a \in A : h(a) < h(b)\} \to 2$$

by:

(α) $p(b) = 1$.
(β) $p \restriction \mathrm{dom}(s) = s$.
(γ) $p(c) = 0$ for every $c \in \mathrm{dom}(p)$ such that $c \neq b$ and $c \notin \mathrm{dom}(s)$.

Clearly, $s \subseteq p$ and $p \in \mathscr{D}_B$.
Let $G \subseteq Fn(A, 2, \kappa)$ be a filter such that $G \cap \mathscr{D}_B \neq \phi$ for every $B \in X$. Pick $\varphi \in \prod_{B \in X}(G \cap \mathscr{D}_B)$ and let $< b_B : B \in X >$ be such that

$$\mathrm{dom}\big(\varphi(B)\big) = \{b_B\} \cup \{a \in A : h(a) < h(b_B)\}.$$

Stipulate that $C = \{b_B : B \in X\}$. Obviously, $C \in H^+ \cap P(A)$. Now suppose $B_0, B_1 \in X$ are such that $h(b_{B_0}) < h(b_{B_1})$. Select $r \in G$ so that $\varphi(B_0) \cup \varphi(B_1) \subseteq r$. Then

$$\big(\varphi(B_1)(b_{B_0})\big) = r(b_{B_0}) = \big(\varphi(B_0)\big)(b_{B_0}) = 1$$

and hence $b_{B_0} \subsetneq b_{B_1}$. Thus C is as desired.

(ii) Assume $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$. Set $S_a = \{b \in A : h(b) > h(a)\}$ for $a \in A$. Then $\bigcap_{a \in x} S_a \in H^+$ for every $x \in P_\kappa(A) - \{\phi\}$. Hence by Lemma 4.1 and Proposition 4.3 there is $D \in H^+ \cap P(A)$ such that $b \in S_a$ whenever $(a, b) \in [D]^2$. □

Proposition 8.2. *Suppose that* $(Q, <)$ *is a* κ*-directed partially ordered set such that* (a) $|\{r \in Q : r < q\}| < \kappa$ *for all* $q \in Q$, *and* (b) $\lambda^{<\kappa}$ *is the least cardinality of any cofinal subset of* $(Q, <)$. *Suppose further that* H *is an ideal on* $P_\kappa(\lambda)$ *such that* $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$ *and* $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. *Then for every* $A \in H^+$, *there exist* $D \in H^+ \cap P(A)$ *and a cofinal subset* T *of* Q *such that* $(D, \subsetneq)$ *and* $(T, <)$ *are isomorphic.*

Proof. Fix $A \in H^+$. Pick $Z \in H^* \cap \mathfrak{K}_{\kappa,\lambda}$ and a cofinal subset R of Q of size $\lambda^{<\kappa}$. Set $A \cap Z = \{a_\delta : \delta < \lambda^{<\kappa}\}$ and $R = \{e_d : d \in Z\}$. Define a one-to-one $h : A \cap Z \to Q$ as follows: suppose $h(a_\xi)$ has already been defined for each $\xi < \delta$. There is $q \in Q$ such that $q \nleq h(a_\xi)$ for all $\xi < \delta$. Select $r \in Q$ so that $q \leq r$ and $e_d \leq r$ for every $d \in Z \cap P(a_\delta)$. Now stipulate that $h(a_\delta) = r$.

Note that h“B is a cofinal subset of Q for every $B \in I^+_{\kappa,\lambda} \cap P(A \cap Z)$. It is readily seen that Q has no maximal element. Hence by Lemma 8.1 there is $C \in H^+ \cap P(A \cap Z)$ such that

$$h(a) < h(b) \to a \subsetneq b$$

for all $a, b \in C$, and $D \in H^+ \cap P(C)$ such that

$$a \subsetneq b \to h(a) < h(b)$$

for all $a, b \in D$. Obviously, $(D, \subsetneq)$ and $(h``D, <)$ are isomorphic. □

COROLLARY 8.3. *Suppose that ν is a cardinal with $\lambda \leq \nu \leq \lambda^{<\kappa}$ and H is an ideal on $P_\kappa(\lambda)$ such that $H^* \cap \mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $\mathrm{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$. Then for all $A \in H^+$ and $B \in I^+_{\kappa,\nu}$, there are $D \in H^+ \cap P(A)$ and $T \in I^+_{\kappa,\nu} \cap P(B)$ such that $(D, \subsetneq)$ and $(T, \subsetneq)$ are isomorphic.*

PROOF. Fix $A \in H^+$ and $B \in I^+_{\kappa,\nu}$. By Proposition 2.3 and Corollary 2.8, there is $C \in \mathfrak{K}_{\kappa,\nu} \cap P(B)$. Using Corollary 3.6, $u(\kappa, \nu) = \nu^{<\kappa} = \lambda^{<\kappa}$, so by Proposition 8.2, there exist $D \in H^+ \cap P(A)$ and a cofinal subset T of $(C, \subsetneq)$ such that $(D, \subsetneq)$ and $(T, \subsetneq)$ are isomorphic. Obviously, $T \in I^+_{\kappa,\nu} \cap P(B)$. □

It easily follows that if $\mathfrak{K}_{\kappa,\lambda} \neq \phi$, $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$, $A \in I^+_{\kappa,\lambda}$ and $B \in I^+_{\kappa,\nu}$, where $\lambda \leq \nu \leq \lambda^{<\kappa}$, then A and B have isomorphic cofinal subsets. Note that if $P_\kappa(\lambda)$ and $P_\kappa(\lambda^{<\kappa})$ have isomorphic subsets, then we must have $u(\kappa, \lambda) = \lambda^{<\kappa}$ and $\mathfrak{K}_{\kappa,\lambda} \neq \phi$.

Next we deal with the problem whether $S_\kappa(\lambda)$ holds. (Recall that $S_\kappa(\lambda)$ asserts that for any $g : P_\kappa(\lambda) \to P_\kappa(\lambda)$, there is a cofinal subset D of $P_\kappa(\lambda)$ such that the image under f of every noncofinal subset of D is noncofinal.)

PROPOSITION 8.4. *Suppose $(Q, <)$ and H are as in the statement of Proposition 8.2, $A \in H^+$ and $f : A \to Q$. Then there is $D \in H^+ \cap P(A)$ such that for every $B \in I_{\kappa,\lambda} \cap P(D), f``B$ is not cofinal in $(Q, <)$.*

PROOF. Fix $Z \in H^* \cap \mathfrak{K}_{\kappa,\lambda}$. A slight modification of the proof of Proposition 8.2 yields the existence of (a) a one-to-one $h : A \cap Z \to Q$ such that $f(a) \leq h(a)$ for all $a \in A \cap Z$, and (b) $D \in H^+ \cap P(A \cap Z)$ such that

$$h(a) < h(b) \to a \subsetneq b$$

for all $a, b \in D$. Given $B \in I_{\kappa,\lambda} \cap P(D)$, there is $a \in D$ such that

$$B \cap \{b \in P_\kappa(\lambda) : a \subseteq b\} = \phi.$$

Then

$$h``B \cap \{q \in Q : h(a) \leq q\} = \phi$$

and hence

$$f``B \cap \{q \in Q : h(a) \leq q\} = \phi. \qquad \square$$

Corollary 8.5. *Suppose $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}})$, $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $g : P_\kappa(\lambda) \to P_\kappa(\lambda)$. Then given $E \in I^+_{\kappa,\lambda}$, there is $D \in I^+_{\kappa,\lambda} \cap P(E)$ such that $g``B \in I_{\kappa,\lambda}$ for any $B \in I_{\kappa,\lambda} \cap P(D)$.*

Proof. Using Proposition 2.3, pick $A \in \mathfrak{K}_{\kappa,\lambda} \cap P(E)$. Define $f : A \to A$ so that $g(a) \subseteq f(a)$ for all $a \in A$. By Proposition 8.4, there is $D \in (I_{\kappa,\lambda}|A)^+ \cap P(A)$ such that for every $B \in I_{\kappa,\lambda} \cap P(D)$, $f``B$ is not cofinal in $(A, \subsetneq)$. It is simple to see that D is as desired. $\square$

To conclude the section, let us show that the two statements "$S_\kappa(\lambda)$ holds" and "$\mathfrak{p}_{\kappa,\lambda} > u(\kappa, \lambda)$" are closely related.

Proposition 8.6. *Suppose that $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $S_\kappa(\lambda)$ holds. Then $\mathfrak{p}_{\kappa,\lambda} > u(\kappa, \lambda)$.*

Proof. Select $A \in I^+_{\kappa,\lambda}$ so that $|A \cap P(e)| < \kappa$ for all $e \in P_\kappa(\lambda)$. Let $S_a \subseteq P_\kappa(\lambda)$ for $a \in A$ be such that $\bigcap_{a \in x} S_a \in I^+_{\kappa,\lambda}$ for every $x \in P_\kappa(A) - \{\phi\}$. Define $g : P_\kappa(\lambda) \to P_\kappa(\lambda)$ so that for every $d \in P_\kappa(\lambda)$, (a) $d \subseteq g(d)$, and (b) $g(d) \in S_a$ for every $a \in A \cap P(d)$. Pick $D \in I^+_{\kappa,\lambda}$ so that $g``B \in I_{\kappa,\lambda}$ for all $B \in I_{\kappa,\lambda} \cap P(D)$. Now set $C = g``D$. Clearly, $C \in I^+_{\kappa,\lambda}$. Moreover for every $a \in A$, $C - S_a \in I_{\kappa,\lambda}$ since $C - S_a \subseteq g``B_a$, where $B_a = \{d \in D : a \nsubseteq d\}$. $\square$

Conservely, $\mathfrak{p}_{\kappa,\lambda} > u(\kappa, \lambda)$ implies that $S_\kappa(\lambda)$ holds.

Proposition 8.7. *Let $E \in I^+_{\kappa,\lambda}$ be such that $\mathfrak{p}_{I_{\kappa,\lambda}|E} > u(\kappa, \lambda)$. Then for every $g : P_\kappa(\lambda) \to P_\kappa(\lambda)$, there is $D \in I^+_{\kappa,\lambda} \cap P(E)$ such that $g``B \in I_{\kappa,\lambda}$ for any $B \in I_{\kappa,\lambda} \cap P(D)$.*

Proof. Select $A \in I^+_{\kappa,\lambda} \cap P(E)$ so that $|A| = u(\kappa, \lambda)$. Set $A = \{a_\alpha : \alpha < u(\kappa, \lambda)\}$. Fix $g : P_\kappa(\lambda) \to P_\kappa(\lambda)$. Define $f : A \to A$ so that for every $\alpha < u(\kappa, \lambda)$, (i) $a_\alpha \cup g(a_\alpha) \subseteq f(a_\alpha)$, and (ii) $f(a_\alpha) \subseteq f(a_\beta)$ for all $\beta < \alpha$. For $a \in A$, set $Z_a = \{f(b) : b \in A$ and $a \subseteq b\}$. It is readily checked that $\bigcap_{a \in x} Z_a \in I^+_{\kappa,\lambda}$ for every $x \in P_\kappa(A) - \{\phi\}$. It follows that there is $C \in I^+_{\kappa,\lambda} \cap P(A)$ such that $C - Z_a \in I_{\kappa,\lambda}$ for every $a \in A$. Now set $D = f^{-1}(C)$. Then $D \in I^+_{\kappa,\lambda}$ since $f^{-1}(C \cap Z_a) \subseteq \{b \in D : a \subseteq b\}$ for all $a \in A$. Given $B \in I_{\kappa,\lambda} \cap P(D)$, select $a \in A$ so that $\{b \in B : a \subseteq b\} = \phi$, and $w \in P_\kappa(\lambda)$ so that $\{c \in C - Z_a : w \subseteq c\} = \phi$. Then $\{z \in g``B : w \subseteq z\} = \phi$ since for every $b \in B$, $g(b) \subseteq f(b) \in C - Z_a$. Thus, $g``B \in I_{\kappa,\lambda}$. $\square$

9. Negative results.

This section presents some negative results concerning combinatorial properties of $P_\kappa(\lambda)$ considered above. Several of these results rely on the fact that restrictions of $I_{\kappa,\lambda}$ may have some degree of normality. It is straightforward to check the following:

Lemma 9.1. *Suppose $C \in I^+_{\kappa,\lambda}$ and θ, ν are two cardinals such that $\omega \leq \theta \leq \kappa \leq \nu \leq \lambda$. Then the following are equivalent:*

(i) *$I_{\kappa,\lambda}|C$ is $[\nu]^{<\theta}$-normal.*

(ii) $I_{\kappa,\lambda}|C = NS_{\kappa,\lambda}^{[\nu]^{<\theta}}|C$.

Let us first describe a situation when $\{A : \mathfrak{p}_{I_{\kappa,\lambda}|A} > \kappa\}$ is not dense in $(I_{\kappa,\lambda}^+, \subseteq)$.

Proposition 9.2. *Suppose $C \in I_{\kappa,\lambda}^+$ is such that $I_{\kappa,\lambda}|C$ is κ-normal. Then $\mathfrak{p}_{I_{\kappa,\lambda}|A} = \kappa$ for every $A \in I_{\kappa,\lambda}^+ \cap P(C)$.*

Proof. Let $B \in (NS_{\kappa,\lambda}^\kappa)^+$ and set $H = NS_{\kappa,\lambda}^\kappa|B$. Then $\mathfrak{p}_H = \kappa$ by Corollary 3.3 and Proposition 3.4 of [**27**]. □

The next results are about the unbalanced partition relation $I_{\kappa,\lambda}^+ \to (I_{\kappa,\lambda}^+, \theta)^2$. The following technical observation is crucial:

Lemma 9.3. *Suppose H is a ν-normal ideal on $P_\kappa(\lambda)$ such that $H^+ \to (H^+, \theta^+)^2$, where θ, ν are two infinite cardinals with $\theta < \kappa \leq \nu \leq \lambda$. Then H is $[\nu]^{<\theta^+}$-normal.*

Proof. By Proposition 5.9 of [**18**] and Lemma 6.4. □

Proposition 9.4. *Suppose H is a κ-normal ideal on $P_\kappa(\lambda)$ such that $H^+ \to (H^+, \theta^+)^2$, where θ is an infinite cardinal $< \kappa$. Then*

$$\{a \in P_\kappa(\lambda) : cf\big(\cup(a \cap \kappa)\big) \leq \theta\} \in H.$$

Proof. By Lemmas 6.3 and 9.3. □

Corollary 9.5. *Suppose $I_{\kappa,\lambda}|C$ is κ-normal for some $C \in \big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$, where θ is a regular infinite cardinal $< \kappa$. Then $I_{\kappa,\lambda}^+ \not\to (I_{\kappa,\lambda}^+, \theta^+)^2$.*

Proof. Setting $A = \{a \in P_\kappa(\lambda) : cf\big(\cup(a \cap \kappa)\big) = \theta\}$, it is simple to check that $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^+$. It follows that $A \cap C \in I_{\kappa,\lambda}^+$. Now by Proposition 9.4,

$$\big(I_{\kappa,\lambda}|(A \cap C)\big)^+ \not\to (I_{\kappa,\lambda}^+, \theta^+)^2. \qquad \square$$

In particular, if $I_{\kappa,\lambda}|C$ is κ-normal for some $C \in NS_{\kappa,\lambda}^*$, then $I_{\kappa,\lambda}^+ \not\to (I_{\kappa,\lambda}^+, \omega_1)^2$. A convenient reformulation of the hypothesis of Corollary 9.5 is supplied by the following:

Lemma 9.6 ([**24**]). *Suppose θ is a regular infinite cardinal $\leq \kappa$ and ν is a cardinal with $\kappa \leq \nu \leq \lambda$. Then the following are equivalent*:

(i) *$I_{\kappa,\lambda}|C$ is ν-normal for some $C \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$.*
(ii) *$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^\nu|A) \leq \lambda$ for some $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$.*

In particular, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^\kappa) \leq \lambda$ implies that $I_{\kappa,\lambda}|C$ is κ-normal for some $C \in NS_{\kappa,\lambda}^*$.

Definition. $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$ is the least size of any $X \subseteq P_{\kappa^+}(\lambda)$ such that $P_{\kappa^+}(\lambda) = \bigcup_{x \in P_\kappa(X)} P(\cup x)$.

Lemma 9.7 ([**26**]). $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^\kappa) = \max\{\overline{\mathrm{cof}}(NS_\kappa), \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)\}$.

By a result of [**24**], it follows that if $\overline{\mathrm{cof}}(NS_\kappa) \leq \lambda < \kappa^{+\kappa}$, then $\overline{\mathrm{cof}}(NS^\kappa_{\kappa,\lambda}) = \lambda$.

If $cf(\lambda) = \kappa$, then for every $C \in NS^*_{\kappa,\lambda}$, $I_{\kappa,\lambda}|C$ is not κ-normal:

PROPOSITION 9.8 ([**24**]). *Suppose that $cf(\lambda) = \kappa$ and θ is an infinite cardinal $< \kappa$ such that $\tau^{<\theta} < \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$. Then for every $C \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^*$, $I_{\kappa,\lambda}|C$ is not κ-normal.*

PROPOSITION 9.9. *Suppose $\mathfrak{d}_\kappa \leq \lambda$ and $u(\kappa^+, \mu) \leq \lambda$ for every cardinal μ with $\kappa < \mu < \lambda$. Then $I^+_{\kappa,\lambda} \not\to (I^+_{\kappa,\lambda})^2$.*

PROOF. For the case $\kappa = \omega = cf(\lambda)$ (respectively, $\kappa = \omega < cf(\lambda)$, $\omega < \kappa = cf(\lambda)$), see [**19**] (respectively, [**22**], [**21**]). Assuming now $\omega < \kappa$ and $cf(\lambda) \neq \kappa$, we have $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda$ since by results of [**31**], (1) if λ is a successor cardinal, say $\lambda = \tau^+$, then $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda \cdot \mathrm{cov}(\tau, \kappa^+, \kappa^+, \kappa)$, and (2) if λ is a limit cardinal, then $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \bigcup_{\kappa<\mu<\lambda} \mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa)$. It follows from Landver's result [**16**] that $\mathrm{cof}(NS_\kappa) = \mathfrak{d}_\kappa$ and Lemma 9.7 that $\overline{\mathrm{cof}}(NS^\kappa_{\kappa,\lambda}) = \lambda$. Hence by Corollary 9.5 and Lemma 9.6, $I^+_{\kappa,\lambda} \not\to (I^+_{\kappa,\lambda}, \omega_1)^2$. □

The results above do not settle the problem whether $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda}, \omega)^2$. The partition relation $H^+ \to (H^+, \omega)^2$ has the following interesting consequence:

PROPOSITION 9.10. *Suppose H is an ideal on $P_\kappa(\lambda)$ such that $H^+ \to (H^+, \omega)^2$ and $A \in H^+$. Then there exists $B \in H^+ \cap P(A)$ with the property that there is no infinite strictly decreasing sequence*

$$a_0 \supsetneq a_1 \supsetneq a_2 \supsetneq \dots$$

of elements of B.

PROOF. Let $j : A \to |A|$ be a bijection. Since $H^+ \to (H^+, \omega)^2$, there is $B \in H^+ \cap P(A)$ such that $j(a) < j(b)$ for every $(a, b) \in [B]^2$. Clearly B is as desired. □

Johnson established the existence of a $C \in I^+_{\kappa,\lambda}$ such that $(I_{\kappa,\lambda}|C)^+ \to (I^+_{\kappa,\lambda}, \omega)^2$ subject to some cardinality assumptions:

PROPOSITION 9.11. *Suppose λ is regular and $\mathrm{cof}(NS^\mu_{\kappa,\lambda}) \leq \lambda$ for every cardinal μ with $\kappa \leq \mu < \lambda$. Then setting $H = \bigcup_{\kappa\leq\delta<\lambda} NS^\delta_{\kappa,\lambda}$, (a) H is weakly selective, (b) $H^+ \to (H^+, \omega + 1)^2$ and (c) For every $A \in H^+$, there is $C \in H^+ \cap P(A)$ such that $H|C = I_{\kappa,\lambda}|C$.*

PROOF. By Theorems 1.7, 1.9 and 1.12 of [**13**] and 1.6 of [**14**]. □

Baumgartner, Carr and Di Prisco have independently shown that $\{P_\kappa(\lambda)\} \to (I^+_{\kappa,\lambda})^3$ implies that κ is mildly λ-ineffable (see [**6**], page 183). This result can be slightly improved:

PROPOSITION 9.12. *Suppose $\{P_\kappa(\lambda)\} \to (I^+_{\kappa,\lambda})^3$. Then κ is mildly $\lambda^{<\kappa}$-ineffable.*

PROOF. Note that κ must be weakly compact and hence inaccessible. Now let $t_x \in {}^x2$ for $x \in P_\kappa(P_\kappa(\lambda))$. Define $F : [P_\kappa(\lambda)]^3 \to 2$ by stipulating that $F(a,b,c) = 0$ if and only if $t_{P(b)} \restriction P(a) = t_{P(c)} \restriction P(a)$. Pick $A \in I^+_{\kappa,\lambda}$ so that F is constant on $[A]^3$. Then clearly F is identically 0 on $[A]^3$. If $(a,b),(a',b') \in [A]^2$, then

$$t_{P(b)} \restriction \big(P(a) \cap P(a')\big) = t_{P(c)} \restriction \big(P(a) \cap P(a')\big) = t_{P(b')} \restriction \big(P(a) \cap P(a')\big),$$

where c is any member of A such that $b \subsetneq c$ and $b' \subsetneq c$. So we can define $g : P_\kappa(\lambda) \to 2$ by $g = \bigcup_{(a,b)\in[A]^2}(t_{P(b)} \restriction P(a))$. Now fix $x \in P_\kappa(P_\kappa(\lambda))$. Given $y \in P_\kappa\big(P_\kappa(\lambda)\big)$, select $(a,b) \in [A]^2$ so that $x \cup y \subseteq P(a)$. Then $y \subseteq P(b)$ and

$$g \restriction x = \big(t_{P(b)} \restriction P(a)\big) \restriction x = t_{P(b)} \restriction x. \qquad \square$$

As was pointed out by the referee, Proposition 9.12 can also be obtained by using Abe's result (see [**1**], Corollary 4.5 (1)) that $\{P_\kappa(\lambda)\} \to (I^+_{\kappa,\lambda})^3$ implies $\{P_\kappa(\lambda^{<\kappa})\} \to (I^+_{\kappa,\lambda^{<\kappa}})^3$.

Our last observation concerns the failure of $S_\kappa(\lambda)$. The following is a straightforward generalization of a result of Galvin (see [**38**], Theorem 5.1, and also Theorem 5.2, Corollary 5.4 and Theorem 5.9.).

PROPOSITION 9.13. *Suppose $\mathfrak{K}_{\kappa,\lambda} \neq \phi$ and $\mathfrak{d}_\kappa \leq u(\kappa,\lambda) = u(\kappa^+,\lambda)$. Then $S_\kappa(\lambda)$ fails.*

PROOF. Fix $Z \in \mathfrak{K}_{\kappa,\lambda}$. Note that $|Z| = u(\kappa,\lambda)$ by Proposition 2.2. Let $Y \in I^+_{\kappa^+,\lambda}$ be such that $Y \subseteq \{y \in P_{\kappa^+}(\lambda) : |y| = \kappa\}$ and $|Y| = u(\kappa,\lambda)$. For $y \in Y$, select a bijection $i_y : \kappa \to y$. Let $F \subseteq {}^\kappa\kappa$ be such that $|F| = \mathfrak{d}_\kappa$ and for every $s \in {}^\kappa\kappa$, there is $t \in F$ with the property that $s(\alpha) \leq t(\alpha)$ for all $\alpha < \kappa$. Pick a bijection $j : Y \times F \to Z$. For $z \in Z$, define $h_z : \kappa \to P_\kappa(\lambda)$ by $h_z(\alpha) = \{i_y(\xi) : \xi < t(\alpha)\}$, where y and t are such that $j(y,t) = z$. It is simple to check that for every $f : \kappa \to P_\kappa(\lambda)$, there is $z \in Z$ such that $f(\delta) \subseteq h_z(\delta)$ for all $\delta \in \kappa$. Define $g : P_\kappa(\lambda) \to P_\kappa(\lambda)$ by

$$g(a) = \bigcup_{z \in Z \cap P(a)} h_z\big((\cup(a \cap \kappa)) + 1\big).$$

Now fix $A \in I^+_{\kappa,\lambda}$. For $\delta \in \kappa$, set $B_\delta = \{a \in A : \delta \notin a\}$. Suppose that $g``B_\delta \in I_{\kappa,\lambda}$ for all $\delta \in \kappa$. Then there is $z \in Z$ such that

$$g``B_\delta \cap \{b \in P_\kappa(\lambda) : h_z(\delta) \subseteq b\} = \phi$$

for every $\delta \in \kappa$. Select $a \in A$ so that $z \subseteq a$, and stipulate that $\delta = (\cup(a \cap \kappa)) + 1$. Then clearly $g(a) \in g``B_\delta$ and $h_z(\delta) \subseteq g(a)$. Contradiction. $\square$

References

[1] Y. Abe, Saturation of fundamental ideals on $\mathscr{P}_\kappa\lambda$, J. Math. Soc. Japan, **48** (1996), 511–524.

[2] Y. Abe, Combinatorics for small ideals on $P_\kappa\lambda$, Mathematical Logic Quarterly, **43** (1997), 541–549.

[3] C. Alves, Partitions of Finite Substructures, Ph. D. Thesis, Pennsylvania State Univ., 1985.

[4] A. W. Apter and M. Gitik, The least measurable can be strongly compact and indestructible, J. Symbolic Logic, **63** (1998), 1404–1412.

[5] J. P. Burgess, Forcing, In: Handbook of Mathematical Logic, (ed. J. Barwise), North-Holland, Amsterdam, 1977, pp. 403–452.

[6] D. M. Carr, $P_\kappa\lambda$ partition relations, Fund. Math., **128** (1987), 181–195.

[7] D. M. Carr, J. P. Levinski and D. H. Pelletier, On the existence of strongly normal ideals over $P_\kappa\lambda$, Arch. Math. Logic, **30** (1990), 59–72.

[8] J. Cummings, M. Foreman and M. Magidor, Squares, scales and stationary reflection, J. Math. Log., **1** (2001), 35–98.

[9] C. A. Di Prisco and W. S. Zwicker, Flipping properties and supercompactness, Fund. Math., **109** (1980), 31–36.

[10] H. D. Donder and P. Matet, Two cardinal versions of diamond, Israel J. Math., **83** (1993), 1–43.

[11] P. Erdös, A. Hajnal, A. Máté and R. Rado, Combinatorial Set Theory: Partition Relations for Cardinals, Stud. Logic Found. Math., **106**, North-Holland, Amsterdam, 1984.

[12] T. Jech and S. Shelah, A partition theorem for pairs of finite sets, J. Amer. Math. Soc., **4** (1991), 647–656.

[13] C. A. Johnson, Seminormal λ-generated ideals on $P_\kappa\lambda$, J. Symbolic Logic, **53** (1988), 92–102.

[14] C. A. Johnson, Some partition relations for ideals on $P_\kappa\lambda$, Acta Math. Hungar., **56** (1990), 269–282.

[15] K. Kunen, Set Theory, North-Holland, Amsterdam, 1980.

[16] A. Landver, Singular Baire numbers and related topics, Ph. D. Thesis, Univ. of Wisconsin-Madison, 1990.

[17] A. Landver, Baire numbers, uncountable Cohen sets and perfect-set forcing, J. Symbolic Logic, **57** (1992), 1086–1107.

[18] P. Matet, Partition relations for strongly normal ideals on $P_\kappa(\lambda)$, Mathematical Logic Quarterly, **46** (2000), 87–103.

[19] P. Matet, Negative partition relations for uncountable cardinals of cofinality ω, Studia Sci. Math. Hungar., **37** (2001), 233–236.

[20] P. Matet, Partition relations for κ-normal ideals on $P_\kappa(\lambda)$, Ann. Pure Appl. Logic, **121** (2003), 89–111.

[21] P. Matet, Weak square bracket relations for $P_\kappa(\lambda)$, preprint.

[22] P. Matet and C. Péan, Distributivity properties on $P_\omega(\lambda)$, Discrete Math., **291** (2005), 143–154.

[23] P. Matet, C. Péan and S. Shelah, Cofinality of normal ideals on $P_\kappa(\lambda)$ I, preprint.

[24] P. Matet, C. Péan and S. Shelah, Cofinality of normal ideals on $P_\kappa(\lambda)$ II, Israel J. Math., to appear.

[25] P. Matet, C. Péan and S. Todorcevic, Prime ideals on $P_\omega(\lambda)$ with the partition property, Arch. Math. Logic, **41** (2002), 743–764.

[26] P. Matet, A. Rosłanowski and S. Shelah, Cofinality of the nonstationary ideal, Trans. Amer. Math. Soc., to appear.

[27] P. Matet and S. Shelah, Positive partition relations for $P_\kappa(\lambda)$, preprint.

[28] A. W. Miller, The Baire category theorem and cardinals of countable cofinality, J. Symbolic Logic, **47** (1982), 275–288.

[29] R. Rado, Axiomatic treatment of rank in infinite sets, Canad. J. Math., **1** (1949), 337–343.

[30] F. Rothberger, Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C, Math. Proc. Cambridge Philos. Soc., **37** (1941), 109–126.

[31] S. Shelah, Cardinal Arithmetic, Oxford Logic Guides, **29**, Oxford Univ. Press, Oxford, 1994.

[32] M. Shioya, Partition properties of subsets of $P_\kappa\lambda$, Fund. Math., **161** (1999), 325–329.

[33] R. M. Solovay, Real-valued measurable cardinals, In: Axiomatic Set Theory, (ed. D. S. Scott), Proc. Sympos. Pure Math., **13**, part 1, Amer. Math. Soc., Providence, 1971, pp. 397–428.

[34] S. Todorcevic, Kurepa families and cofinal similarities, handwritten notes 1989.

[35] S. Todorcevic, Cofinal Kurepa families, handwritten notes, 1990.

[36] S. Todorcevic, Partitioning pairs of countable sets, Proc. Amer. Math. Soc., **111** (1991), 841–844.

[37] S. Todorcevic, Coherent sequences, In: Handbook of Set Theory, (eds. M. Foreman, A. Kanamori and M. Magidor), Kluwer, Dordrecht, to appear.

[38] W. Zwicker, A beginning for structural properties of ideals on $P_\kappa\lambda$, In: Set Theory and its Applications (eds. J. Steprāns and S. Watson), Lecture Notes in Math., **1401**, Springer, Berlin, 1989, pp. 201–217.

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