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Cartan matrices and Brauer's $k(B)$-conjecture IV

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Abstract. In this note we give applications of recent results coming mostly from the third paper of this series. It is shown that the number of irreducible characters in a p-block of a finite group with abelian defect group D is bounded by $|D|$ (Brauer's $k(B)$-conjecture) provided D has no large elementary abelian direct summands. Moreover, we verify Brauer's $k(B)$-conjecture for all blocks with minimal non-abelian defect groups. This extends previous results by various authors.

1. Introduction.

Let p be a prime and let G be a finite group. We consider p-blocks B of G with respect to a p-modular system which is "large enough" in the usual sense. In two recent articles [**32**], [**43**] properties of the Cartan matrix C of B have been expressed in terms of the defect group D of B. In the present paper we apply these results in order to prove the inequality $k(B) \leq |D|$ (Brauer's $k(B)$-conjecture) in certain cases where $k(B)$ denotes the number of irreducible characters in B. Continuing former work by several authors [**3**], [**6**], [**10**], [**11**], [**28**], [**44**], we verify Brauer's $k(B)$-conjecture for all blocks with minimal non-abelian defect groups. Here a group is called *minimal non-abelian* if all its proper subgroups are abelian, but the group itself is non-abelian. This leads also to a proof of Brauer's conjecture for the 5-blocks of defect 3.

In the last part of the paper we revisit a theorem of Watanabe [**40**], [**39**] about blocks with abelian defect groups. Watanabe has studied a certain correspondence of blocks whenever the inertial group has non-trivial fixed points on D (similar to the Z^*-Theorem). We will show that this correspondence often preserves Cartan matrices up to basic sets (this means up to a transformation of the form $C \mapsto SCS^{\mathrm{T}}$ for some $S \in GL(l, \mathbb{Z})$ where S^{T} denotes the transpose of S). As another tool we show that a coprime action on an abelian p-group without elementary abelian direct summands always has a regular orbit. This is used to give a proof of Brauer's $k(B)$-conjecture for abelian defect groups D such that D has no elementary abelian direct summand of order p^3. Improvements of this result for small primes are also presented. In particular, we verify Brauer's conjecture for 2-blocks with abelian defect groups of rank at most 7. This greatly generalizes some results in [**31**]. Some of the proofs rely implicitly on the classification of the finite simple groups.

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Most of our notation is standard and can be found in [**4**], [**22**], [**30**] for example. The number of irreducible Brauer characters of B is denoted by $l(B)$. Moreover, we denote the inertial quotient of B by $I(B)$. Its order $e(B) := |I(B)|$ is the inertial index of B. A cyclic group of order n is denoted by Z_n, and for convenience, $Z_n^m := Z_n \times \cdots \times Z_n$ (m copies). Commutators are defined as $[x, y] := xyx^{-1}y^{-1}$ and $H' := [H, H]$ is the commutator subgroup of $H \leq G$. Moreover, groups act from the left as ${}^a x$. We say that a finite group A acts *freely* on a finite group H if $\mathrm{C}_A(x) = 1$ for all $1 \neq x \in H$. For an abelian p-group P we set $\Omega_i(P) := \{x \in P : x^{p^i} = 1\}$ and $\Omega(P) := \Omega_1(P)$.

2. Fusion systems.

We start by recalling some notation from the theory of fusion systems. Details can be found in [**1**]. Our fusion systems will always be saturated.

Definition 1. Let $\mathcal{F}$ be a fusion system on a finite p-group P.

(i) A subgroup $Q \leq P$ is called *fully $\mathcal{F}$-centralized* if $|\mathrm{C}_P(\varphi(Q))| \leq |\mathrm{C}_P(Q)|$ for all morphisms $\varphi : Q \to P$ in $\mathcal{F}$.

(ii) If Q is fully $\mathcal{F}$-centralized, then there is a fusion system $\mathrm{C}_{\mathcal{F}}(Q)$ on $\mathrm{C}_P(Q)$ defined as follows: a group homomorphism $\varphi : R \to S$ $(R, S \leq \mathrm{C}_P(Q))$ belongs to $\mathrm{C}_{\mathcal{F}}(Q)$ if there exists a morphism $\psi : QR \to QS$ in $\mathcal{F}$ such that $\psi_{|Q} = \mathrm{id}_Q$ and $\psi_{|R} = \varphi$.

(iii) If Q is abelian and fully $\mathcal{F}$-centralized, then there is a fusion system $\mathrm{C}_{\mathcal{F}}(Q)/Q$ on $\mathrm{C}_P(Q)/Q$ defined as follows: a group homomorphism $\varphi : R/Q \to S/Q$ $(Q \leq R, S \leq \mathrm{C}_P(Q))$ belongs to $\mathrm{C}_{\mathcal{F}}(Q)/Q$ if there exists a morphism $\psi : R \to S$ in $\mathrm{C}_{\mathcal{F}}(Q)$ such that $\psi(u)Q = \varphi(uQ)$ for all $u \in R$.

If in the situation of Definition 1 the group Q is cyclic, say $Q = \langle u \rangle$, then we write $\mathrm{C}_{\mathcal{F}}(u)$ instead of $\mathrm{C}_{\mathcal{F}}(\langle u \rangle)$.

Let B be a block of a finite group G with defect group D. Recall that a (B-)*subsection* is a pair (u, b_u) such that $u \in D$ and b_u is a Brauer correspondent of B in $\mathrm{C}_G(u)$. If b_u and B have the same defect, the subsection is called *major*. This holds for example for the *trivial* subsection $(1, B)$. More generally, a (B-)*subpair* is a pair (Q, b_Q) such that $Q \leq D$ and b_Q is a Brauer correspondent of B in $\mathrm{C}_G(Q)$. In case $Q = D$, we say (D, b_D) is a Sylow B-subpair. It is well-known that every block B of a finite group with defect group D determines a fusion system $\mathcal{F}$ on D which describes the conjugation of subpairs. In this setting, $I(B) \cong \mathrm{Out}_{\mathcal{F}}(D)$. By the Schur–Zassenhaus Theorem we can consider $I(B)$ as a subgroup of $\mathrm{Aut}(D)$.

The next lemma might be already known, but we were unable to find a reference (cf. [**24**, Theorem 1.5]). Therefore a proof is given.

Lemma 2. *Let B be a p-block of a finite group G with defect group D and fusion system $\mathcal{F}$. Let $Z \leq \mathrm{Z}(G)$ be a p-subgroup. Then B dominates a unique block $\overline{B}$ of G/Z with defect group D/Z and fusion system $\mathcal{F}/Z$.*

Proof. Since $Z \trianglelefteq G$, we have $Z \leq D$. Moreover, it is easy to see that $\mathcal{F} = \mathrm{C}_{\mathcal{F}}(Z)$. Hence, $\mathcal{F}/Z$ is well defined. The uniqueness of $\overline{B}$ and its defect group can be found in

[**22**, Theorem 5.8.11]. It remains to determine the fusion system of $\overline{B}$. For $H \le G$ we write $\overline{H} := HZ/Z$. We fix a Sylow B-subpair (D, b_D). For every subgroup $Z \le Q \le D$ there exists a unique B-subpair (Q, b_Q) such that $(Q, b_Q) \le (D, b_D)$. Let $\mathrm{C}_{\overline{G}}(\overline{Q}) = \overline{C_Q}$ with $\mathrm{C}_G(Q) \le C_Q \le \mathrm{N}_G(Q)$. By the definition of the Brauer correspondence (see [**22**, Section 5.3]), $\beta_Q := b_Q^{C_Q}$ is well defined. Let $\overline{\beta_Q}$ be the unique block of $\mathrm{C}_{\overline{G}}(\overline{Q})$ dominated by β_Q. We claim that $(\overline{Q}, \overline{\beta_Q})$ is a $\overline{B}$-subpair. To prove this, we need to show that $\overline{\beta_Q}^{\overline{G}} = \overline{B}$. Let e_B be the block idempotent of B with respect to an algebraically closed field F of characteristic p. Let $\theta : FG \to F\overline{G}$ be the canonical epimorphism. Then $\theta(e_B) = e_{\overline{B}}$ by [**22**, Theorem 5.8.11]. Let ω_{β_Q} be the central character of β_Q. Then, by [**22**, Lemma 5.8.5], the central character $\omega_{\overline{\beta_Q}}$ of $\overline{\beta_Q}$ satisfies $\omega_{\beta_Q} = \omega_{\overline{\beta_Q}} \circ \theta$ where θ is identified with its restriction to $\mathrm{Z}(FC_Q)$. Let

$$\eta : \mathrm{Z}(FG) \to \mathrm{Z}(FC_Q), \ \sum_{g \in G} \alpha_g g \mapsto \sum_{g \in C_Q} \alpha_g g \qquad (\alpha_g \in F).$$

Then the analogous map $\overline{\eta} : \mathrm{Z}(F\overline{G}) \to \mathrm{Z}(F\,\mathrm{C}_{\overline{G}}(\overline{Q}))$ is the Brauer homomorphism. Moreover,

$$\omega_{\overline{\beta_Q}}(\overline{\eta}(e_{\overline{B}})) = \omega_{\overline{\beta_Q}}(\overline{\eta}(\theta(e_B))) = \omega_{\overline{\beta_Q}}(\theta(\eta(e_B))) = \omega_{\beta_Q}(\eta(e_B)) = \omega_B(e_B) = 1.$$

This shows that $\overline{\beta_Q}^{\overline{G}} = \overline{B}$ and $(\overline{Q}, \overline{\beta_Q})$ is a $\overline{B}$-subpair. In particular, $(\overline{D}, \overline{\beta_D})$ is a Sylow $\overline{B}$-subpair. Suppose that $(R, b_R) \trianglelefteq (S, b_S)$ for some subgroups $Z \le R \trianglelefteq S \le D$. Then $b_R^{\mathrm{C}_G(R)S} = b_S^{\mathrm{C}_G(R)S}$. As we have seen above,

$$\overline{\beta_R}^{\mathrm{C}_{\overline{G}}(\overline{R})\overline{S}} = \overline{\beta_R}^{\overline{C_R S}} = \overline{\beta_R^{C_R S}} = \overline{b_R^{C_R S}} = \overline{b_S^{C_R S}} = \overline{\beta_S^{C_R S}} = \overline{\beta_S}^{\overline{C_R S}} = \overline{\beta_S}^{\mathrm{C}_{\overline{G}}(\overline{R})\overline{S}}$$

(observe that $\mathrm{C}_G(R)S \le C_R S \le G$). This implies $(\overline{R}, \overline{\beta_R}) \trianglelefteq (\overline{S}, \overline{\beta_S})$. Therefore the poset of B-subpairs $(Q, b_Q) \le (D, b_D)$ such that $Z \le Q$ is in one-to-one correspondence with the poset of $\overline{B}$-subpairs via Brauer correspondence and θ. Let $\mathcal{F}'$ be the fusion system of $\overline{B}$. Suppose that $\overline{\varphi} : \overline{R} \to \overline{S}$ is a morphism in $\mathcal{F}'$ for $Z \le R, S \le D$. Then there exists a $g \in G$ such that $\overline{g}(\overline{R}, \overline{\beta_R})\overline{g}^{-1} \le (\overline{S}, \overline{\beta_S})$ and $\overline{\varphi}(\overline{x}) = \overline{gxg}^{-1}$ for all $\overline{x} \in \overline{R}$. Obviously, we have $gRg^{-1} \le S$. Moreover, $\overline{g\beta_R g^{-1}} = \overline{g}\overline{\beta_R}\overline{g}^{-1} = \overline{\beta_{gRg^{-1}}}$ and

$$(gb_R g^{-1})^{C_{gRg^{-1}}} = g(b_R^{C_R})g^{-1} = g\beta_R g^{-1} = \beta_{gRg^{-1}} = b_{gRg^{-1}}^{C_{gRg^{-1}}}.$$

It follows that there exists an element $h \in C_{gRg^{-1}} \le \mathrm{N}_G(gRg^{-1})$ such that $hgb_R g^{-1}h^{-1} = b_{gRg^{-1}}$ and $\overline{\varphi}(\overline{x}) = \overline{hgxg^{-1}h^{-1}}$ for $\overline{x} \in \overline{R}$. Therefore, $hg(R, b_R)g^{-1}h^{-1} \le (S, b_S)$ and the map $\varphi : R \to S$ such that $\varphi(x) := hgxg^{-1}h^{-1}$ for $x \in R$ is a morphism in $\mathcal{F}$. Conversely, if $\varphi : R \to S$ is given in $\mathcal{F}$, then it is easy to see that the corresponding map $\overline{\varphi}$ lies in $\mathcal{F}'$. Consequently, $\mathcal{F}' = \mathcal{F}/Z$. □

LEMMA 3. *Let B be a block of a finite group G with defect group D and fusion system $\mathcal{F}$. Let (u, b) be a B-subsection such that $\langle u \rangle$ is fully $\mathcal{F}$-centralized. Then b has defect group $\mathrm{C}_D(u)$ and fusion system $\mathrm{C}_{\mathcal{F}}(u)$. Moreover, b dominates a unique block $\overline{b}$ of $\mathrm{C}_G(u)/\langle u \rangle$ with defect group $\mathrm{C}_D(u)/\langle u \rangle$ and fusion system $\mathrm{C}_{\mathcal{F}}(u)/\langle u \rangle$. In particular,* *we have canonical isomorphisms*

$$I(\overline{b}) \cong I(b) \cong \mathrm{C}_{\mathrm{Out}_{\mathcal{F}}(\mathrm{C}_D(u))}(u).$$

If $\overline{b}$ *has Cartan matrix* $\overline{C}$*, then* b *has Cartan matrix* $|\langle u\rangle|\overline{C}$*. In particular,* $l(b) = l(\overline{b})$*.*

PROOF. The first claim follows from [**1**, Theorem IV.3.19]. The uniqueness of $\overline{b}$ and the claim about the Cartan matrices can be found in [**22**, Theorem 5.8.11]. The fusion system of $\overline{b}$ was determined in Lemma 2. It is well-known that the inertial quotient $I(b) \cong \mathrm{Out}_{\mathrm{C}_{\mathcal{F}}(u)}(\mathrm{C}_D(u))$ is a p'-group. Thus, [**20**, Theorem 6.3(i)] implies $\mathrm{Out}_{\mathrm{C}_{\mathcal{F}}(u)}(\mathrm{C}_D(u)) \cong \mathrm{Out}_{\mathrm{C}_{\mathcal{F}}(u)/\langle u\rangle}(\mathrm{C}_D(u)/\langle u\rangle) \cong I(\overline{b})$. Finally, the isomorphism $I(b) \cong \mathrm{C}_{\mathrm{Out}_{\mathcal{F}}(\mathrm{C}_D(u))}(u)$ follows from the definition of $\mathrm{C}_{\mathcal{F}}(u)$. □

We also recall two important subgroups related to fusion systems.

DEFINITION 4. Let $\mathcal{F}$ be a fusion system on a finite p-group P.

(i) $\mathfrak{foc}(\mathcal{F}) := \langle f(x)x^{-1} : x \in Q \le P,\ f \in \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$ is called the *focal subgroup* of $\mathcal{F}$.

(ii) $\mathrm{Z}(\mathcal{F}) := \{x \in P : x$ is fixed by every morphism in $\mathcal{F}\}$ is called the *center* of $\mathcal{F}$.

If B is a block with fusion system $\mathcal{F}$ and defect group D, then we set $\mathfrak{foc}(B) := \mathfrak{foc}(\mathcal{F})$ (but $\mathrm{Z}(B)$ is usually used for the center of the block algebra). We say that B is *controlled* if all morphisms of $\mathcal{F}$ are generated by restrictions from $\mathrm{Aut}_{\mathcal{F}}(D)$. In this case, $\mathfrak{foc}(B) = [D, I(B)]$ and $\mathrm{Z}(\mathcal{F}) = \mathrm{C}_D(I(B))$. If D is abelian, then B is controlled and $D = [D, I(B)] \oplus \mathrm{C}_D(I(B))$ (see [**8**, Theorem 2.3]).

3. Non-abelian defect groups.

THEOREM 5. *Let* B *be a* p*-block of a finite group with non-abelian defect group* D*. Suppose that* $D/\langle z\rangle$ *is abelian of rank* 2 *for some* $z \in \mathrm{Z}(D)$*. Then* $k(B) \le |D|$*.*

PROOF. Let $x, y \in D$ such that $D = \langle x, y, z\rangle$. Since D is non-abelian, $1 \ne [x, y] \in D' \subseteq \langle z\rangle$. Let $\alpha \in \mathrm{C}_{\mathrm{Aut}(D)}(z)$ be a p'-automorphism. We write $\alpha(x) \equiv x^i y^j \pmod{\langle z\rangle}$ and $\alpha(y) \equiv x^k y^l \pmod{\langle z\rangle}$ with $i, j, k, l \in \mathbb{Z}$. By [**12**, III.1.2, III.1.3],

$$[x, y] = \alpha([x, y]) = [x^i y^j, x^k y^l] = [x, y]^{il-jk}$$

and therefore $il - jk \equiv 1 \pmod p$. Hence, α corresponds to a matrix with determinant 1 under the isomorphism $\mathrm{Aut}(D/\langle x^p, y^p, z\rangle) \cong \mathrm{Aut}(Z_p^2) \cong GL(2, p)$. If x and y have the same order modulo $\langle z\rangle$, then α also corresponds to a matrix with determinant 1 under the isomorphism $\mathrm{Aut}(\Omega(D/\langle z\rangle)) \cong GL(2, p)$. Now assume, without loss of generality, that x has larger order than y modulo $\langle z\rangle$. Then $p \mid k$, since $\alpha(y)$ and y have the same order. In particular $il \equiv 1 \pmod p$. Let p^n be the order of x modulo $\langle z\rangle$. Then obviously, $\alpha(x^{p^{n-1}}) \equiv x^{ip^{n-1}} \pmod{\langle z\rangle}$. This shows that α induces an upper triangular matrix with determinant 1 in $\mathrm{Aut}(\Omega(D/\langle z\rangle))$. Hence, in any case α corresponds to an element of $SL(\Omega(D/\langle z\rangle))$.

Now suppose that α has a non-trivial fixed point in $D/\langle z\rangle$. Then there is also a non-trivial fixed point in $\Omega(D/\langle z\rangle)$. It follows that α is conjugate to a unitriangular matrix under $\mathrm{Aut}(\Omega(D/\langle z\rangle)) \cong GL(2,p)$. However, then α acts trivially on $\Omega(D/\langle z\rangle)$, since α is a p'-element. By [**8**, Theorem 5.2.4], α also acts trivially on $D/\langle z\rangle$. This forces $\alpha = 1$ by [**8**, Theorem 5.3.2]. Therefore we have shown that every p'-automorphism of $\mathrm{C}_{\mathrm{Aut}(D)}(z)$ acts freely on $D/\langle z\rangle$.

Now let $\mathcal{F}$ be the fusion system of B. Let (z, b_z) be a (major) subsection of B. Since $z \in \mathrm{Z}(D)$, the subgroup $\langle z\rangle$ is fully $\mathcal{F}$-centralized. By Lemma 3, b_z dominates a block $\overline{b_z}$ of $C_G(z)/\langle z\rangle$ with abelian defect group $\overline{D} := D/\langle z\rangle$ and inertial quotient $I(\overline{b_z}) \cong \mathrm{C}_{I(B)}(z)$. As we have seen above, $I(\overline{b_z})$ acts freely on $\overline{D}$. In particular, all non-trivial $\overline{b_z}$-subsections (u, β_u) have inertial index 1. This implies $l(\beta_u) = 1$, since $\overline{D}$ is abelian (see [**4**, Theorem V.9.13]). Let $\overline{C}$ be the Cartan matrix of $\overline{b_z}$. Then we deduce from a result of Fujii [**5**, Corollary 1] that $\det \overline{C} = |\overline{D}|$. Since $|\langle z\rangle|\overline{C}$ is the Cartan matrix of b_z, the claim follows from [**32**, Theorem 11]. □

Corollary 6. *Brauer's $k(B)$-conjecture holds for all blocks with minimal non-abelian defect groups.*

Proof. The minimal non-abelian p-groups were classified by Rédei (see [**12**, Aufgabe III.7.22]), but the present proof can go without detailed structure knowledge. Let D be a minimal non-abelian defect group of a block B. Then there are non-commuting elements $x, y \in D$. Since $\langle x, y\rangle$ is non-abelian, we have $D = \langle x, y\rangle$. Now let $u \in \Phi(D)$ and $v \in D$ be arbitrary. Then v lies in a maximal subgroup $M < D$ and so does u. Since M is abelian, it follows that $[u, v] = 1$. This shows that $\Phi(D) \subseteq \mathrm{Z}(D)$. In particular $z := [x, y] \in D' \subseteq \Phi(D) \subseteq \mathrm{Z}(D)$. Since $D/\langle z\rangle$ is abelian of rank 2, the claim follows from Theorem 5. □

Corollary 6 includes the non-abelian defect groups of order p^3. In particular, this extends results by Hendren [**11**, Theorem 4.10]. Apart from minimal non-abelian groups, Theorem 5 also applies to other groups like the central product $D_8 * Z_{2^n}$ for some $n \geq 2$ where D_8 is the dihedral group of order 8.

In [**29**, Corollary 1] we have proved that Brauer's $k(B)$-conjecture holds for the 3-blocks of defect 3. Now we can do the same for $p = 5$.

Corollary 7. *Brauer's $k(B)$-conjecture holds for the* 5*-blocks of defect at most* 3.

Proof. The abelian defect groups of order at most 5^3 have been handled in [**30**, Theorem 14.17] (see also Proposition 22 below). In the non-abelian case, Corollary 6 applies. □

Out next results concern a larger class of p-groups, but introduces restrictions on p. The proof makes use of a recent result by Watanabe [**43**].

Theorem 8. *Let $p \leq 5$, and let B be a p-block of a finite group with defect group D. Suppose that $D/\langle z\rangle$ is metacyclic for some $z \in \mathrm{Z}(D)$. Then $k(B) \leq |D|$.*

Proof. The case $p = 2$ is already known (see [**30**, Theorem 13.8]). Thus, let

$p \in \{3,5\}$. If D is abelian, then the rank of D is at most 3 and the result follows from [**30**, Theorems 14.16 and 14.17]. Now assume that D is non-abelian. If $D/\langle z\rangle$ is abelian, then Theorem 5 applies. Thus, we may assume that $D/\langle z\rangle$ is non-abelian. If $p = 3$, then the claim follows from [**30**, Proposition 8.16]. Therefore, let $p = 5$. Let (z, b_z) be a B-subsection. As before, b_z dominates a block $\overline{b_z}$ with non-abelian, metacyclic defect group $D/\langle z\rangle$. By a result of Stancu [**35**] the fusion system $\overline{\mathcal{F}_z}$ of $\overline{b_z}$ is controlled. Moreover, the possible automorphism groups $I(\overline{b_z})$ are described in a paper by Sasaki [**33**]. It follows that $\mathfrak{foc}(\overline{b_z}) = [D/\langle z\rangle, I(\overline{b_z})]$ is cyclic (for details see [**30**, proof of Theorem 8.8]). Hence, by the main result of [**43**], $l(b_z) = l(\overline{b_z}) \mid 4$. In case $l(b_z) \le 2$, the claim follows from [**30**, Theorem 4.9]. Finally, let $l(b_z) = 4$. Let $|\langle z\rangle| = 5^n$, and let C be the Cartan matrix of b_z. By [**43**, Corollary on p. 181], C has elementary divisors 5^a and $|D|$ where $|D|$ occurs with multiplicity 1 and $a \ge n$. Choose a basic set such that C has block form

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

where $C_1 \in \mathbb{Z}^{r\times r}$ does not split further (for any basic set) and $r \le 4$ (possibly $r = 4$). Without loss of generality, $|D|$ is an elementary divisor of C_1. By way of contradiction, we may assume that there is a vector $0 \ne x \in \mathbb{Z}^4$ such that $x|D|C^{-1}x^{\mathrm{T}} < 4$ (see [**4**, Theorem V.9.17]). Looking into the proof of [**4**, Theorem V.9.17] more closely, reveals that there is a character $\chi \in \mathrm{Irr}(B)$ such that the row of generalized decomposition numbers $d_\chi := (d^u_{\chi\varphi} : \varphi \in \mathrm{IBr}(b_z))$ satisfies

$$\mathrm{tr}\big(d_\chi |D| C^{-1} \overline{d_\chi}^{\mathrm{T}}\big) < 4[\mathbb{Q}(\zeta) : \mathbb{Q}] = 16 \cdot 5^{n-1}$$

where ζ is a primitive 5^n-th root of unity and tr is the trace of the Galois extension $\mathbb{Q}(\zeta)|\mathbb{Q}$. We may write $d_\chi = (d_1, d_2)$ where $d_1 \in \mathbb{C}^r$ and $d_2 \in \mathbb{C}^{4-r}$. Then

$$\mathrm{tr}\big(d_\chi |D| C^{-1} \overline{d_\chi}^{\mathrm{T}}\big) = \mathrm{tr}\big(d_1 |D| C_1^{-1} \overline{d_1}^{\mathrm{T}}\big) + \mathrm{tr}\big(d_2 |D| C_2^{-1} \overline{d_2}^{\mathrm{T}}\big).$$

Since all entries of $|D|C_2^{-1}$ are divisible by 5, it follows that $\mathrm{tr}\big(d_2 |D| C_2^{-1} \overline{d_2}^{\mathrm{T}}\big) \ge 5\varphi(5^n) = 20 \cdot 5^{n-1}$ or $\mathrm{tr}\big(d_2 |D| C_2^{-1} \overline{d_2}^{\mathrm{T}}\big) = 0$. The first case is impossible. Hence, $d_2 = 0 \in \mathbb{Z}^{4-r}$. Since d_χ consists of algebraic integers, we may write

$$d_\chi = \sum_{i=0}^{\varphi(5^n)-1} a_i \zeta^i$$

for some $a_i \in \mathbb{Z}^4$. Let us write $Q(x, y) := x|D|C^{-1}\overline{y}^{\mathrm{T}}$ for $x, y \in \mathbb{C}^4$. Then Q is a positive definite Hermitian form. Moreover,

$$\alpha := Q(d_\chi, d_\chi) = a_0^* + \sum_{i=1}^{2\cdot 5^{n-1}-1} a_i^* (\zeta^i + \zeta^{-i})$$

for some $a_i^* \in \mathbb{Z}$. Since $\zeta^{2\cdot 5^{n-1}} + \zeta^{-2\cdot 5^{n-1}} = -1 - \zeta^{5^{n-1}} - \zeta^{-5^{n-1}}$, we get

$$a_0^* = \sum_{i=0}^{\varphi(5^n)-1} Q(a_i, a_i) - \sum_{\substack{0 \le s < t < \varphi(5^n), \\ t-s \equiv \pm 2\cdot 5^{n-1} \pmod{5^n}}} Q(a_s, a_t) > 0$$

and

$$a_{5^{n-1}}^* = \sum_{\substack{0 \le s < t < \varphi(5^n), \\ t-s \equiv 5^{n-1} \pmod{5^n}}} Q(a_s, a_t) - \sum_{\substack{0 \le s < t < \varphi(5^n), \\ t-s \equiv \pm 2\cdot 5^{n-1} \pmod{5^n}}} Q(a_s, a_t).$$

Suppose for the moment that χ has positive height. Then the 5-adic valuation of α is strictly larger than 1 (see [**30**, Proposition 1.36]). In particular, $\alpha/5$ is an algebraic integer (this can be seen by going over to the cyclotomic field over the 5-adic numbers, see [**23**, Proposition II.7.13]). Since $1, \zeta + \zeta^{-1}, \ldots, \zeta^{2\cdot 5^{n-1}-1} + \zeta^{-2\cdot 5^{n-1}+1}$ is a basis for the ring of real algebraic integers, we have $5 \mid a_i^*$ for all i. Moreover,

$$\operatorname{tr}(\alpha) = a_0^*\varphi(5^n) + \sum_{i=1}^{2\cdot 5^{n-1}-1} a_i^* \operatorname{tr}(\zeta^i + \zeta^{-i}) = a_0^*\varphi(5^n) - 2\cdot 5^{n-1} a_{5^{n-1}}^*.$$

If $a_{5^{n-1}}^* \le 0$, then we obtain the contradiction $\operatorname{tr}(\alpha) \ge a_0^*\varphi(5^n) \ge 20\cdot 5^{n-1}$. Thus, $a_{5^{n-1}}^* > 0$. Observe that

$$\begin{aligned} a_{5^{n-1}}^* &\le \frac{1}{2}\sum_{i=0}^{5^{n-1}-1} Q(a_i, a_i) + \sum_{i=5^{n-1}}^{3\cdot 5^{n-1}-1} Q(a_i, a_i) \\ &\quad + \frac{1}{2}\sum_{i=3\cdot 5^{n-1}}^{4\cdot 5^{n-1}-1} Q(a_i, a_i) - \sum_{\substack{0 \le s < t < \varphi(5^n), \\ t-s \equiv \pm 2\cdot 5^{n-1} \pmod{5^n}}} Q(a_s, a_t) \\ &= a_0^* - \frac{1}{2}\sum_{i=0}^{5^{n-1}-1} Q(a_i, a_i) - \frac{1}{2}\sum_{i=3\cdot 5^{n-1}}^{4\cdot 5^{n-1}-1} Q(a_i, a_i). \end{aligned}$$

Now it is easy to see that $a_0^* > a_{5^{n-1}}^*$ and thus $a_0^* \ge a_{5^{n-1}}^* + 5$. This gives the contradiction $\operatorname{tr}(\alpha) \ge 20\cdot 5^{n-1} + 2\cdot 5^{n-1} a_{5^{n-1}}^* \ge 20\cdot 5^{n-1}$. Therefore, we have shown that χ has height 0.

In particular, $d_\chi |D| C^{-1} \overline{d_\psi}^{\mathrm{T}} \ne 0$ for all $\psi \in \operatorname{Irr}(B)$ (see [**30**, Proposition 1.36]). Since $d_2 = 0$, it follows that the first r components of d_ψ cannot all be zero. Hence, in order to bound $k(B)$ by the number of rows d_ψ, we may work with the matrix C_1 instead of C. This means it suffices to show

$$\min\{x|D|C_1^{-1}x^{\mathrm{T}} : 0 \ne x \in \mathbb{Z}^r\} \ge r$$

(cf. [**25**, Proposition 2.2]).

The integral matrix $\overline{C_1} := 5^{-a}C_1$ has elementary divisors 1 and $5^{-a}|D|$ where $5^{-a}|D|$ occurs with multiplicity 1. In particular, $\det \overline{C_1} = 5^{-a}|D|$. Since $r \le 4$, it is known that $\overline{C_1}$ can be factorized in the form

$$\overline{C_1} = Q_1^{\mathrm{T}} Q_1$$

where $Q_1 \in \mathbb{Z}^{k\times r}$ for some $k \in \mathbb{N}$ (see [**21**]). We may assume that Q_1 has no vanishing rows. By the choice of C_1, the matrix Q_1 is indecomposable with the notation of [**32**, Definition 1]. Now [**32**, Lemma 4] implies

$$\min\{x|D|C_1^{-1}x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^r\} = \min\{\det(\overline{C_1})x\overline{C_1}^{-1}x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^r\} \geq r.$$

This completes the proof. □

Most parts of the proof above also work for any odd prime p. However, the splitting theorem by Mordell [**21**] is no longer true for matrices of larger dimension. Consider for example the following situation: $p = 7$, $z = 1$, $l(B) = 6$ and

$$C = C_1 = 7^2 \begin{pmatrix} 3 & . & 1 & . & . & . \\ . & 2 & . & 1 & . & . \\ 1 & . & 2 & 1 & . & . \\ . & 1 & 1 & 2 & 1 & . \\ . & . & . & 1 & 2 & 1 \\ . & . & . & . & 1 & 2 \end{pmatrix}$$

(the matrix is a modified version of the E_6 lattice). Then $\det(7^{-2}C) = 7$ and there is no factorization of the form $7^{-2}C = Q^{\mathrm{T}}Q$ for some integral matrix Q. In fact

$$\min\{x7^3C^{-1}x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^6\} = 4 < 6.$$

However, we do not know if C can actually occur as a Cartan matrix of a block.

4. Abelian defect groups.

We begin with a remark about a theorem of Watanabe [**40**].

LEMMA 9. *Let B be a p-block of a finite group G with abelian defect group, and let Z be a central p-subgroup of G. Then $k(B) = |Z|k(\overline{B})$ where $\overline{B}$ is the unique block of G/Z dominated by B.*

PROOF. Let D be a defect group of B. Obviously, $Z \subseteq \mathrm{C}_D(I(B))$. Let $\mathcal{R}$ be a set of representatives for the $I(B)$-conjugacy classes of $[D, I(B)]$. Then $\{(uz, b_{uz}) : u \in \mathcal{R},\ z \in \mathrm{C}_D(I(B))\}$ is a set of representatives of the G-conjugacy classes of B-subsections. By [**40**, Corollary 1], we have $l(b_{uz}) = l(b_u)$ for all $z \in \mathrm{C}_D(I(B))$. This shows

$$k(B) = \sum_{u\in\mathcal{R}} \sum_{z\in \mathrm{C}_D(I(B))} l(b_{uz}) = |\mathrm{C}_D(I(B))| \sum_{u\in\mathcal{R}} l(b_u).$$

Now we consider the block $\overline{B}$. For $H \leq G$ and $x \in D$ we write $\overline{H} := HZ/Z$ and $\overline{x} := xZ$. Let $\mathrm{C}_{\overline{G}}(\overline{x}) = \overline{C_x}$ with $\mathrm{C}_G(\langle x\rangle Z) = \mathrm{C}_G(x) \leq C_x \leq \mathrm{N}_G(\langle x\rangle Z)$. Moreover, let $\overline{b_x^{C_x}}$ be the unique block of $\mathrm{C}_{\overline{G}}(\overline{x})$ dominated by $b_x^{C_x}$. Choose a transversal $\mathcal{S} \subseteq G$ for the cosets $\overline{\mathrm{C}_D(I(B))}$. Since $I(B) \cong I(\overline{B})$, the set

$$\{(\overline{uz}, \overline{b_{uz}^{C_{uz}}}) : u \in \mathcal{R},\ z \in \mathcal{S}\}$$

represents the $\overline{B}$-subsections up to $\overline{G}$-conjugacy (cf. proof of Lemma 2). By [**22**, Theorem 5.8.11], $l(\overline{b_{uz}^{C_{uz}}}) = l(b_{uz}^{C_{uz}})$. Since C_{uz} acts trivially on $\langle \overline{uz} \rangle$ and on Z, it follows that $C_{uz}/\mathrm{C}_G(uz)$ is a p-group. From the properties of fusion systems it is clear that $\mathrm{N}_G(\langle uz \rangle Z, b_{uz})/\mathrm{C}_G(uz)$ is a p'-group. Hence, $\mathrm{N}_G(\langle uz \rangle Z, b_{uz}) \cap C_{uz} = \mathrm{C}_G(uz)$ and the Fong–Reynolds Theorem implies $l(b_{uz}^{C_{uz}}) = l(b_{uz}) = l(b_u)$. Consequently,

$$k(\overline{B}) = \sum_{u \in \mathcal{R}} \sum_{z \in \mathcal{S}} l(\overline{b_{uz}^{C_{uz}}}) = |\overline{\mathrm{C}_D(I(B))}| \sum_{u \in \mathcal{R}} l(b_u).$$

This proves the claim. □

The statement of Lemma 9 is not true for non-abelian defect groups, as it can be seen from the principal 2-block of $SL(2,3)$ with $Z := \mathrm{Z}(SL(2,3))$.

Next, we need a result about the so-called $*$-construction introduced in [**2**].

Lemma 10. *Let B be a p-block of a finite group with defect group D. Let $u \in D$ and let (u,b) be a B-subsection. Let $\chi \in \mathrm{Irr}(B)$, $\varphi \in \mathrm{IBr}(b)$, and let $\lambda \in \mathrm{Irr}(D/\mathfrak{foc}(B)) \subseteq \mathrm{Irr}(D)$. Then $\lambda * \chi \in \mathrm{Irr}(B)$ and*

$$d^u_{\lambda * \chi, \varphi} = \lambda(u) d^u_{\chi\varphi}.$$

Proof. We use the approach from [**27**, Section 1]. Our first claim is already proved there. Let $\mathcal{R}$ be a set of representatives for the G-conjugacy classes of B-subsections such that $(u,b) \in \mathcal{R}$. For $(v, b_v) \in \mathcal{R}$, $\psi \in \mathrm{IBr}(b_v)$ and $x \in \mathrm{C}_G(v)$ let

$$\widetilde{\psi}(x) := \begin{cases} \psi(s) & \text{if } x = vs \text{ where } s \in \mathrm{C}_G(v)_{p'}, \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{C}_G(v)_{p'}$ denotes the set of p-regular elements of $\mathrm{C}_G(v)$. Then $\widetilde{\psi}$ is a class function on $\mathrm{C}_G(v)$, and it is well-known (as a consequence of Brauer's second main theorem) that

$$\chi = \sum_{(v,b_v) \in \mathcal{R}} \sum_{\psi \in \mathrm{IBr}(b_v)} d^v_{\chi\psi} \widetilde{\psi}^G.$$

By [**27**] we have

$$\lambda * \chi = \sum_{(v,b_v) \in \mathcal{R}} \sum_{\psi \in \mathrm{IBr}(b_v)} \lambda(v) d^v_{\chi\psi} \widetilde{\psi}^G.$$

Therefore, it suffices to show that the functions $\{\widetilde{\psi}^G : (v, b_v) \in \mathcal{R}, \psi \in \mathrm{IBr}(b_v)\}$ are linearly independent over $\mathbb{C}$. Thus, assume that

$$\Phi := \sum_{(v,b_v)\in\mathcal{R}} \sum_{\psi\in\mathrm{IBr}(b_v)} \alpha_\psi \widetilde{\psi}^G = 0$$

for some $\alpha_\psi \in \mathbb{C}$. Let $(v, b_v), (v', b_{v'}) \in \mathcal{R}$ such that v and v' are not conjugate in G. Then the functions $\widetilde{\psi}^G$ and $\widetilde{\psi'}^G$ for $\psi \in \mathrm{IBr}(b_v)$ and $\psi' \in \mathrm{IBr}(b_{v'})$ have disjoint support. Hence, it suffices to consider partial sums of Φ corresponding to subsets $\mathcal{S}$ of the form

$$\mathcal{S} := \{(v, b_v) \in \mathcal{R} : v \text{ is conjugate to } u \text{ in } G\}.$$

Choose $1 = x_1, \ldots, x_n \in G$ such that $\mathcal{S} = \{(x_i u x_i^{-1}, b_{x_i u x_i^{-1}}) : i = 1, \ldots, n\}$. Then $\{x_i^{-1} b_{x_i u x_i^{-1}} x_i : i = 1, \ldots, n\}$ is the set of Brauer correspondents of B in $\mathrm{C}_G(u)$. Moreover, for $s \in \mathrm{C}_G(u)_{p'}$ we have

$$\begin{aligned}\Phi(us) &= \sum_{(v,b_v)\in\mathcal{S}} \sum_{\psi\in\mathrm{IBr}(b_v)} \alpha_\psi \widetilde{\psi}^G(us) = \sum_{i=1}^{n} \sum_{\psi\in\mathrm{IBr}(b_{x_i u x_i^{-1}})} \alpha_\psi \big({}^{x_i^{-1}}\psi\big)(s) \\ &= \sum_{\substack{b\in\mathrm{Bl}(\mathrm{C}_G(u)),\\ b^G = B}} \sum_{\psi\in\mathrm{IBr}(b)} \alpha^*_\psi \psi(s)\end{aligned}$$

where $\alpha^*_\psi := \alpha_{\psi'}$ if ${}^{x_i}\psi = \psi'$ for some $i \in \{1, \ldots, n\}$. Since the irreducible Brauer characters of $\mathrm{C}_G(u)$ are linearly independent as functions on $\mathrm{C}_G(u)_{p'}$ (see [**4**, Lemma IV.3.4]), the claim follows. □

The following result generalizes [**32**, Corollary 13].

PROPOSITION 11. *Let B be a block of a finite group with abelian defect group D. Suppose that there is an element $u \in D$ such that $\mathrm{C}_{I(B)}(u)$ acts freely on $[D, \mathrm{C}_{I(B)}(u)]$. Then $k(B) \le |D|$. This applies in particular, if $[D, \mathrm{C}_{I(B)}(u)]$ is cyclic or if $\mathrm{C}_{I(B)}(u)$ has prime order.*

PROOF. Let (u, b) be a B-subsection. We will determine the shape of the Cartan matrix C_u of b. By Lemma 3, b has defect group D and inertial quotient $I(b) \cong \mathrm{C}_{I(B)}(u)$. Let $Z := \mathrm{C}_D(I(b))$, and let b_Z be the Brauer correspondent of b in $\mathrm{C}_G(Z)$ $(\subseteq \mathrm{C}_G(u))$. By [**39**, Corollary] (applied repeatedly), the elementary divisors of the Cartan matrices of b and b_Z coincide (counting multiplicities). Let $\overline{b_Z}$ be the block of $\mathrm{C}_G(Z)/Z$ dominated by b_Z with defect group $\overline{D} := D/Z$. Then $I(\overline{b_Z}) \cong I(b)$ acts freely on $\overline{D} \cong [D, \mathrm{C}_{I(B)}(u)]$. Hence, a result by Fujii [**5**] implies that the elementary divisors of the Cartan matrix of $\overline{b_Z}$ are 1 and $|\overline{D}|$ where $|\overline{D}|$ occurs with multiplicity 1. Consequently, the elementary divisors of C_u are $|Z|$ and $|D|$ where $|D|$ occurs with multiplicity 1. In particular, $\widetilde{C}_u := |Z|^{-1} C_u$ is an integral matrix with determinant $|\overline{D}|$. Let Q_u be the decomposition matrix of b. By the proof of [**27**, Theorem 2] we have $\lambda * \chi \ne \chi$ for every $\chi \in \mathrm{Irr}(b)$ and $1 \ne \lambda \in \mathrm{Irr}(D/[D, I(b)]) \cong \mathrm{Irr}(Z)$ (this is related to the fact that decomposition numbers corresponding to major subsections do not vanish). Therefore, by Lemma 10, every row of Q_u appears $|Z|$ times. Taking only every $|Z|$-th row of Q_u, we obtain an indecomposable matrix $\widetilde{Q}_u \in \mathbb{Z}^{k \times l(b)}$ of rank $l(b)$ without vanishing rows such that

$\widetilde{C}_u = \widetilde{Q}_u^{\mathrm{T}}\widetilde{Q}_u$ and $k := k(b)/|Z|$ (see [**32**, Definition 1 and Proposition 2]). Lemma 4 in [**32**] gives

$$\min\{|D|xC_u^{-1}x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^{l(b)}\} = \min\{\det(\widetilde{C}_u)x\widetilde{C}_u^{-1}x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^{l(b)}\} \geq l(b).$$

Hence, a result by Brauer (see [**30**, Theorem 4.4]) implies the first claim. The second claim is trivial. □

Since every abelian coprime linear group has a regular orbit, we obtain the following (cf. [**30**, Lemma 14.6]).

Corollary 12. *Let B be a block of a finite group with abelian defect group D. Suppose that $I(B)$ contains an abelian subgroup of prime index or of index* 4. *Then $k(B) \leq |D|$.*

A recent paper by Keller–Yang [**13**] provides a dual version.

Corollary 13. *Let B be a block of a finite group with abelian defect group D. Suppose that the commutator subgroup $I(B)'$ of $I(B)$ has prime order or order* 4. *Then $k(B) \leq |D|$.*

Now we prove a result about the number of irreducible Brauer characters.

Proposition 14. *Let B be a block of a finite group with abelian defect group D such that $e(B)$ is a prime. Then $l(B) \leq e(B)$.*

Proof. By [**40**] we may assume that $\mathrm{C}_D(I(B)) = 1$. Then for every non-trivial B-subsection (u, b) we have $l(b) = 1$. Since $I(B)$ acts freely on D, the number of conjugacy classes of these subsections is $(|D| - 1)/e(B)$. In particular,

$$k(B) = \frac{|D| - 1}{e(B)} + l(B).$$

Let C be the Cartan matrix of B. By [**5**], $\det(C) = |D|$. Hence, [**32**, Theorem 5] implies

$$k(B) \leq \frac{|D| - 1}{l(B)} + l(B).$$

The claim follows. □

Observe that Alperin's weight conjecture predicts that $l(B) = e(B)$ in the situation of Proposition 14. This has been shown for principal blocks in [**34**]. Our next result covers a special case of Usami [**38**]. This is of interest, since the proof in case $(p, e(B)) = (2, 3)$ was announced in [**26**, Introduction], but never appeared in print (to the author's knowledge).

Theorem 15. *Let B be a* 2*-block of a finite group with abelian defect group D such that $e(B) \leq 7$. Then B is perfectly isometric* (*even isotypic*) *to the principal* 2*-block of $D \rtimes I(B)$.*

Proof. In order to determine $l(B)$, we may assume that $\mathrm{C}_D(I(B)) = 1$. In case $e(B) = 1$ the block is nilpotent and the claim is well-known. Thus, let $e(B) > 1$. Since $e(B)$ is odd, we must have $e(B) \in \{3, 5, 7\}$. In particular, Proposition 14 implies $l(B) \le e(B)$. Let (u, b) be a B-subsection such that u has order 2. Since $l(b) = 1$, the generalized decomposition numbers $d^u_{\chi\varphi}$ ($\chi \in \mathrm{Irr}(B)$, $\mathrm{IBr}(b) = \{\varphi\}$) form a column of $k(B)$ non-zero integers whose sum of squares equals $|D|$. By the Kessar–Malle Theorem [**15**] about Brauer's height zero conjecture, we know that all irreducible characters in B have height 0. It follows that the numbers $d^u_{\chi\varphi}$ are odd (see for example [**30**, Lemma 1.38]). Hence, $k(B) \equiv |D| \pmod 8$. Since $1 \le l(B) \le e(B) \le 7$ and

$$\frac{|D| - 1}{e(B)} + l(B) = k(B) \equiv |D| \equiv \frac{|D| - 1}{e(B)} + e(B) \pmod 8,$$

we get $l(B) = e(B)$. Now the claim follows from the main theorem of [**41**]. □

We remark that the Cartan matrix of B in the situation of Theorem 15 is given by

$$|\mathrm{C}_D(I(B))| \left(\frac{|[D, I(B)]| - 1}{e(B)} + \delta_{ij} \right)_{i,j=1}^{e(B)}$$

up to basic sets where δ_{ij} is the Kronecker delta (see [**32**, Proposition 6]).

Now we present an extended version of [**30**, Theorem 13.2] in the spirit of [**40**].

Proposition 16. *Let B be a 2-block of a finite group with abelian defect group D such that $|[D, I(B)]| \le 16$. Then one of the following holds*:

(i) *B is nilpotent. Then $e(B) = l(B) = 1$ and $k(B) = |D|$.*

(ii) *$e(B) = l(B) = 3$, $|[D, I(B)]| = 4$, $k(B) = |D|$ and the Cartan matrix of B is $(1/4)|D|(1 + \delta_{ij})$ up to basic sets.*

(iii) *$e(B) = l(B) = 3$, $|[D, I(B)]| = 16$, $k(B) = (1/2)|D|$ and the Cartan matrix of B is $(1/16)|D|(5 + \delta_{ij})$ up to basic sets.*

(iv) *$e(B) = l(B) = 5$, $k(B) = (1/2)|D|$ and the Cartan matrix of B is $(1/16)|D|(3+\delta_{ij})$ up to basic sets.*

(v) *$e(B) = l(B) = 7$, $k(B) = |D|$ and the Cartan matrix of B is $(1/8)|D|(1 + \delta_{ij})$ up to basic sets.*

(vi) *$e(B) = l(B) = 9$, $k(B) = |D|$ and the Cartan matrix of B is $(1/16)|D|(1+\delta_{ij})^3_{i,j=1} \otimes (1 + \delta_{ij})^3_{i,j=1}$ up to basic sets where $\otimes$ denotes the Kronecker product.*

(vii) *$e(B) = 9$, $l(B) = 1$ and $k(B) = (1/2)|D|$.*

(viii) *$e(B) = l(B) = 15$, $k(B) = |D|$ and the Cartan matrix of B is $(1/16)|D|(1+\delta_{ij})$ up to basic sets.*

(ix) *$e(B) = 21$, $l(B) = 5$, $k(B) = |D|$ and the Cartan matrix of B is*

$$\frac{|D|}{8}\begin{pmatrix} 2 & . & . & . & 1 \\ . & 2 & . & . & 1 \\ . & . & 2 & . & 1 \\ . & . & . & 2 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

up to basic sets.

Proof. In case $[D, I(B)] = 1$, the block B is nilpotent and the first case applies. Thus, we may assume that B is non-nilpotent for the rest of the proof. Since the action of $I(B)$ on $[D, I(B)]$ is coprime, we need to discuss the following cases $[D, I(B)] \in \{Z_2^2, Z_2^3, Z_2^4, Z_4^2\}$. The different actions on these groups can be determined easily. As usual $D = \mathrm{C}_D(I(B)) \times [D, I(B)]$. Let $Z := \mathrm{C}_D(I(B))$, and let b_Z be a Brauer correspondent of B in $\mathrm{C}_G(Z)$. Then by [**40**] (applied repeatedly), $l(B) = l(b_Z)$ and $k(B) = k(b_Z)$. Moreover, b_Z dominates a block $\overline{b_Z}$ of $\mathrm{C}_G(Z)/Z$ with defect group $D/Z \cong [D, I(B)]$ and $l(\overline{b_Z}) = l(b_Z)$. Using [**30**, Theorems 8.1, 13.1 and 13.2] and Lemma 9 it is easy to determine $l(B) = l(\overline{b_Z})$ and $k(B) = |Z|k(\overline{b_Z})$. Therefore, it remains to compute the Cartan matrix of B.

The case $e(B) \le 7$ is covered by Theorem 15 and the subsequent remark. The same argument also works for $e(B) = 15$, since here $I(B)$ acts freely on $[D, I(B)]$. Therefore, we may assume that $[D, I(B)] \in \{Z_2^3, Z_2^4\}$. We explain our general method for these cases. Let $\mathcal{R}$ be a set of representatives for the $I(B)$-conjugacy classes of $[D, I(B)]$. For $x \in \mathcal{R}$ let Q_x be the part of the generalized decomposition matrix of B corresponding to the subsection (x, b_x). Then by Lemma 10 (together with [**27**, Theorem 2]), every row of Q_x appears $|Z|$ times. This holds in particular for the ordinary decomposition matrix Q_1. Hence, in order to compute Q_1 we may divide the Cartan matrices C_x of b_x by $|Z|$. So, let $\widetilde{C}_x := (1/|Z|)C_x$ for $1 \ne x \in \mathcal{R}$. Since x has order at most 2, the matrices Q_x are all integral. Assume that we have found matrices $\widetilde{Q}_x \in \mathbb{Z}^{k(\overline{b_Z}) \times l(b_x)}$ $(1 \ne x \in \mathcal{R})$ such that

$$\widetilde{Q}_x^{\mathrm{T}}\widetilde{Q}_y = \begin{cases} \widetilde{C}_x & \text{if } x = y, \\ 0 & \text{if } x \ne y \end{cases}$$

for $x, y \in \mathcal{R}\backslash\{1\}$. This means we are actually constructing the generalized decomposition matrix of $\overline{b_Z}$. Let

$$\Gamma := \{v \in \mathbb{Z}^{k(\overline{b_Z})} : v\widetilde{Q}_x = 0 \in \mathbb{Z}^{l(b_x)}\ \forall x \in \mathcal{R} \setminus \{1\}\}.$$

We choose a basis for the $\mathbb{Z}$-module Γ and we write the basis vectors as columns of a matrix $\widetilde{Q}_1 \in \mathbb{Z}^{k(\overline{b_Z}) \times l(B)}$ (cf. [**30**, Section 4.2]). Finally, set

$$Q_1 := \begin{pmatrix} \widetilde{Q}_1 \\ \vdots \\ \widetilde{Q}_1 \end{pmatrix} \in \mathbb{Z}^{k(B)\times l(B)}.$$

Then the orthogonality relations for the group $[D, I(B)]$ guarantee that Q_1 is orthogonal to any column of generalized decomposition numbers corresponding to a non-trivial subsection (provided a suitable ordering of $\operatorname{Irr}(B)$). Since the elementary divisors of Q_1 are equal 1, the Cartan matrix of B is given by $Q_1^{\mathrm{T}} Q_1$ up to basic sets.

Now we have to deal with the various cases according to the action of $I(B)$ on $[D, I(B)]$. As mentioned above, we may assume that $e(B) \in \{9, 21\}$. If $[D, I(B)] \cong Z_2^3$, it follows that $e(B) = 21$ and $I(B) \cong Z_7 \rtimes Z_3$. Here there is only one matrix $\widetilde{C}_x$ for $x \in \mathcal{R} \setminus \{1\}$ given by $\widetilde{C}_x = 2(1 + \delta_{ij})_{i,j=1}^3$ (see Theorem 15 and the subsequent remark). Since $k(\overline{b_Z}) = 8$ (see [**30**, Theorem 13.1]), there is essentially only one choice for $\widetilde{Q}_x$, namely

$$\widetilde{Q}_x = \begin{pmatrix} 1\,1\,1\,1\,.\,.\,.\,. \\ 1\,1\,.\,.\,1\,1\,.\,. \\ 1\,1\,.\,.\,.\,.\,1\,1 \end{pmatrix}^{\mathrm{T}}.$$

This makes it easy to compute $\widetilde{Q}_1$ and $Q_1^{\mathrm{T}} Q_1$. Now let $[D, I(B)] \cong Z_2^4$ and $e(B) = 9$. Then $I(B)$ is elementary abelian. In the proof of [**30**, Theorem 13.7] we used extensive computer calculations to enumerate the matrices $\widetilde{Q}_x$ for $1 \neq x \in \mathcal{R}$. Here we use the opportunity to give a computer-free argument. Let $\mathcal{R} = \{1, x, y, xy\}$ such that $l(b_x) = l(b_y) = 3$ and $l(b_{xy}) = 1$. As usual one has $\widetilde{C}_x = \widetilde{C}_y = 4(1 + \delta_{ij})_{i,j=1}^3$. We may choose a basic set for b_x such that

$$\widetilde{Q}_x = \begin{pmatrix} 1\,1\,1\,1\,1\,1\,1\,1\,.\,.\,.\,.\,.\,.\,.\,. \\ 1\,1\,1\,1\,.\,.\,.\,.\,1\,1\,1\,1\,.\,.\,.\,. \\ 1\,1\,1\,1\,.\,.\,.\,.\,.\,.\,.\,.\,1\,1\,1\,1 \end{pmatrix}^{\mathrm{T}}.$$

Let $M_x := 16\widetilde{Q}_x \widetilde{C}_x^{-1} \widetilde{Q}_x^{\mathrm{T}} = |D| \widetilde{Q}_x C_x^{-1} \widetilde{Q}_x^{\mathrm{T}}$ be a part of the contribution matrix of B with respect to (x, b_x). Then

$$M_x = \begin{pmatrix} 3J & J & J & J \\ J & 3J & -J & -J \\ J & -J & 3J & -J \\ J & -J & -J & 3J \end{pmatrix}$$

where J is the 4×4 matrix whose entries are all 1. Up to permutations and signs, the (part of the) contribution matrix M_y has the same shape. There exists an $I(B)$-stable generalized character λ of $[D, I(B)]$ (and of D) such that $\lambda(1) = \lambda(xy) = 0$ and $\lambda(x) = -\lambda(y) = 4$. Hence, for $\chi \in \operatorname{Irr}(B)$, $\chi * \lambda$ is a generalized character of B. This implies $(1/4)M_x - (1/4)M_y \in \mathbb{Z}^{16\times 16}$. Thus, $M_x \equiv M_y \pmod 4$. Moreover, by the orthogonality relations we have $M_xM_y = 0$. Since we can still permute the first four characters and the next four and so on, we may assume that the first row of M_y has the form $(3,-1,-1,-1,-3,1,1,1,-3,1,1,1,-3,1,1,1)$. After changing the basic set of b_y if necessary, we may assume that the first row of $\widetilde{Q}_y$ is $(1,1,1)$. By symmetry reason, it is easy to see that we may assume

$$\widetilde{Q}_y = \begin{pmatrix} 1 & -1 & . & . & -1 & 1 & . & . & -1 & 1 & . & . & -1 & 1 & . & . \\ 1 & . & -1 & . & -1 & . & 1 & . & -1 & . & 1 & . & -1 & . & 1 & . \\ 1 & . & . & -1 & -1 & . & . & 1 & -1 & . & . & 1 & -1 & . & . & 1 \end{pmatrix}^{\mathrm{T}}.$$

Again by the $*$-construction, $M_x \equiv -M_{xy} \pmod 4$. It follows that $\widetilde{Q}_{xy} = (1,1,1,1,-1,\ldots,-1)^{\mathrm{T}}$. This gives $\widetilde{Q}_1$ and finally C_1. We have given the Cartan matrix of the principal block of the group $D \rtimes I(B) \cong \mathrm{C}_D(I(B)) \times A_4^2$ where A_4 is the alternating group of degree 4. $\square$

We note that part (viii) of Proposition 16 relies on the classification of the finite simple groups.

The argument of Proposition 16 also works for other situations. However, it is not clear if in general B and b_z for $z \in \mathrm{C}_D(I(B))$ have the same Cartan matrix up to basic sets. This depends on the question whether the knowledge of the number $l(B)$ and the Cartan matrices of b_x for $1 \neq x \in [D, I(B)]$ determine the Cartan matrix of B. It is conjectured in general that the blocks B and b_z are perfectly isometric or even Morita equivalent (see for example [**14**], [**19**]).

Corollary 17. *Let B be a 2-block of a finite group with abelian defect group D. Suppose that there is an element $u \in D$ such that $|[D, \mathrm{C}_{I(B)}(u)]| \leq 16$. Then $k(B) \leq |D|$.*

Proof. Let (u, b) be a B-subsection. Then Proposition 16 applies for b. If $9 \neq e(b) \neq 21$, then the action of $I(b)$ on $[D, I(b)]$ is free, and the claim follows from Proposition 11. Now let $e(b) = 21$. Here one can apply [**30**, Theorem 4.2] with the quadratic form corresponding to the positive definite matrix

$$\frac{1}{2}\begin{pmatrix} 2 & 1 & . & . & -1 \\ 1 & 2 & . & . & -1 \\ . & . & 2 & . & -1 \\ . & . & . & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

Finally, let $e(b) = 9$. Let C_b be the Cartan matrix of b given by Proposition 16. In order to apply [**30**, Theorem 4.4] we consider the quadratic form corresponding to the matrix $|D|C_b^{-1}$. For this let $M := (1 + \delta_{ij})_{i,j=1}^3$. Then $4M^{-1} = (-1 + 4\delta_{ij})$. For $0 \neq x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ we have

$$4xM^{-1}x^{\mathrm{T}} = x_1^2 + x_2^2 + x_3^2 + (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 \geq 3.$$

This shows that $\min\{4xM^{-1}x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^3\} = 3$. In general the minimum of a tensor product of quadratic forms does not need to coincide with the product of the minima of its factors. However, in this case it is true by [**16**, Theorem 7.1.1]. For the convenience of the reader, we give an elementary argument. First observe that $|D|C_b^{-1} = 16(M \otimes M)^{-1} = 4M^{-1} \otimes 4M^{-1}$. Now let $0 \neq x = (x_1, x_2, x_3) \in \mathbb{Z}^9$ with $x_i \in \mathbb{Z}^3$. Then

$$\begin{aligned} 16x(M \otimes M)^{-1}x^{\mathrm{T}} &= \sum_{i=1}^{3} 4x_iM^{-1}x_i^{\mathrm{T}} + \sum_{i<j} 4(x_i - x_j)M^{-1}(x_i - x_j)^{\mathrm{T}} \\ &\geq 3\min\{4yM^{-1}y^{\mathrm{T}} : 0 \neq y \in \mathbb{Z}^3\} \geq 9. \end{aligned}$$

Hence, $\min\{x|D|C_b^{-1}x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^9\} = 9$, and the claim follows from [**30**, Theorem 4.4]. □

PROPOSITION 18. *Let B be a 3-block of a finite group with abelian defect group D. Suppose that there is an element $u \in D$ such that $|[D, \mathrm{C}_{I(B)}(u)]| \leq 9$. Then $k(B) \leq |D|$.*

PROOF. By Proposition 11 we may assume that $[D, \mathrm{C}_{I(B)}(u)]$ is elementary abelian of order 9. Let (u, b) be a B-subsection. Then $I(b) \cong \mathrm{C}_{I(B)}(u) \leq \mathrm{Aut}([D, \mathrm{C}_{I(B)}(u)]) \cong GL(2, 3)$. Therefore, $I(b)$ lies in a Sylow 2-subgroup of $GL(2, 3)$ which is isomorphic to the semidihedral group SD_{16} of order 16. By [**30**, Lemma 14.5], we may assume that $e(b) \geq 8$. If $I(b) \in \{Z_8, Q_8\}$ where Q_8 is the quaternion group of order 8, then the action of $I(b)$ on $[D, I(b)]$ is free (even regular). Hence, these cases are handled by Proposition 11. It remains to deal with the cases $I(b) \in \{D_8, SD_{16}\}$. In order to do so, we may consider a block $\overline{b}$ with defect group Z_3^2 and inertial quotient $I(b)$. The numbers $k(\overline{b})$ and $l(\overline{b})$ were determined in [**17**], [**42**]. The case $l(b) = l(\overline{b}) = 2$ can be ignored by [**30**, Theorem 4.9]. Hence, we have $k(\overline{b}) = 9$ and $l(\overline{b}) \in \{5, 7\}$ according to the two possibilities for $I(b)$. In the proof of [**30**, Theorem 13.7] we have computed the possible Cartan matrices for $\overline{b}$:

$$\begin{pmatrix} 3 & . & 1 & . & 1 \\ . & 3 & 1 & . & 1 \\ 1 & 1 & 3 & 1 & . \\ . & . & 1 & 3 & 1 \\ 1 & 1 & . & 1 & 3 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 2 & 1 & . & . & . & . & 1 \\ 1 & 2 & . & . & . & . & 1 \\ . & . & 2 & 1 & . & . & 1 \\ . & . & 1 & 2 & . & . & 1 \\ . & . & . & . & 2 & 1 & 1 \\ . & . & . & . & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

Since the construction of these matrices was carried out by enumerating the generalized decomposition numbers as in the proof of Proposition 16, the Cartan matrix of b is just a scalar multiple of one of these matrices. Now we can apply [**30**, Theorem 4.2] with the quadratic form corresponding to the positive definite matrix

$$\frac{1}{2}\begin{pmatrix} 2 & . & -1 & . & -1 \\ . & 2 & -1 & 1 & -1 \\ -1 & -1 & 2 & -1 & 1 \\ . & 1 & -1 & 2 & -1 \\ -1 & -1 & 1 & -1 & 2 \end{pmatrix} \qquad \text{or} \qquad \frac{1}{2}\begin{pmatrix} 2 & -1 & . & . & . & . & -1 \\ -1 & 2 & . & . & . & . & . \\ . & . & 2 & -1 & . & . & -1 \\ . & . & -1 & 2 & . & 1 & . \\ . & . & . & . & 2 & -1 & -1 \\ . & . & . & 1 & -1 & 2 & . \\ -1 & . & -1 & . & -1 & . & 2 \end{pmatrix}$$

respectively. This completes the proof. □

The following result about regular orbits under coprime actions might be of general interest.

PROPOSITION 19. *Let P be an abelian p-group such that $\Omega(P) \subseteq \Phi(P)$. Then every p'-automorphism group of P has a regular orbit on P.*

PROOF. Let $A \leq \mathrm{Aut}(P)$ be a p'-group. By [**8**, Theorem 5.2.4], we may assume that P has exponent p^2. Following [**37**, Lemma 1.7], we will show that the action of A on P is isomorphic to the componentwise action of A on $\Omega(P)\times\Omega(P)$. Let $x\Omega(P) \in P/\Omega(P)$. Since A acts on $P/\Omega(P)$, we can define a subgroup $A_1 := \mathrm{C}_A(x\Omega(P)) \leq A$ which fixes $x\Omega(P)$ as a set. By [**18**, 8.2.1], there exists a representative $r(x\Omega(P))$ of $x\Omega(P)$ such that $r(x\Omega(P)) \in \mathrm{C}_P(A_1)$. Now for any $a \in A$ we set $r({}^a x\Omega(P)) := {}^a r(x\Omega(P))$. This is well defined, since ${}^a x \equiv {}^b x \pmod{\Omega(P)}$ implies $b^{-1}a \in A_1$ and ${}^a r(x\Omega(P)) = {}^b r(x\Omega(P))$ for $a, b \in A$. Repeating this with the other orbits of cosets we end up with an A-invariant transversal $\mathcal{R}$ for $P/\Omega(P)$. Now let

$$\begin{aligned} \varphi : P &\longrightarrow \Omega(P) \times \Omega(P), \\ \widetilde{x}y &\longmapsto (\widetilde{x}^p, y) \qquad (\widetilde{x} \in \mathcal{R},\ y \in \Omega(P)). \end{aligned}$$

It is easy to see that φ is a bijection and

$${}^a\varphi(\widetilde{x}y) = ({}^a\widetilde{x}^p, {}^a y) = \varphi({}^a\widetilde{x}{}^a y) = \varphi({}^a(\widetilde{x}y))$$

for $a \in A$, $\widetilde{x} \in \mathcal{R}$ and $y \in \Omega(P)$. Hence, P is A-isomorphic to $\Omega(P) \times \Omega(P)$.

By [**9**], there exist $x, y \in \Omega(P)$ such that $\mathrm{C}_A(x) \cap \mathrm{C}_A(y) = 1$. Hence, the A-orbit of $(x, y) \in \Omega(P) \times \Omega(P)$ is regular. The claim follows. □

We are now in a position to generalize other theorems from [**30**, Chapter 14].

THEOREM 20. *Let B be a block of a finite group with abelian defect group D such that D has no elementary abelian direct summand of order p^3. Then $k(B) \leq |D|$.*

PROOF. We can decompose $D = \prod_{i=1}^n D_i$ into indecomposable $I(B)$-invariant summands D_i. By [**8**, Theorem 5.2.2], each D_i is homocyclic, i. e. a direct product of isomorphic cyclic groups. If D_i is not elementary abelian, then we choose $x_i \in D_i$ such that $\mathrm{C}_{I(B)}(x_i) = \mathrm{C}_{I(B)}(D_i)$ by Proposition 19. Now assume that D_i is elementary abelian. Then by hypothesis, $|D_i| \le p^2$. Here we choose any $1 \ne x_i \in D_i$. If all elementary abelian components D_i have order p, then it is easy to see that $\mathrm{C}_{I(B)}(x) = 1$ for $x := x_1 \cdots x_n$. In this case the claim has already been known to Brauer (see [**30**, Proposition 4.7] for example). Now suppose that only D_1 is elementary abelian and of order p^2. Then $[D, \mathrm{C}_{I(B)}(x)]$ is cyclic, and the claim follows from Proposition 11. □

As usual, we can say slightly more if p is small.

PROPOSITION 21. *Let B be a 2-block of a finite group with abelian defect group D such that D has no elementary abelian direct summand of order 2^8. Then $k(B) \le |D|$.*

PROOF. Using the arguments in the proof of Theorem 20, we may assume that D is elementary abelian of order at most 2^7. We will choose an element $x \in D$ such that $|[D, \mathrm{C}_{I(B)}(x)]|$ is small. By Corollary 17, we may assume that $32 \le |[D, \mathrm{C}_{I(B)}(x)]| < |D|$. Let $|D| = 64$. If D decomposes as $D = D_1 \oplus D_2$ with $I(B)$-invariant subgroups D_i, then we can take $1 \ne x_i \in D_i$ and $x := x_1x_2$. It follows that $|[D, \mathrm{C}_{I(B)}(x)]| \le 16$. Hence, we may assume that $I(B)$ acts irreducibly on D. By the Feit–Thompson Theorem, $I(B)$ is solvable. Thus, we can use the GAP package IRREDSOL [**7**] to find all possibilities for $I(B)$. It turns out that in all cases we find elements $x \in D$ such that $|[D, \mathrm{C}_{I(B)}(x)]| \le 16$. Finally, let $|D| = 2^7$. Here, it can happen that $D = D_1 \oplus D_2$ with irreducible $I(B)$-invariant subgroups of order 2^4 and 2^3 respectively. However, there is always an element $x_1 \in D_1$ such that $\mathrm{C}_{I(B)}(x_1) = \mathrm{C}_{I(B)}(D_1)$. Therefore, it remains to handle the case where $I(B)$ acts irreducibly on D. It turns out that we only need to deal with the case $I(B) \cong Z_{127} \rtimes Z_7$ (cf. [**36**, Remark 4 on p. 168]). For this case, Proposition 11 applies. □

Apart from the elementary abelian defect group of order 64, the proof of Proposition 21 also works for some non-abelian defect groups of order 64. Thus, referring to the list in [**30**, p. 200], Brauer's $k(B)$-conjecture is still open for the defect groups SmallGroup(64,q) where

$$q \in \{134, 135, 136, 137, 138, 139, 202, 224, 229,\\ 230, 231, 238, 239, 242, 254, 255, 257, 258, 259, 262\}.$$

Speaking of abelian defect groups for $p = 2$, the next challenge is $D \cong Z_2^8$ with $I(B) \cong (Z_{31} \rtimes Z_5) \times (\mathrm{Z}_7 \rtimes Z_3)$ acting reducibly.

PROPOSITION 22. *Let $p \in \{3, 5\}$, and let B be a p-block of a finite group with abelian defect group D such that D has no elementary abelian direct summand of order p^4. Then $k(B) \le |D|$.*

PROOF. The case $p = 3$ follows easily from Proposition 18. Now let $p = 5$. As before, let $D = D_1 \oplus D_2$ be an $I(B)$-invariant decomposition such that D_1 is elementary abelian and $\mathrm{C}_{I(B)}(x_2) = \mathrm{C}_{I(B)}(D_2)$ for some $x_2 \in D_2$. By Theorem 20, we may assume that $|D_1| = p^3$. Since $I(B)/\mathrm{C}_{I(B)}(D_1) \le \mathrm{Aut}(D_1) \cong GL(3, 5)$, one can show that there is an element $x_1 \in D_1$ such that $|\mathrm{C}_{I(B)}(x_1)/\mathrm{C}_{I(B)}(D_1)| \le 4$ or $\mathrm{C}_{I(B)}(x_1)/\mathrm{C}_{I(B)}(D_1) \cong S_3$ where S_3 is the symmetric group of degree 3 (cf. [**30**, proofs of 14.16 and 14.17]). Let

$x := x_1 x_2$. Then

$$\mathrm{C}_{I(B)}(x) = \mathrm{C}_{I(B)}(x_1) \cap \mathrm{C}_{I(B)}(x_2) = \mathrm{C}_{I(B)}(x_1) \cap \mathrm{C}_{I(B)}(D_2) \hookrightarrow \mathrm{C}_{I(B)}(x_1)/\,\mathrm{C}_{I(B)}(D_1)$$

where the inclusion comes from the canonical map $g \mapsto g\,\mathrm{C}_{I(B)}(D_1)$. The claim follows from [**30**, Lemma 14.5]. □

The new method does not suffice to overcome the next problems for $p \in \{3, 5, 7\}$ already described in [**30**, Chapter 14].

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References

[1] M. Aschbacher, R. Kessar and B. Oliver, Fusion systems in algebra and topology, London Math. Soc. Lecture Note Series, **391**, Cambridge University Press, Cambridge, 2011.

[2] M. Broué and L. Puig, A Frobenius theorem for blocks, Invent. Math., **56** (1980), 117–128.

[3] C. W. Eaton, B. Külshammer and B. Sambale, 2-Blocks with minimal nonabelian defect groups II, J. Group Theory, **15** (2012), 311–321.

[4] W. Feit, The representation theory of finite groups, North-Holland Mathematical Library, **25**, North-Holland Publishing Co., Amsterdam, 1982.

[5] M. Fujii, On determinants of Cartan matrices of p-blocks, Proc. Japan Acad. Ser. A Math. Sci., **56** (1980), 401–403.

[6] S. Gao and J. Zeng, On the number of ordinary irreducible characters in a p-block with a minimal nonabelian defect group, Comm. Algebra, **39** (2011), 3278–3297.

[7] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.8; 2015, (http://www.gap-system.org).

[8] D. Gorenstein, Finite groups, Harper & Row Publishers, New York, 1968.

[9] Z. Halasi and K. Podoski, Every coprime linear group admits a base of size two, Trans. Amer. Math. Soc., **368** (2016), 5857–5887.

[10] S. Hendren, Extra special defect groups of order p^3 and exponent p^2, J. Algebra, **291** (2005), 457–491.

[11] S. Hendren, Extra special defect groups of order p^3 and exponent p, J. Algebra, **313** (2007), 724–760.

[12] B. Huppert, Endliche Gruppen. I, Grundlehren der Mathematischen Wissenschaften, **134**, Springer-Verlag, Berlin, 1967.

[13] T. M. Keller and Y. Yang, Abelian quotients and orbit sizes of finite groups, arXiv:1407.6436v1.

[14] R. Kessar and M. Linckelmann, On perfect isometries for tame blocks, Bull. London Math. Soc., **34** (2002), 46–54.

[15] R. Kessar and G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, Ann. of Math. (2), **178** (2013), 321–384.

[16] Y. Kitaoka, Arithmetic of quadratic forms, Cambridge Tracts in Mathematics, **106**, Cambridge University Press, Cambridge, 1993.

[17] M. Kiyota, On 3-blocks with an elementary abelian defect group of order 9, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **31** (1984), 33–58.

[18] H. Kurzweil and B. Stellmacher, The theory of finite groups, Universitext, Springer-Verlag, New York, 2004.

[19] B. Külshammer and T. Okuyama, On centrally controlled blocks of finite groups, unpublished.
[20] M. Linckelmann, Introduction to fusion systems, In: Group representation theory, 79–113, EPFL Press, Lausanne, 2007. Revised version: http://web.mat.bham.ac.uk/C.W.Parker/Fusion/fusion-intro.pdf.
[21] L. J. Mordell, The representation of a definite quadratic form as a sum of two others, Ann. of Math. (2), **38** (1937), 751–757.
[22] H. Nagao and Y. Tsushima, Representations of finite groups, Academic Press Inc., Boston, MA, 1989.
[23] J. Neukirch, Algebraic number theory, Grundlehren der Mathematischen Wissenschaften, **322**, Springer-Verlag, Berlin, 1999.
[24] J. B. Olsson, On 2-blocks with quaternion and quasidihedral defect groups, J. Algebra, **36** (1975), 212–241.
[25] W. Plesken, Solving $XX^{\mathrm{tr}} = A$ over the integers, Linear Algebra Appl., **226/228** (1995), 331–344.
[26] L. Puig and Y. Usami, Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order 4, J. Algebra, **172** (1995), 205–213.
[27] G. R. Robinson, On the focal defect group of a block, characters of height zero, and lower defect group multiplicities, J. Algebra, **320** (2008), 2624–2628.
[28] B. Sambale, 2-Blocks with minimal nonabelian defect groups, J. Algebra, **337** (2011), 261–284.
[29] B. Sambale, Cartan matrices and Brauer's $k(B)$-conjecture II, J. Algebra, **337** (2011), 345–362.
[30] B. Sambale, Blocks of finite groups and their invariants, Springer Lecture Notes in Math., **2127**, Springer-Verlag, Berlin, 2014.
[31] B. Sambale, On the Brauer–Feit bound for abelian defect groups, Math. Z., **276** (2014), 785–797.
[32] B. Sambale, Cartan matrices and Brauer's $k(B)$-conjecture III, Manuscripta Math., **146** (2015), 505–518.
[33] H. Sasaki, The mod p cohomology algebras of finite groups with metacyclic Sylow p-subgroups, J. Algebra, **192** (1997), 713–733.
[34] M. Sawabe and A. Watanabe, On the principal blocks of finite groups with abelian Sylow p-subgroups, J. Algebra, **237** (2001), 719–734.
[35] R. Stancu, Control of fusion in fusion systems, J. Algebra Appl., **5** (2006), 817–837.
[36] D. A. Suprunenko, Matrix groups, Amer. Math. Soc., Providence, R.I., 1976.
[37] A. Turull, Fixed point free action with regular orbits, J. Reine Angew. Math., **371** (1986), 67–91.
[38] Y. Usami, On p-blocks with abelian defect groups and inertial index 2 or 3. I, J. Algebra, **119** (1988), 123–146.
[39] A. Watanabe, Note on a p-block of a finite group with abelian defect group, Osaka J. Math., **26** (1989), 829–836.
[40] A. Watanabe, Notes on p-blocks of characters of finite groups, J. Algebra, **136** (1991), 109–116.
[41] A. Watanabe, On perfect isometries for blocks with abelian defect groups and cyclic hyperfocal subgroups, Kumamoto J. Math., **18** (2005), 85–92.
[42] A. Watanabe, Appendix on blocks with elementary abelian defect group of order 9, In: Representation Theory of Finite Groups and Algebras, and Related Topics (Kyoto, 2008), 9–17, Kyoto University Research Institute for Mathematical Sciences, Kyoto, 2010.
[43] A. Watanabe, The number of irreducible Brauer characters in a p-block of a finite group with cyclic hyperfocal subgroup, J. Algebra, **416** (2014), 167–183.
[44] S. Yang and S. Gao, On the control of fusion in the local category for the p-block with a minimal nonabelian defect group, Sci. China Math., **54** (2011), 325–340.

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