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Dimension formulas of paramodular forms of squarefree level and comparison with inner twist

By Tomoyoshi Ibukiyama and Hidetaka Kitayama

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Abstract. In this paper, we give an explicit dimension formula for the spaces of Siegel paramodular cusp forms of degree two of squarefree level. As an application, we propose a conjecture on symplectic group version of Eichler–Jacquet–Langlands type correspondence. It is a generalization of the previous conjecture of the first named author for prime levels published in 1985, where inner twists corresponding to binary quaternion hermitian forms over definite quaternion algebras were treated. Our present study contains also the case of indefinite quaternion algebras. Additionally, we give numerical examples of L functions which support the conjecture. These comparisons of dimensions and examples give also evidence for conjecture on a certain precise lifting theory. This is related to the lifting theory from pairs of elliptic cusp forms initiated by Y. Ihara in 1964 in the case of compact twist, but no such construction is known in the case of non-split symplectic groups corresponding to quaternion hermitian groups over indefinite quaternion algebras and this is new in that sense.

1. Introduction.

The purpose of this paper is to give two main theorems and two conjectures on Siegel paramodular cusp forms of degree two. The first theorem (Main Theorem 1.1) is an explicit dimension formula for the spaces of Siegel cusp forms with respect to the paramodular group of squarefree level, which was announced in [**Kit14**]. The second theorem (Main Theorem 1.2) is a comparison of dimensions of paramodular forms and those of automorphic forms belonging to non-split $\mathbb{Q}$-forms of the symplectic group of rank 2. This is derived from the first theorem and other explicit dimension formulas of the authors. We propose Conjecture 1.3 on correspondence between different symplectic $\mathbb{Q}$ forms which is expected from Main Theorem 1.2. This is closely related to Conjecture 1.4 on liftings given next. These two conjectures are not only based on dimension formulas, but also on many explicit examples. These are given as Theorem 4.6 in Section 4.

This study is motivated by our attempts to generalize Eichler–Jacquet–Langlands type correspondence to the case of symplectic groups of higher degree. Originally, Eichler [**Eic56**], [**Eic58**] proved the correspondence between automorphic forms on $SL(2;\mathbb{R})$ and

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on $SU(2)$, and Jacquet–Langlands [**JL72**] developed it from the viewpoint of representation theory. Langlands [**Lan70**] gave a general conjecture for any reductive algebraic groups. On the other hand, Ihara [**Iha64**] raised independently a problem of generalizing Eichler correspondence to symplectic groups of degree two in 1960s. About this problem, the first named author gave several precise conjectures, specifying corresponding discrete subgroups of prime level which are parahoric locally, defining new forms for such cases. These are based on numerical examples [**Ibu84**] and explicit dimension formulas in [**Ibu85**], [**HI85**] and [**Ibu07a**]. In particular we found there good dimensional relations of automorphic forms on the real symplectic group $Sp(2;\mathbb{R}) \subset M_4(\mathbb{R})$ of rank two and its compact twist which support the conjectures. However, the conjectures were restricted to the case of prime level because explicit dimension formulas for paramodular forms were not known except for prime level. In this paper, we achieve a generalization of the dimension formula of paramodular cusp forms to squarefree level and give further consideration to the conjecture. The generalization to the squarefree level case from the prime level case is not just a patch work of the prime levels. Firstly, the general case contains the case of quaternion hermitian groups over indefinite quaternion algebras, which has not been treated before since each level (discriminant) is a product of an even number of different primes. The nature of automorphic forms in these cases is very different from the compact twist case when the definite quaternion algebras appear, since in the latter, automorphic forms are essentially harmonic polynomials invariant by finite groups, while in the former, they are functions on the Siegel upper half space. Secondly, a global description of new forms in the general case demands more careful study of liftings. Indeed, the new comparison predicts a new conjectural *injective* liftings in detail.

Our first main theorem is as follows. We denote by $S_{k,j}(K(N))$ the space of Siegel cusp forms of degree two of weight $\det^k \otimes \mathrm{Sym}_j$ for the paramodular group $K(N)$ of level N. (See Subsection 2.1 for definitions).

Main Theorem 1.1 (Theorem 3.1). *We assume that N is a squarefree positive integer. If $j \geq 0$ is an even integer and $k \geq 5$, then we have*

$$\dim S_{k,j}(K(N)) = \sum_{i=1}^{12} H_i(k,j,N) + \sum_{i=1}^{10} I_i(k,j,N),$$

where $H_i(k,j,N)$ and $I_i(k,j,N)$ are explicit elementary functions of k, j and N, which are given in Theorem 3.1.

Remark. (i) If $k = 3, 4$ and $j = 0$, then the above formula is still applicable as follows:

$$\dim S_{k,0}(K(N)) = \sum_{i=1}^{12} H_i(k,0,N) + \sum_{i=1}^{10} I_i(k,0,N) + \begin{cases} 1 \cdots \text{ if } k = 3, \\ 0 \cdots \text{ if } k = 4. \end{cases}$$

See Ibukiyama [**Ibu07b**]. For $j > 0$ and $k = 3, 4$, we conjecture that the above formula in Theorem 1.1 is valid without any correction terms.

(ii) Main Theorem 1.1 is a generalization of the previous formulas in [**Ibu85**], [**Ibu07a**] for prime level, and also [**Wak12**, Theorem 7.1] for $K(1) = Sp(2;\mathbb{Z})$.

We prove Main Theorem 1.1 in Section 7 and 8 by using the method which is based on Selberg trace formula. The condition $k \geq 5$ comes from the condition for convergence of the trace formula. Roughly speaking, in order to obtain an explicit dimension formula from Selberg trace formula, we have to calculate the contribution of all $K(N)$-conjugacy classes and sum them up. As for semisimple elements, we can achieve it by careful use of local data in [**HI80**] and [**Ibu85**]. On the other hand, for non-semisimple elements, we have to classify $K(N)$-conjugacy classes by direct calculation and evaluate the contributions by using formulas in [**Has83**] and [**Wak12**]. For further details, see Section 6.

Our second main theorem is explained briefly as follows. Let B be a quaternion algebra over $\mathbb{Q}$ and let N be the product of primes $p < \infty$ such that B_p is division. We call N the discriminant of B. We denote by $\omega(N)$ the number of prime divisors of N. Then B is definite if $\omega(N)$ is odd and indefinite if even. We define $G = \{g \in M(2;B) \mid g\,{}^t\overline{g} = n(g)1_2, n(g) \in \mathbb{Q}^\times\}$, where ${}^t\overline{g} = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix}$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\bar{*}$ is the canonical involution of B. When B is definite, let G_A be its adelization and $U(N)$ be a certain open subgroup of G_A which is defined in subsection 2.2. We denote by $\mathfrak{M}_{k+j-3,k-3}(U(N))$ the space of automorphic forms on $U(N)$ of weight $\rho_{k+j-3,k-3}$ corresponding to the Young diagram $(k+j-3, k-3)$ which parametrizes the irreducible representation of the compact symplectic group $Sp(2)$ of rank 2. This space of automorphic forms can be regarded as a direct sum of some harmonic polynomials of eight variables invariant by certain finite groups which are the groups of automorphisms of some binary quaternion hermitian lattices in a fixed genus (See [**II87**]). On the other hand, when B is indefinite, $G^1 = \{g \in G \mid n(g) = 1\}$ is a non-split $\mathbb{Q}$-form of $Sp(2;\mathbb{R})$. Let $U'(N)$ be a certain discrete subgroup of $Sp(2;\mathbb{R})$ which is defined in subsection 2.3. We denote by $S_{k,j}(U'(N))$ the space of Siegel cusp forms with respect to $U'(N)$ of weight $\det^k \otimes Sym_j$. Then our second main theorem is as follows. We denote by $S_k^{new}(\Gamma_0^{(1)}(N))$ the space of new cusp forms of weight k of level N of one variable with respect to $\Gamma_0^{(1)}(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2;\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}$ in the usual sense.

Main Theorem 1.2. (i) *We assume that N is a squarefree positive integer and $\omega(N)$ is odd. If $j \geq 2$ is even and $k \geq 5$, or $j = 0$ and $k \geq 3$, then we have*

$$\begin{aligned}
&\sum_{M|N} (-1)^{\omega(M)} 2^{\omega(M)} \dim S_{k,j}(K(N/M)) \\
&= \dim \mathfrak{M}_{k+j-3,k-3}(U(N)) \\
&\quad - \sum_{M|N,\omega(M)=odd} (\dim S_{j+2}^{new}(\Gamma_0^{(1)}(M)) + \delta_{j0}) \times \dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)) - \delta_{j0}\delta_{k3},
\end{aligned}$$

where δ_{j0} and δ_{k3} is the Kronecker delta.

(ii) *We assume that N is a squarefree positive integer and $\omega(N)$ is even. If j is an even non-negative integer and $k \geq 5$, then we have*

$$\sum_{M|N}(-1)^{\omega(M)}2^{\omega(M)}\dim S_{k,j}(K(N/M))$$
$$=\dim S_{k,j}(U'(N))-\sum_{M|N,\omega(M)=odd}(\dim S_{j+2}^{new}(\Gamma_0^{(1)}(M))+\delta_{j0})\times\dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)).$$

We conjecture that the same equality holds also for $k=3$, 4 for any even $j\geq 0$ in both cases.

This theorem makes us expect a correspondence between $S_{k,j}(K(N))$ and $\mathfrak{M}_{k+j-3,k-3}(U(N))$ or $S_{k,j}(U'(N))$ according to the parity of $\omega(N)$. The first named author proved the above equality (i) already in the case of $\omega(N)=1$ and proposed a conjecture of correspondence between spaces of "new forms" of $S_{k,j}(K(p))$ and $\mathfrak{M}_{k+j-3,k-3}(U(p))$ in [**Ibu85**] and [**Ibu07a**]. Main Theorem 1.2 makes us expect that we can propose the following generalized conjecture for subspaces of suitably defined new forms of $S_{k,j}(K(N))$, $\mathfrak{M}_{k+j-3,k-3}(U(N))$, and $S_{k,j}(U'(N))$.

Conjecture 1.3. *Let N be a squarefree positive integer. We denote subspaces of "new forms" by $S_{k,j}^{new}(K(N))$, $\mathfrak{M}_{k+j-3,k-3}^{new}(U(N))$, and $S_{k,j}^{new}(U'(N))$. Then for any $k\geq 3$ and even $j\geq 0$, we have isomorphisms*

$$S_{k,j}^{new}(K(N))\xrightarrow{\sim}\begin{cases}\mathfrak{M}_{k+j-3,k-3}^{new}(U(N)) & \text{if } \omega(N) \text{ is odd,}\\ S_{k,j}^{new}(U'(N)) & \text{if } \omega(N) \text{ is even}\end{cases}$$

which preserve (the spinor) L functions.

Definitions of new forms has been given in [**Ibu85**] when N is a prime. The general definition for squarefree N will be explained later in Section 4. Here, to describe such precise isomorphisms, we note that the choice of $U(N)$ or $U^{'}(N)$ is quite essential (See section 2 for definition.)

The relation in Main Theorem 1.2 strongly suggests also that there should exist liftings to automorphic forms of both sides from elliptic modular forms. What to expect on such liftings is closely related to Conjecture 1.3 above. To describe the conjecture precisely on such liftings, we must specify the action of the Atkin–Lehner involution. We define

$$S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(N))=\left\{f\in S_{2k-2}^{new}(\Gamma_0^{(1)}(N))\;\middle|\;u_Nf=\mp(-1)^kf\right\},$$

where u_N is the Atkin–Lehner involution defined by the action of $\begin{bmatrix}0&-1\\N&0\end{bmatrix}$. In fact, the sign of the superscript $\pm$ of $S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(N))$ is equal to the sign of the functional equation (and *not* the eigenvalue of the Atkin–Lehner involution), that is, if we put $\Lambda(s,f)=(2\pi/\sqrt{N})^{-s}\Gamma(s)L(s,f)$ for $f\in S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(N))$, then $\Lambda(2k-2-s)=\pm\Lambda(s,f)$.

Careful check of examples and dimensions suggest the existence of the following liftings.

Conjecture 1.4. *The spaces $\mathfrak{M}_{k-3,k-3}(U(N))$ and $S_{k,0}(U'(N))$ for $k\geq 3$ have* *liftings from*

$$\begin{cases} S_2^{new}(\Gamma_0^{(1)}(M)) \times S_{2k-2}^{new}(\Gamma_0^{(1)}(N/M)) : M \mid N,\ \omega(M) = odd, \\ S_{2k-2}^{new,+}(\Gamma_0^{(1)}(N/M)) \qquad : M \mid N,\ \omega(M) = odd, \\ S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N/M)) \qquad : M \mid N,\ \omega(M) = even. \end{cases}$$

If $j \geq 2$, the spaces $\mathfrak{M}_{k+j-3,k-3}(U(N))$ and $S_{k,j}(U'(N))$ for $k \geq 3$ have liftings from

$$S_{j+2}^{new}(\Gamma_0^{(1)}(M)) \times S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)) \quad : \quad M \mid N,\ \omega(M) = odd.$$

All the above lifting maps are injective.

In this Conjecture, the relation of the L functions should be as follows. For $g \in S_{j+2}^{new}(\Gamma_0^{(1)}(M))$ and $f \in S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M))$, the above lift $\iota(f,g) \in \mathfrak{M}_{k+j-3,k-3}(U(N))$ or $S_{k,j}(U^{'}(N))$ from the pair (f,g) should satisfy

$$L(s, \iota(f,g), Sp) = L(s,f)L(s-k+2,g),$$

up to bad Euler factors where the LHS is the spinor L function. For $f \in S_{2k-2}^{new,\pm}(\Gamma_0(M))$ for an appropriate sign and a lift $\iota(f) \in \mathfrak{M}_{k-3,k-3}(U(N))$ or $S_{k,0}(U^{'}(N))$, we should have

$$L(s, \iota(f), Sp) = \zeta(s-k+1)\zeta(s-k+2)L(s,f)$$

up to bad Euler factors.

The last claim on injectivity is important since this is suggested by dimensional relations and a key point for the definition of new forms.

By the way, we recall that for $\mathfrak{M}_{k+j-3,k-3}(U(N))$, there has been a certain theory of Saito–Kurokawa type or Yoshida type lifting initiated by Ihara [**Iha64**] and generalized in [**II87**], but these constructions do not cover all the above cases. Besides, in the case of $S_{k,j}(U^{'}(N))$, any construction of Yoshida type lifting is not known as far as the authors know, so it seems very interesting to explore such a concrete theory in this case.

We organize this paper as follows. This paper consists of nine sections. In Section 2, we give preliminaries on Siegel cusp forms, paramodular groups, compact twists and non-split $\mathbb{Q}$-forms of $Sp(2;\mathbb{R})$. In Section 3, we give Main Theorem 1.1 precisely (Theorem 3.1) and give the proof of Main Theorem 1.2. We postpone the proof of Main Theorem 1.1 until Sections 6–8 because it is very lengthy. Section 6 gives preliminaries to describing the proof, and Sections 7 and 8 give the proof in detail. In Section 4, we explain details on Main Theorem 1.2 and Conjecture 1.3. In particular, we give definitions of new forms and we explain more in detail why our Conjecture 1.3 and 1.4 seem reasonable. Numerical evidence of L functions which support Conjecture 1.3 and 1.4 is given as Theorem 4.6. In Section 5, we give some details how to calculate the results in Theorem 4.6 and explain more about liftings. Finally, we give numerical tables of dimensions in Section 9.

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showing their interest in our works.

2. Preliminaries.

2.1. Siegel cusp forms.

Let $Sp(2;\mathbb{R})$ be the real symplectic group of degree two:

$$Sp(2;\mathbb{R}) = \left\{ g \in GL(4,\mathbb{R}) \;\middle|\; g \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix} {}^t g = \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix} \right\}.$$

Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two:

$$\mathfrak{H}_2 = \{ Z \in M(2;\mathbb{C}) \mid {}^t Z = Z,\ \mathrm{Im}(Z) \text{ is positive definite } \}.$$

The group $Sp(2;\mathbb{R})$ acts on $\mathfrak{H}_2$ by

$$g\langle Z \rangle := (AZ+B)(CZ+D)^{-1}$$

for any $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2;\mathbb{R})$ and $Z \in \mathfrak{H}_2$.

Let Γ be a discrete subgroup of $Sp(2;\mathbb{R})$ such that $\mathrm{vol}(\Gamma\backslash\mathfrak{H}_2) < \infty$. Let $\tau_{k,j} : GL(2;\mathbb{C}) \to GL(j+1;\mathbb{C})$ be the irreducible rational representation of the Young diagram parameter $(j+k,k)$ for $k, j \in \mathbb{Z}_{\geq 0}$, i.e. $\tau_{k,j} = \det^k \otimes Sym_j$, where Sym_j is the symmetric j-th tensor representation of $GL(2;\mathbb{C})$. We denote by $S_{k,j}(\Gamma)$ the space of Siegel cusp forms of weight $\tau_{k,j}$ with respect to Γ, i.e. the space which consists of holomorphic function $f : \mathfrak{H}_2 \to \mathbb{C}^{j+1}$ satisfying the following two conditions:

(i) $f(\gamma\langle Z \rangle) = \tau_{k,j}(CZ+D) f(Z)$, for $\forall \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma$, $\forall Z \in \mathfrak{H}_2$,

(ii) $\left| \tau_{k,j}(\mathrm{Im}(Z)^{1/2}) f(Z) \right|_{\mathbb{C}^{j+1}}$ is bounded on $\mathfrak{H}_2$,

where we define $|u|_{\mathbb{C}^{j+1}} = ({}^t u \overline{u})^{1/2}$ for $u \in \mathbb{C}^{j+1}$.

Let N be a positive integer. We denote the paramodular group of level N by $K(N)$, namely

$$K(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap Sp(2;\mathbb{Q}).$$

2.2. Compact twist of $Sp(2,\mathbb{R})$.

Let B be a definite quaternion algebra over $\mathbb{Q}$ with discriminant N. We put

$$G = \left\{ g \in M(2;B) \mid g \cdot {}^t\overline{g} = n(g) 1_2, n(g) \in \mathbb{Q}^\times \right\},$$
$$G_\infty = \left\{ g \in M(2;\mathbb{H}) \mid g \cdot {}^t\overline{g} = n(g) 1_2, n(g) \in \mathbb{R}_+^\times \right\}$$

($\mathbb{H}$: the division quaternion algebra over $\mathbb{R}$). Then $G^1_\infty = \{g \in G_\infty; n(g) = 1\}$ is the compact twist of $Sp(2;\mathbb{R})$. Let G_A be the adelization and G_p the local component at a prime p. We fix a maximal order $\mathfrak{O}$ of B and let $\mathfrak{O}_p$ be the local completion. For $p \mid N$, we take $\xi \in GL(2;\mathfrak{O}_p) = M_2(\mathfrak{O}_p)^\times$ such that $\xi \cdot {}^t\overline{\xi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and let π be a prime element of $\mathfrak{O}_p$. Put

$$U_p = \xi^{-1} \begin{bmatrix} \mathfrak{O}_p & \pi^{-1}\mathfrak{O}_p \\ \pi\mathfrak{O}_p & \mathfrak{O}_p \end{bmatrix}^\times \xi \cap G_p.$$

For any prime $p \nmid N$, we put $U_p = GL(2;\mathfrak{O}_p) \cap G_p$. We define an open subgroup $U(N)$ of G_A by

$$U(N) = G_\infty \prod_p U_p.$$

We take a rational representation (ρ_{f_1,f_2}, V) of G^1_∞ corresponding to the Young diagram parameter (f_1, f_2) with $f_1 \geq f_2 \geq 0$. We assume that $f_1 \equiv f_2 \bmod 2$. Then ρ_{f_1,f_2} factors through $G^1_\infty/\{\pm 1_2\}$. We define a representation of G_A by

$$G_A \to G_\infty \to G_\infty/\mathbb{R}^\times 1_2 \simeq G^1_\infty/\{\pm 1_2\} \to GL(V)$$

and denote this also by ρ_{f_1,f_2}. A V-valued function $f(g)$ on G_A is defined to be an automorphic form of weight ρ_{f_1,f_2} belonging to $U(N)$ if $f(uga) = \rho_{f_1,f_2}(u)f(g)$ for any $a \in G$, $u \in U(N)$, and $g \in G_A$. We denote the space of these automorphic forms by $\mathfrak{M}_{f_1,f_2}(U(N))$.

2.3. Non-split $\mathbb{Q}$-forms of $Sp(2,\mathbb{R})$.

Let B be an indefinite division quaternion algebra over $\mathbb{Q}$ with discriminant N. We define an algebraic group G^* by

$$G^* = \left\{ g \in M(2;B) \mid g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} {}^t\overline{g} = n(g) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, n(g) \in \mathbb{Q}^\times \right\}.$$

Then $G^{*1} = \{g \in G^* \mid n(g) = 1\}$ is a non-split $\mathbb{Q}$ form of $Sp(2,\mathbb{R})$ and $G^{*1}_\infty \cong Sp(2,\mathbb{R})$. Let $\mathfrak{O}$ be a maximal order of B and $\mathfrak{P}$ the maximal ideal of $\mathfrak{O}_p$ when $p \mid N$. We take the two-sided ideal $\mathfrak{A}$ of $\mathfrak{O}$ such that $\mathfrak{A}_p = \mathfrak{P}$ for $p \mid N$ and $\mathfrak{A}_p = \mathfrak{O}_p$ for $p \nmid N$. We define a discrete subgroup $U'(N)$ by

$$U'(N) = \begin{bmatrix} \mathfrak{O} & \mathfrak{A}^{-1} \\ \mathfrak{A} & \mathfrak{O} \end{bmatrix}^\times \cap G^{*1}.$$

We can regard this as a discrete subgroup of $Sp(2;\mathbb{R})$ with covolume finite through the isomorphism $G^{*1}_\infty \cong Sp(2,\mathbb{R})$.

2.4. Dimension formulas for compact twist and non-split forms.

Explicit formulas of $\dim\mathfrak{M}_{k+j-3,k-3}(U)$ and $\dim S_{k,j}(U')$ were given in [**HI82**] and [**Kit11**] for more general U and $U^{'}$. Here for $U(N)$ and $U^{'}(N)$, we rewrite the formulas below in slightly easier way than the previous papers, which is more suitable for our comparison. We use the following notations. We denote by $\omega(N)$ the number of prime divisors of N. For natural number m and n, we denote by $[a_0,\ldots,a_{m-1};m]_n$ the function of n which takes the value a_i if $n\equiv i \bmod m$. We define the set $N(m;n):=\{p\mid N; p\equiv m \bmod n\}$. We use the Legendre symbols:

$$\left(\frac{-1}{p}\right)=\begin{cases}0 & \cdots \text{ if } p=2,\\ 1 & \cdots \text{ if } p\equiv 1 \bmod 4,\\ -1 & \cdots \text{ if } p\equiv 3 \bmod 4,\end{cases}\qquad \left(\frac{-3}{p}\right)=\begin{cases}0 & \cdots \text{ if } p=3,\\ 1 & \cdots \text{ if } p\equiv 1 \bmod 3,\\ -1 & \cdots \text{ if } p\equiv 2 \bmod 3.\end{cases}$$

We define the following functions:

$$C_3(k,j)=[(k-2)(-1)^{j/2},-(j+k-1),-(k-2)(-1)^{j/2},j+k-1;4]_k,$$
$$C_4(k,j)=(j+k-1)\cdot[1,-1,0;3]_k+(k-2)\cdot[1,0,-1;3]_{j+k},$$
$$C_5(k,j)=(j+k-1)\cdot[-1,-1,0,1,1,0;6]_k+(k-2)\cdot[1,0,-1,-1,0,1;6]_{j+k},$$

$$C_8(k,j)=\begin{cases}[1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]_k\cdots & \text{if } j\equiv 0 \bmod 12,\\ [-1,1,0,1,1,0,1,-1,0,-1,-1,0;12]_k\cdots & \text{if } j\equiv 2 \bmod 12,\\ [1,-1,0,0,-1,1,-1,1,0,0,1,-1;12]_k\cdots & \text{if } j\equiv 4 \bmod 12,\\ [-1,0,0,-1,1,-1,1,0,0,1,-1,1;12]_k\cdots & \text{if } j\equiv 6 \bmod 12,\\ [1,1,0,1,-1,0,-1,-1,0,-1,1,0;12]_k\cdots & \text{if } j\equiv 8 \bmod 12,\\ [-1,-1,0,0,1,1,1,1,0,0,-1,-1;12]_k\cdots & \text{if } j\equiv 10 \bmod 12,\end{cases}$$

$$C_9(k,j)=\begin{cases}[1,0,0,-1,0,0;6]_k\ \cdots & \text{if } j\equiv 0 \bmod 6,\\ [-1,1,0,1,-1,0;6]_k\cdots & \text{if } j\equiv 2 \bmod 6,\\ [0,-1,0,0,1,0;6]_k\ \cdots & \text{if } j\equiv 4 \bmod 6,\end{cases}$$

$$C_{10}(k,j)=\begin{cases}[1,0,0,-1,0;5]_k\cdots & \text{if } j\equiv 0 \bmod 10,\\ [-1,1,0,0,0;5]_k\cdots & \text{if } j\equiv 2 \bmod 10,\\ 0\ \cdots & \text{if } j\equiv 4 \bmod 10,\\ [0,0,0,1,-1;5]_k\cdots & \text{if } j\equiv 6 \bmod 10,\\ [0,-1,0,0,1;5]_k\cdots & \text{if } j\equiv 8 \bmod 10,\end{cases}$$

$$C_{11}(k,j)=\begin{cases}[1,0,0,-1;4]_k\cdots & \text{if } j\equiv 0 \bmod 8,\\ [-1,1,0,0;4]_k\cdots & \text{if } j\equiv 2 \bmod 8,\\ [-1,0,0,1;4]_k\cdots & \text{if } j\equiv 4 \bmod 8,\\ [1,-1,0,0;4]_k\cdots & \text{if } j\equiv 6 \bmod 8.\end{cases}$$

Theorem 2.1. *Let N be a squarefree positive integer and j an even non-negative integer. Then we have*

$$\dim \mathfrak{M}_{k+j-3,k-3}(U(N)) = \sum_{i=1}^{12} H_i'(k,j,N) \qquad \text{if } \omega(N) \text{ is odd and } k \geq 3,$$

$$\dim S_{k,j}(U'(N)) = \sum_{i=1}^{12} H_i'(k,j,N) + \sum_{i=1}^{10} I_i'(k,j,N) \qquad \text{if } \omega(N) \text{ is even and } k \geq 5,$$

where $H_i'(k,j,N)$ *and* $I_i'(k,j,N)$ *are given as follows:*

$$H_1'(k,j,N) = \prod_{p|N}(p^2-1)\cdot 2^{-7}3^{-3}5^{-1}\cdot(j+1)(k-2)(j+k-1)(j+2k-3),$$

$$H_2'(k,j,N) = -2^{-7}3^{-2}(j+k-1)(k-2)(-1)^k \times \begin{cases} 3 & \text{if } N=2, \\ 0 & \text{if } N\neq 2, \end{cases}$$

$$H_3'(k,j,N) = 2^{-5}3^{-1}C_3(k,j) \times \begin{cases} 3 & \text{if } N=2, \\ 0 & \text{if } N\neq 2, \end{cases}$$

$$H_4'(k,j,N) = 2^{-3}3^{-3}C_4(k,j) \times \begin{cases} 8 & \text{if } N=3, \\ 0 & \text{if } N\neq 3, \end{cases}$$

$$H_5'(k,j,N) = 0,$$

$$H_6'(k,j,N) = H_{6,1}'(N)\times(-1)^{j/2}(2k+j-3) + H_{6,2}'(N)\times(-1)^{j/2+k}(j+1), \text{ where}$$

$$H_{6,1}'(N) = \frac{1}{2^5\cdot 3}\prod_{p|N}\left(p-\left(\frac{-1}{p}\right)\right) + \frac{1}{2^7\cdot 3}\prod_{p|N}\left(p\left(\frac{-1}{p}\right)-1\right),$$

$$H_{6,2}'(N) = \frac{(-1)^{\omega(N)}}{2^5\cdot 3}\prod_{p|N}\left(p-\left(\frac{-1}{p}\right)\right) - \frac{1}{2^7\cdot 3}\prod_{p|N}\left(p\left(\frac{-1}{p}\right)-1\right),$$

$$H_7'(k,j,N) = H_{7,1}'(N)\times[1,-1,0;3]_j(2k+j-3) + H_{7,2}'(N)\times[0,1,-1;3]_{j+2k}(j+1), \text{ where}$$

$$H_{7,1}'(N) = \frac{1}{2^3\cdot 3^2}\prod_{p|N}\left(p-\left(\frac{-3}{p}\right)\right) + \frac{1}{2^3\cdot 3^3}\prod_{p|N}\left(p\left(\frac{-3}{p}\right)-1\right),$$

$$H_{7,2}'(N) = \frac{(-1)^{\omega(N)}}{2^3\cdot 3^2}\prod_{p|N}\left(p-\left(\frac{-3}{p}\right)\right) - \frac{1}{2^3\cdot 3^3}\prod_{p|N}\left(p\left(\frac{-3}{p}\right)-1\right),$$

$$H_8'(k,j,N) = 0,$$

$$H_9'(k,j,N) = -2^{-1}3^{-2}C_9(k,j) \times \begin{cases} 3 & \text{if } N=2, \\ 0 & \text{if } N\neq 2, \end{cases}$$

$$H_{10}'(k,j,N) = (-1)^{\omega(N)}2^{-1}5^{-1}C_{10}(k,j)\prod_{p|N} 2 \times \begin{cases} 0 & \text{if } N(1;5)\cup N(4;5)\neq\emptyset, \\ 1 & \text{if } N(1;5)\cup N(4;5)=\emptyset \text{ and } 5\mid N, \\ 2 & \text{otherwise}, \end{cases}$$

$$H_{11}'(k,j,N) = (-1)^{\omega(N)}2^{-3}C_{11}(k,j)\prod_{p|N, p\neq 2} 2 \times \begin{cases} 0 & \text{if } N(1;8)\cup N(7;8)\neq\emptyset, \\ 1 & \text{if } N(1;8)\cup N(7;8)=\emptyset, \end{cases}$$

$$H_{12}'(k,j,N) = H_{12,1}'(N)\times[1,-1,0;3]_j(-1)^{j/2+k} + H_{12,2}'(N)\times[0,-1,1;3]_{j+2k}(-1)^{j/2},$$

where

$$H'_{12,1}(N)=\frac{(-1)^{\omega(N)}}{2^3\cdot 3}\prod_{p|N}\left(1-\left(\frac{3}{p}\right)\right)-\frac{1}{2^3\cdot 3}\prod_{p|N}\left(\left(\frac{-1}{p}\right)-\left(\frac{-3}{p}\right)\right),$$

$$H'_{12,2}(N)=\frac{(-1)^{\omega(N)}}{2^3\cdot 3}\prod_{p|N}\left(1-\left(\frac{3}{p}\right)\right)+\frac{(-1)^{\omega(N)}}{2^3\cdot 3}\prod_{p|N}\left(\left(\frac{-1}{p}\right)-\left(\frac{-3}{p}\right)\right),$$

$$I'_1(k,j,N)=2^{-3}3^{-1}(j+1)\prod_{p|N}(p-1),$$

$$\begin{aligned}I'_2(k,j,N)&=I'_3(k,j,N)=I'_4(k,j,N)=I'_5(k,j,N)\\&=I'_6(k,j,N)=I'_7(k,j,N)=I'_8(k,j,N)=0,\end{aligned}$$

$$I'_9(k,j,N)=-2^{-3}(-1)^{j/2}\prod_{p|N}\left(\left(\frac{-1}{p}\right)-1\right),$$

$$I'_{10}(k,j,N)=-2^{-1}3^{-1}[1,-1,0;3]_j\cdot\prod_{p|N}\left(\left(\frac{-3}{p}\right)-1\right).$$

We conjecture that the formula for $\dim S_{k,j}(U^{'}(N))$ is valid also for $k=3$ and 4 for any even $j\geq 0$, if we add a correction term $+1$ to the RHS in case $(k,j)=(3,0)$.

3. Main Theorem 1.1 and the proof of Main Theorem 1.2.

In this section, we give Main Theorem 1.1 precisely and give the proof of Main Theorem 1.2. We postpone the proof of Main Theorem 1.1 until Sections 6 – 8 because it is very lengthy.

3.1. Main Theorem 1.1.

We give Main Theorem 1.1 precisely (Theorem 3.1 below). We use the same notations as in Theorem 2.1 and we denote the cardinality of $N(m;n)$ by $\sharp N(m;n)$. We use the Jacobi symbol for squarefree integers N:

$$\left(\frac{-1}{N}\right)=\prod_{p|N}\left(\frac{-1}{p}\right).$$

When we use the following formula for $N=1$, we should read $\prod_{p|N}(\cdots)=1$ and $N(m;n)=\emptyset$. We note that, if j is odd, then we have $S_{k,j}(K(N))=\{0\}$ for any k since $K(N)$ contains -1_4.

Theorem 3.1 (Main Theorem 1.1). *We assume N is a squarefree positive integer. If $j\geq 2$ is even and $k\geq 5$, or $j=0$ and $k\geq 3$, then we have*

$$\dim S_{k,j}(K(N))=\sum_{i=1}^{12}H_i(k,j,N)+\sum_{i=1}^{10}I_i(k,j,N)+\delta_{j0}\delta_{k3},$$

where $H_i(k,j,N)$ and $I_i(k,j,N)$ are defined in Section 6 and are evaluated explicitly as follows:

$$H_1(k,j,N)=\prod_{p|N}(p^2+1)\cdot 2^{-7}3^{-3}5^{-1}\cdot(j+1)(k-2)(j+k-1)(j+2k-3),$$

$$H_2(k,j,N)=2^{\omega(N)-8}3^{-2}\cdot(j+k-1)(k-2)(-1)^k\cdot\begin{cases}11\cdots \textit{ if } 2\,|\,N,\\ 14\cdots \textit{ if } 2\nmid N,\end{cases}$$

$$H_3(k,j,N)=2^{\omega(N)-6}3^{-1}\cdot C_3(k,j)\cdot\begin{cases}5\cdots \textit{ if } 2\,|\,N,\\ 2\cdots \textit{ if } 2\nmid N,\end{cases}$$

$$H_4(k,j,N)=2^{\omega(N)-3}3^{-3}\cdot C_4(k,j)\cdot\begin{cases}5\cdots \textit{ if } 3\,|\,N,\\ 1\cdots \textit{ if } 3\nmid N,\end{cases}$$

$$H_5(k,j,N)=2^{\omega(N)-3}3^{-2}\cdot C_5(k,j),$$

$$H_6(k,j,N)=H_{6,1}(N)\times(-1)^{j/2}(2k+j-3)+H_{6,2}(N)\times(-1)^{j/2+k}(j+1), \textit{ where}$$

$$H_{6,1}(N)=\frac{1}{2^5\cdot 3}\prod_{p|N}\left(p+\left(\frac{-1}{p}\right)\right)+\frac{1}{2^7\cdot 3}\prod_{p|N}\left(p\left(\frac{-1}{p}\right)+1\right),$$

$$H_{6,2}(N)=\frac{1}{2^5\cdot 3}\prod_{p|N}\left(p+\left(\frac{-1}{p}\right)\right)-\frac{1}{2^7\cdot 3}\prod_{p|N}\left(p\left(\frac{-1}{p}\right)+1\right),$$

$$H_7(k,j,N)=H_{7,1}(N)\times[1,-1,0;3]_j(2k+j-3)+H_{7,2}(N)\times[0,1,-1;3]_{j+2k}(j+1),$$

where

$$H_{7,1}(N)=\frac{1}{2^3\cdot 3^2}\prod_{p|N}\left(p+\left(\frac{-3}{p}\right)\right)+\frac{1}{2^3\cdot 3^3}\prod_{p|N}\left(p\left(\frac{-3}{p}\right)+1\right),$$

$$H_{7,2}(N)=\frac{1}{2^3\cdot 3^2}\prod_{p|N}\left(p+\left(\frac{-3}{p}\right)\right)-\frac{1}{2^3\cdot 3^3}\prod_{p|N}\left(p\left(\frac{-3}{p}\right)+1\right),$$

$$H_8(k,j,N)=2^{\omega(N)-2}3^{-1}\cdot C_8(k,j),$$

$$H_9(k,j,N)=2^{\omega(N)-2}3^{-2}\cdot C_9(k,j)\cdot\begin{cases}1\cdots \textit{ if } 2\,|\,N,\\ 4\cdots \textit{ if } 2\nmid N,\end{cases}$$

$$H_{10}(k,j,N)=5^{-1}\cdot C_{10}(k,j)\cdot\begin{cases}2^{\sharp N(\pm1;5)}\cdots \textit{ if } N(2;5)\cup N(3;5)=\emptyset,\\ \quad 0 \qquad\cdots \textit{ otherwise,}\end{cases}$$

$$H_{11}(k,j,N)=2^{-3}\cdot C_{11}(k,j)\cdot\begin{cases}2^{\sharp N(\pm1;8)}\cdots \textit{ if } N(3;8)\cup N(5;8)=\emptyset,\\ \quad 0 \qquad\cdots \textit{ otherwise,}\end{cases}$$

$$H_{12}(k,j,N)=H_{12,1}(N)\times[1,-1,0;3]_j(-1)^{j/2+k}+H_{12,2}(N)\times[0,-1,1;3]_{j+2k}(-1)^{j/2},$$

where

$$H_{12,1}(N)=\frac{1}{2^3\cdot 3}\prod_{p|N}\left(1+\left(\frac{3}{p}\right)\right)-\frac{1}{2^3\cdot 3}\prod_{p|N}\left(\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right)\right),$$

$$H_{12,2}(N)=\frac{1}{2^3\cdot 3}\prod_{p|N}\left(1+\left(\frac{3}{p}\right)\right)+\frac{1}{2^3\cdot 3}\prod_{p|N}\left(\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right)\right),$$

$$I_1(k,j,N)=2^{-3}3^{-1}\cdot(j+1)\cdot\prod_{p|N}(p+1),$$
$$I_2(k,j,N)=-2^{\omega(N)-4}3^{-1}\cdot(j+1),$$
$$I_3(k,j,N)=-2^{\omega(N)-5}3^{-2}\cdot N\cdot(j+1)(2k+j-3),$$
$$I_4(k,j,N)=2^{\omega(N)-5}\cdot(-1)^k\cdot\left(4-\left(\frac{-1}{N}\right)\right),$$
$$I_5(k,j,N)=-2^{\omega(N)-4}\cdot 3^{-1}\cdot(-1)^k\cdot(2k+j-3),$$
$$I_6(k,j,N)=-2^{\omega(N)-3}\cdot[(-1)^{j/2},-1,-(-1)^{j/2},1;4]_k,$$
$$I_7(k,j,N)=-2^{\omega(N)-2}\cdot 3^{-2}\cdot\begin{cases}[3,-3,0;3]_k+[3,0,-3;3]_{j+k} & \cdots \text{ if } N\equiv 0 \bmod 3,\\ [5,-1,4;3]_k+[1,-4,-5;3]_{j+k} & \cdots \text{ if } N\equiv 1 \bmod 3,\\ [1,-5,-4;3]_k+[5,4,-1;3]_{j+k} & \cdots \text{ if } N\equiv 2 \bmod 3,\end{cases}$$
$$I_8(k,j,N)=-2^{\omega(N)-2}\cdot 3^{-1}\cdot([-1,-1,0,1,1,0;6]_k+[1,0,-1,-1,0,1;6]_{j+k}),$$
$$I_9(k,j,N)=-2^{-3}\cdot(-1)^{j/2}\cdot\prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right),$$
$$I_{10}(k,j,N)=-2^{-1}3^{-1}\cdot[1,-1,0;3]_j\cdot\prod_{p|N}\left(1+\left(\frac{-3}{p}\right)\right).$$

Proof. We prove Theorem 3.1 in Sections 6 – 8 by using the method based on the Selberg trace formula. □

3.2. Proof of Main Theorem 1.2.

We can prove Main Theorem 1.2 as follows. Let $H_i(k,j,N)$, $H_i'(k,j,N)$, $I_i(k,j,N)$ and $I_i'(k,j,N)$ be the same as in Theorem 2.1 and Theorem 3.1. We put

$$H_i^*(k,j,N)=\sum_{M|N}(-1)^{\omega(M)}2^{\omega(M)}H_i(k,j,N/M)-H_i'(k,j,N),$$
$$I_i^*(k,j,N)=\sum_{M|N}(-1)^{\omega(M)}2^{\omega(M)}I_i(k,j,N/M)-\delta(N)\cdot I_i'(k,j,N),$$

where the sums are taken over positive divisors of N , and we put $\delta(N)=0$ if $\omega(N)$ is odd and $\delta(N)=1$ if $\omega(N)$ is even.

Lemma 3.2. *We have*

$$\begin{aligned}H_1^*(j,k,N)&=H_2^*(j,k,N)=H_3^*(j,k,N)=H_4^*(j,k,N)=H_5^*(j,k,N)\\&=H_8^*(j,k,N)=H_9^*(j,k,N)=H_{10}^*(j,k,N)=H_{11}^*(j,k,N)=0,\\ I_2^*(j,k,N)&=I_5^*(j,k,N)=I_6^*(j,k,N)=I_8^*(j,k,N)=0,\end{aligned}$$

regardless of the parity of $\omega(N)$.

Proof. The proof can be obtained by straightforward calculations, using the following formula

$$\sum_{M|N}(-1)^{\omega(M)}a^{\omega(M)}\prod_{p|(N/M)}x_p=\prod_{p|N}(x_p-a),$$

which is almost trivial. □

As for the others, situations depend on the parity of $\omega(N)$.

Case 1. The case $\omega(N)=\text{odd}$: We can obtain the other parts as follows by straightforward calculation.

$$H_6^*(k,j,N)=H_{6,1}^*(N)\times(-1)^{j/2}(2k+j-3)+H_{6,2}^*(N)\times(-1)^{j/2+k}(j+1),\text{where}$$
$$H_{6,1}^*(N)=\frac{1}{2^4\cdot 3}\sum_{M|N,\omega(M)=odd}\prod_{p|M}\left(\left(\frac{-1}{p}\right)-1\right)\prod_{p|(N/M)}(p-1),$$
$$H_{6,2}^*(N)=\frac{1}{2^4\cdot 3}\sum_{M|N,\omega(M)=even}\prod_{p|M}\left(\left(\frac{-1}{p}\right)-1\right)\prod_{p|(N/M)}(p-1),$$
$$H_7^*(k,j,N)=H_{7,1}^*(N)\times[1,-1,0;3]_j(2k+j-3)+H_{7,2}^*(N)\times[0,1,-1;3]_{j+2k}(j+1),\text{where}$$
$$H_{7,1}^*(N)=\frac{1}{2^2\cdot 3^2}\sum_{M|N,\omega(M)=odd}\prod_{p|M}\left(\left(\frac{-3}{p}\right)-1\right)\prod_{p|(N/M)}(p-1),$$
$$H_{7,2}^*(N)=\frac{1}{2^2\cdot 3^2}\sum_{M|N,\omega(M)=even}\prod_{p|M}\left(\left(\frac{-3}{p}\right)-1\right)\prod_{p|(N/M)}(p-1),$$
$$H_{12}^*(k,j,N)=H_{12,1}^*(N)\times[1,-1,0;3]_j(-1)^{j/2+k}+H_{12,2}^*(N)\times[0,1,-1;3]_{j+2k}(-1)^{j/2},$$

where

$$H_{12,1}^*(N)=-\frac{1}{2^2\cdot 3}\sum_{M|N,\omega(M)=even}\prod_{p|M}\left(\left(\frac{-1}{p}\right)-1\right)\prod_{p|(N/M)}\left(\left(\frac{-3}{p}\right)-1\right),$$
$$H_{12,2}^*(N)=-\frac{1}{2^2\cdot 3}\sum_{M|N,\omega(M)=odd}\prod_{p|M}\left(\left(\frac{-1}{p}\right)-1\right)\prod_{p|(N/M)}\left(\left(\frac{-3}{p}\right)-1\right),$$
$$I_1^*(k,j,N)=\frac{1}{2^3\cdot 3}(j+1)\prod_{p|N}(p-1),$$
$$I_3^*(k,j,N)=-\frac{2^{\omega(N)}}{2^5\cdot 3^2}\prod_{p|N}(p-1)\times(j+1)(2k+j-3),$$
$$I_4^*(k,j,N)=-\frac{2^{\omega(N)}}{2^5}\prod_{p|N}\left(\left(\frac{-1}{p}\right)-1\right)\times(-1)^k,$$
$$I_7^*(k,j,N)=-\frac{2^{\omega(N)}}{2\cdot 3^2}\times[1,-1,0;3]_j\times[0,1,-1;3]_{j+2k}\times\prod_{p|N}\left(\left(\frac{-3}{p}\right)-1\right),$$
$$I_9^*(k,j,N)=-\frac{1}{2^3}\times\prod_{p|N}\left(\left(\frac{-1}{p}\right)-1\right)\times(-1)^{j/2},$$
$$I_{10}^*(k,j,N)=-\frac{1}{2\cdot 3}\times\prod_{p|N}\left(\left(\frac{-3}{p}\right)-1\right)\times[1,-1,0;3]_j.$$

By using the following well-known formulas for $M>1$:

$$\begin{aligned}\dim S_{j+2}^{new}(\Gamma_0^{(1)}(M)) =& \frac{j+1}{12}\prod_{p|M}(p-1)-\frac{(-1)^{j/2}}{4}\prod_{p|M}\left(\left(\frac{-1}{p}\right)-1\right)\\ &-\frac{[1,-1,0;3]_j}{3}\prod_{p|M}\left(\left(\frac{-3}{p}\right)-1\right)+\delta_{j,0}\cdot(-1)^{\omega(M)},\\ \dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)) =& \frac{2k+j-3}{12}\prod_{p|(N/M)}(p-1)-\frac{(-1)^{j/2+k}}{4}\prod_{p|(N/M)}\left(\left(\frac{-1}{p}\right)-1\right)\\ &-\frac{[0,1,-1;3]_{2k+j}}{3}\prod_{p|(N/M)}\left(\left(\frac{-3}{p}\right)-1\right)+\delta_{2k+j-2,2}\cdot(-1)^{\omega(N/M)}\\ &-\begin{cases}0 & \cdots \text{ if } N/M>1\\ 1/2 & \cdots \text{ if } N/M=1\end{cases},\end{aligned}$$

where $\delta_{*,*}$ means the Kronecker delta, we can verify Main Theorem 1.2 (i) directly from these when $(k,j)\neq(3,0)$. If $(k,j)=(3,0)$, the formula for $\dim S_{3,0}(K(M))$ is given by adding one to the usual formula ([**Ibu07b**]). This means that the formula for $\sum_{M|N}(-2)^{\omega(M)}\dim S_{3,0}(K(N/M))$ is given by adding $\sum_{M|N}(-2)^{\omega(M)}=(-1)^{\omega(N)}$ to the usual formula. Since $\omega(N)$ is odd now, we should add -1. On the other hand, in the comparison of the theorem, we are subtracting 1 from $\dim \mathfrak{M}_{0,0}$ in this case. The formula for $\dim S_2^{new}(\Gamma_0^{(1)}(M))$ is again given by adding $(-1)^{\omega(M)}$ to the usual formula for weight bigger than 2, but here we took $\omega(M)$ to be odd so this is -1, and this cancels with the correction term $\delta_{j0}=1$, and the formula coming from elliptic modular forms are unchanged. So we obtain the claim also in this case.

Case 2. The case $\omega(N)=$ even: In this case, we obtain the other parts as follows.

$$H_6^*(k,j,N)=H_{6,1}^*(N)\times(-1)^{j/2}(2k+j-3)+H_{6,2}^*(N)\times(-1)^{j/2+k}(j+1),\ \text{where}$$

$$H_{6,1}^*(N)=H_{6,2}^*(N)=\frac{1}{2^4\cdot 3}\sum_{M|N,\omega(M)=odd}\prod_{p|M}\left(\left(\frac{-1}{p}\right)-1\right)\prod_{p|(N/M)}(p-1),$$

$$H_7^*(k,j,N)=H_{7,1}^*(N)\times[1,-1,0;3]_j(2k+j-3)+H_{7,2}^*(N)\times[0,1,-1;3]_{j+2k}(j+1),\ \text{where}$$

$$H_{7,1}^*(N)=H_{7,2}^*(N)=\frac{1}{2^2\cdot 3^2}\sum_{M|N,\omega(M)=odd}\prod_{p|M}\left(\left(\frac{-3}{p}\right)-1\right)\prod_{p|(N/M)}(p-1),$$

$$H_{12}^*(k,j,N)=H_{12,1}^*(N)\times[1,-1,0;3]_j(-1)^{j/2+k}+H_{12,2}^*(N)\times[0,1,-1;3]_{j+2k}(-1)^{j/2},$$

where

$$\begin{aligned}&H_{12,1}^*(N)=H_{12,2}^*(N)\\ &=-\frac{1}{2^2\cdot 3}\sum_{M|N,\omega(M)=odd}\prod_{p|M}\left(\left(\frac{-1}{p}\right)-1\right)\prod_{p|(N/M)}\left(\left(\frac{-3}{p}\right)-1\right),\end{aligned}$$

$$I_1^*(k,j,N)=0,$$

$$
\begin{aligned}
I_3^*(k,j,N) &= -\frac{2^{\omega(N)}}{2^5\cdot 3^2}\prod_{p|N}(p-1)\times(j+1)(2k+j-3),\\
I_4^*(k,j,N) &= -\frac{2^{\omega(N)}}{2^5}\prod_{p|N}\left(\left(\frac{-1}{p}\right)-1\right)\times(-1)^k,\\
I_7^*(k,j,N) &= -\frac{2^{\omega(N)}}{2\cdot 3^2}\times[1,-1,0;3]_j\times[0,1,-1;3]_{j+2k}\times\prod_{p|N}\left(\left(\frac{-3}{p}\right)-1\right),\\
I_9^*(k,j,N) &= 0,\\
I_{10}^*(k,j,N) &= 0.
\end{aligned}
$$

We can verify Main Theorem 1.2 (ii) by almost the same calculation as in Case 1.

Remark. In case 2 for $(k,j)=(3,0)$, as in case 1, the dimension of LHS is given by adding $(-1)^{\omega(N)}=1$ to the usual formula. This supports us to conjecture that the formula for $S_{3,0}(U'(N))$ is valid also for $k=3$, 4 for any even $j\geq 0$ if we add one to the usual formula in case $(k,j)=(3,0)$ and the relation in Theorem 1.2 should be valid for any $k\geq 3$ and even $j\geq 0$.

4. New forms and details of Main Theorem 1.2 and Conjecture 1.3, 1.4.

In this section, we explain how to interpret Main Theorem 1.2 into Conjecture 1.3 and 1.4 in detail. First, we explain the definition of paramodular old forms and new forms and give an explicit formula of dimensions of paramodular new forms under a certain natural assumption. Then we explain liftings naturally expected by this formula and define new forms belonging to other twists. By these consideration, we explain the precise meaning of Conjecture 1.3 and 1.4 on correspondence between $S_{k,j}(K(N))$ and $\mathfrak{M}_{k+j-3,k-3}(U(N))$ and $S_{k,j}(U'(N))$, and the Ihara–Saito–Kurokawa type lifting and the Yoshida type lifting to these spaces, and also make it clear why these are natural, based on our Main Theorem 1.2.

4.1. Paramodular new forms: prime level case.

The definition of paramodular new forms was originally given by the first named author in [**Ibu85**] for prime level case and recently considered by Roberts and Schmidt from the viewpoint of local representation theory (cf. [**RS07**]). For convenience of readers, we review the prime level case first in detail from [**Ibu85**] since this case is easy to understand. We explain the general squarefree level case later. For any non-negative integer m, we write

$$
U_m=\begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & m & 0 & 0\\ m & 0 & 0 & 0\end{pmatrix}.
$$

By the Iwahori–Bruhat–Tits theory, for a prime p, the local split symplectic group $Sp(2,\mathbb{Q}_p)$ of rank 2 has three standard maximal compact subgroups (up to $Sp(2,\mathbb{Q}_p)$ conjugation), one is the localization of $K(p)$ and the other two are $Sp(2,\mathbb{Z}_p)$ and $U_p^{-1}Sp(2,\mathbb{Z}_p)U_p$. The global discrete subgroups corresponding to these are $K(p)$, $K(1) = Sp(2,\mathbb{Z})$ and $U_p^{-1}Sp(2,\mathbb{Z})U_p = U_p^{-1}K(1)U_p$, respectively. There are no inclusion relations between these groups at all. But we can define a mapping between Siegel cusp forms belonging to these discrete groups by taking the traces. We write $\Gamma_0'(p) = K(p)\cap Sp(2,\mathbb{Z})$ and $\Gamma_0''(p) = K(p)\cap U_p^{-1}Sp(2,\mathbb{Z})U_p$. Then since U_p normalizes $K(p)$, we have $\Gamma_0''(p) = U_p^{-1}\Gamma_0'(p)U_p$. We can define two mappings $T(1,p)$ and $T'(1,p)$ to $S_{k,j}(K(p))$, one is from $S_{k,j}(Sp(2,\mathbb{Z}))$ and the other from $S_{k,j}(U_p^{-1}Sp(2,\mathbb{Z})U_p)$, by

$$F|T(1,p) = \sum_{g\in\Gamma_0'(p)\backslash K(p)} F|_{k,j}[g], \qquad G|T'(1,p) = \sum_{g\in\Gamma_0''(p)\backslash K(p)} G|_{k,j}[g]$$

for any $F \in S_{k,j}(Sp(2,\mathbb{Z}))$ and $G \in S_{k,j}(U_p^{-1}Sp(2,\mathbb{Z})U_p)$. We obviously have $S_{k,j}(U_p^{-1}Sp(2,\mathbb{Z})U_p) = S_{k,j}(Sp(2,\mathbb{Z}))|_{k,j}U_p$ as spaces. If $K(p) = \bigcup_i \Gamma_0'(p)\gamma_i$, then we have $K(p) = \bigcup_i \Gamma_0''(p)U_p^{-1}\gamma_i U_p$. So we have $T'(1,p) = U_p^{-1}T(1,p)U_p$ and

$$S_{k,j}(U_p^{-1}Sp(2,\mathbb{Z})U_p)|_{k,j}T'(1,p) = S_{k,j}(Sp(2,\mathbb{Z}))|_{k,j}T(1,p)U_p.$$

So it is natural to define the space of old forms $S_{k,j}^{old}(K(p))$ of $S_{k,j}(K(p))$ by the subspace

$$S_{k,j}^{old}((K(p)) = S_{k,j}(K(1))|T(1,p) + S_{k,j}(K(1))|T(1,p)U_p \subset S_{k,j}(K(p)),$$

where we note that $K(1) = Sp(2,\mathbb{Z})$. We define the space $S_{k,j}^{new}(K(p))$ of new forms as the orthogonal complement of $S_{k,j}^{old}(K(p))$ in $S_{k,j}(K(p))$ with respect to the usual Petersson inner product. The dimensional relation of Theorem 1.2 for prime level case reads

$$\begin{aligned}\dim S_{k,j}(K(p)) &- 2\dim S_{k,j}(K(1)) \\ &= \dim \mathfrak{M}_{k+j-3,k-3}(U(p)) - \left(\dim S_{j+2}^{new}(\Gamma_0^{(1)}(p)) + \delta_{j0}\right) \\ &\quad \times \dim S_{2k+j-2}(SL_2(\mathbb{Z})) - \delta_{j0}\delta_{k3}. \end{aligned} \tag{1}$$

Now we will see how this fits our Conjecture 1.3. For $j = 0$ and even $k \geq 10$, we know that there exists the Saito–Kurokawa lifting from $S_{2k-2}(SL_2(\mathbb{Z}))$ to $S_k(Sp(2,\mathbb{Z}))$. This lifting is known to be injective, that is, if we denote the image of this lifting by SK_k, then we have $\dim SK_k = \dim S_{2k-2}(SL_2(\mathbb{Z}))$. Since the two trace maps coincide on SK_k (which was a conjecture in [**Ibu85**] and now it is a fact by [**RS06**]), we can expect that the correct dimension of old paramodular forms should be

$$\begin{aligned}\dim S_{k,j}(Sp(2,\mathbb{Z})) + \dim S_{k,j}(U_p^{-1}Sp(2,\mathbb{Z})U_p) - \delta_{j0}\dim SK_k \\ = 2\dim S_{k,j}(K(1)) - \delta_{j0}\delta_{k,even}\dim S_{2k-2}(SL_2(\mathbb{Z}))\end{aligned}$$

where we put $\delta_{k,even} = 1$ or 0 according to the case when k is even or odd. When $j = 0$ and k is even, it is more natural to move $-\dim S_{2k-2}(SL_2(\mathbb{Z}))$ in RHS of (1) to $+\dim S_{2k-2}(SL_2(\mathbb{Z}))$ in LHS of (1), expecting that the two trace maps are independent on non Saito–Kurokawa liftings. For $\epsilon = \pm$, we denote by $S_{2k-2}^{new,\epsilon}(SL_2(\mathbb{Z}))$ the space of forms whose sign of the functional equation is ϵ. It is well known that $S_{2k-2}(SL_2(\mathbb{Z})) = S_{2k-2}^{new,+}(SL_2(\mathbb{Z}))$ (resp. $= S_{2k-2}^{new,-}(SL_2(\mathbb{Z}))$), if and only if k is odd (resp. even). So we have $\delta_{k,even}\dim S_{2k-2}(SL_2(\mathbb{Z})) = \dim S_{2k-2}^{new,-}(SL_2(\mathbb{Z}))$.

Next we see the RHS of (1). First of all, there exists a theory of lifting from $S_{j+2}(\Gamma_0^{(1)}(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$ to $\mathfrak{M}_{k+j-3,k-3}(U(p))$ for general $j \geq 0$ and from $S_{2k-2}(SL_2(\mathbb{Z}))$ in case $j = 0$, as is shown in [**Iha64**] and [**II87**]. We will call these Ihara liftings. The former lift is a kind of Yoshida-type, and the latter one is a kind of Saito–Kurokawa type. The above relation suggests that the Yoshida type is always injective, and the Saito–Kurokawa type is injective when k is odd but vanishes when k is even. Though these conjectural injectivities have never been proved in general, all the examples that we have support this property. So when $(k,j) \neq (3,0)$, we define the space of old forms of $\mathfrak{M}_{k+j-3,k-3}(U(p))$ by the space obtained by the above Ihara liftings. If $(k,j) = (3,0)$, then $\mathfrak{M}_{k+j-3,k-3}(U(p)) = \mathfrak{M}_{0,0}(U(p))$ contains the constant functions, and we have $S_4(SL_2(\mathbb{Z})) = 0$, so we define old forms to be the constant functions. The space $\mathfrak{M}_{k+j-3,k-3}^{new}(U(p))$ of new forms is defined to be the orthogonal complement of old forms in $\mathfrak{M}_{k+j-3,k-3}(U(p))$.

Summing up, we should have

$$\begin{aligned}\dim S_{k,j}^{new}(K(p)) &= \dim S_{k,j}(K(p)) - 2\dim S_{k,j}(K(1)) + \delta_{j0}\dim S_{2k-2}^{new,-}(SL(2,\mathbb{Z})).\\ \dim \mathfrak{M}_{k+j-3,k-3}^{new}(U(p)) &= \dim \mathfrak{M}_{k+j-3,k-3}(U(p)) - \dim S_{j+2}^{new}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))\\ &\quad - \delta_{j0}\dim S_{2k-2}^{new,+}(SL_2(\mathbb{Z})) - \delta_{j0}\delta_{k3}.\end{aligned}$$

We must note that our definition of new forms does *not* mean that new forms are not lifts. In fact, if $j = 0$, there exists an injective Saito–Kurokawa type lifting from $S_{2k-2}^{new,-}(\Gamma_0^{(1)}(p))$ to $S_k(K(p))$ defined through Jacobi forms of index p by Gritsenko and these are new forms in the above sense. We are not specifying the local representations at p corresponding to paramodular new forms and in fact several different local representations appear. Our conjecture also indicates that there should be the corresponding injective lifting from $S_{2k-2}^{new,-}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k+j-3,k-3}(U(p))$. There are many experimental examples of such liftings, but we cannot give a general construction of this lift by the known method of Ihara liftings. So this gives an interesting new problems. But anyway, we are saying that new forms of both groups contain certain liftings also and that liftings should correspond to liftings in our conjecture.

Roberts and Schmidt also considered some operators from $S_{k,j}(K(N))$ to $S_{k,j}(K(pN))$ in [**RS06**]. In next section, we show that these are really the same operators as ours at least for the squarefree level case.

4.2. Paramodular old forms: squarefree level case.

Next we consider the general squarefree level case. Throughout this section, we assume that N is squarefree for simplicity, though some part can be easily generalized for non squarefree case. First, for any $0 < N_1 | N$, we define the trace operator $T(N_1, N)$:

$S_{k,j}(K(N_1)) \to S_{k,j}(K(N))$ by

$$F|T(N_1,N) = \sum_{g \in K(N_1)\cap K(N)\backslash K(N)} F|_{k,j}[g]$$

for any $F \in S_{k,j}(K(N_1))$. It is easy to see that a complete set of representatives of $K(N_1) \cap K(N)\backslash K(N)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & N_1^{-1}b \\ 0 & 0 & 1 & 0 \\ 0 & cN & 0 & d \end{pmatrix}$$

where $(a:b) \bmod (N/N_1)$ runs over elements of

$$\mathbb{P}^1(\mathbb{Z}/(N/N_1)\mathbb{Z}) = \prod_{p|(N/N_1), p:prime} \mathbb{P}^1(\mathbb{F}_p)$$

where $\mathbb{P}^1$ means the projective space of lines and $\mathbb{F}_p$ is the finite field of p elements. We also define the Atkin–Lehner operator u_m on $S_{k,j}(K(N))$ as in [**RS06**] as follows. For any $0 < m|N$, we put

$$\eta_m = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and let γ_m be any element in $Sp(2,\mathbb{Z})$ such that

$$\gamma_m \equiv \begin{cases} 1_4 & \bmod N/m \\ \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \bmod m. \end{cases}$$

Then the action $F|_{k,j}[\gamma_m\eta_m]$ for $F \in S_{k,j}(K(N))$ does not depend on the choice of γ_m since the principal congruence subgroup $\Gamma(N)$ of level N is contained in $K(N)$. We write this operator on $S_{k,j}(K(N))$ by u_m. In particular, on $S_{k,j}(K(N))$, the action of u_N and U_N are the same. The operator u_m apparently depends on the choice of N, but we omit N in the notation since this does not matter so much.

PROPOSITION 4.1. *We have the following relations.*
(i) *For any non-negative integers N_1, N_2 such that $N_1|N_2|N$, we have*

$$T(N_1,N_2)T(N_2,N) = T(N_1,N)$$

as operators on $S_{k,j}(K(N_1))$.
(ii) *If* m *and* n *are coprime, then we have* $u_m u_n = u_{mn}$.
(iii) *If* p, q *are distinct primes such that* $p|N$ *and* $q \nmid N$, *then as operators on* $S_{k,j}(K(N))$, *we have*

$$u_p T(N, qN) = T(N, qN) u_p.$$

(iv) *If* $p \nmid N$, *then as operators on* $S_{k,j}(K(N))$, *we have*

$$T(N, pN) u_p = U_N^{-1} T(N, pN) U_{pN}.$$

In particular, we have the equality of the following images of the mappings.

$$S_{k,j}(K(N))|T(N, pN)u_p = S_{k,j}(K(N))|T(N, pN)U_{pN}.$$

The proof needs slightly lengthy arguments but essentially elementary and we omit the proofs here. See also [**RS06**] for a similar argument. There they defined two operators θ_p and θ_p' from $S_k(K(N))$ to $S_k(K(pN))$. We see easily from (2) that $\theta_p' = T(N, pN) = U_N \theta_p U_{pN}$.

The following corollary is obvious.

Corollary 4.2. *For any* $N_1|N$, *write* $N/N_1 = p_1 \cdots p_r$, *where* p_i *are distinct primes. We assume that* $e_i = 0$ *or* 1 *for each* i *with* $1 \le i \le r$, *Then we have*

$$T(N_1, p_1N_1)u_{p_1}^{e_1} T(p_1N_1, p_1p_2N_1)u_{p_2}^{e_2} \cdots T(N_1p_1 \cdots p_{r-1}, N)u_{p_r}^{e_r} = T(N_1, N)u_m$$

where $m = p_1^{e_1} \cdots p_r^{e_r}$. *In particular, this depends only on* e_i *for each* p_i *and does not depend on the order of the action of* $u_{p_i}^{e_i}$.

Then by induction, we can easily see that the space of old forms can be written by

$$S_{k,j}^{old}(K(N)) = \sum_{\substack{M|N \\ M \neq N}} \sum_{m|(N/M)} S_{k,j}^{new}(K(M))T(M, N)u_m. \tag{3}$$

Here we must note that the sum is not necessary the direct sum. To figure out what is likely to happen for this sum and to make a necessary assumption to get the conjectural dimension formulas of new forms from the formula for the total dimensions, we need explanation on liftings, which we do in the next subsection.

4.3. Dimensions of new paramodular forms and liftings.

A kind of Saito–Kurokawa lifting is constructed to $S_k(K(N))$ (i.e. the case $j = 0$) as a lifting from the space of Jacobi forms $J_{k,N}(\Gamma_1^J)$ of weight k and of index N for the Jacobi modular group Γ_1^J by Gritsenko [**Gri95**] (See also Schmidt [**S07**]). By Skoruppa and Zagier [**SZ88**], the space of Jacobi cusp forms $J_{k,N}^{cusp}(\Gamma_1^J)$ is isomorphic to $\bigoplus_{M|N} S_{2k-2}^{new,-}(\Gamma_0^{(1)}(M))$ for squarefree N, where the latter notation with $\pm$ is as in the introduction. So it seems natural to expect that the multiplicity of the lifting from $S_{2k-2}^{new,-}(\Gamma_0^{(1)}(M))$ to $S_k^{new}(K(M))$ is one for each $M|N$. Actually, if $F \in S_k(K(M))$ is a

Gritsenko lift, then $F|T(M,pM) = F|T(M,pM)u_p$ by [**RS06**], so this means that the RHS of (3) is not a direct sum. We denote by $S_k^{new,L}(K(N))$ the space of Gritsenko lifts from Jacobi forms corresponding to $S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N))$.

We consider the following assumption.

Assumption 4.3. (i) *For each N, we have the direct sum decomposition*

$$\dim S_{k,j}^{old}(K(N)) = \bigoplus_{\substack{M|N \\ M\neq N}} \Bigg(\sum_{m|(N/M)} S_{k,j}^{new}(K(M))|T(M,N)u_m \Bigg).$$

(ii) *For each $M|N$, $T(M,N)$ is injective on $S_{k,j}^{new}(K(M))$.*

(iii) *For each $M|N$ and for any m, $m'|(N/M)$, we have*

$$S_{k,j}^{new}(K(M))T(M,N)u_m \cap S_{k,j}^{new}(K(M))T(M,N)u_{m'} = \delta_{j0}\, S_k^{new,L}(K(M))|T(M,N).$$

We believe that it is reasonable to expect that this assumption is always satisfied.

Proposition 4.4. *Under the above assumption* 4.3, *we have*

$$\begin{aligned} &\dim S_{k,j}(K(N)) \\ &= \sum_{M|N} \left\{ 2^{\omega(M)} \dim S_{k,j}^{new}(K(N/M)) - (2^{\omega(M)}-1)\delta_{j0} \dim S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N/M)) \right\}. \end{aligned} \tag{4}$$

$$\begin{aligned} \dim S_{k,j}^{new}(K(N)) = &\sum_{M|N} (-2)^{\omega(M)} \dim S_{k,j}(K(N/M)) \\ &- \delta_{j0} \sum_{M|N, M\neq 1} (-1)^{\omega(M)} \dim S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N/M)). \end{aligned} \tag{5}$$

Proof. The formula (4) is a direct consequence from Assumption (4.3). The formula (5) is proved easily by Möbius inversion formula applied to (4). Indeed, if we put

$$\begin{aligned} f(M) =& 2^{-\omega(M)} \Bigg(\dim S_{k,j}(K(M)) - \delta_{j0} \sum_{d|M} \dim S_{2k-2}^{new,-}(\Gamma_0^{(1)}(d)) \Bigg), \\ g(M) =& 2^{-\omega(M)} \Bigg(\dim S_{k,j}^{new}(K(M)) - \delta_{j0} \dim S_{2k-2}^{new,-}(\Gamma_0^{(1)}(M)) \Bigg), \end{aligned}$$

then the formula (4) reads

$$f(N) = \sum_{M|N} g(M).$$

So we have $g(N) = \sum_{M|N} \mu(N/M) f(M)$ where $\mu(*)$ is the Möbius function, and since $\sum_{M|N} (-2)^{\omega(N/M)} \sum_{d|M} \dim S_{2k-2}^{new,-}(\Gamma_0^{(1)}(d)) = \sum_{d|N} (-1)^{\omega(N/d)} \dim S_{2k-2}^{new,-}(\Gamma_0^{(1)}(d))$, we have (5). □

Now we see what is the consequence of this result, compared with our dimensional relation Main Theorem 1.2. If we modify the LHS of Main Theorem 1.2 (i) and (ii) to $\dim S_{k,j}^{new}(K(N))$ by using the above formula, then the RHS should be the conjectural dimension of new forms of the non-split group. This gives an idea how to define new forms for non-split group and how is the lifting there. So we should have

$$\begin{aligned} \dim \mathfrak{M}_{k+j-3,k-3}^{new}(U(N)) &= \dim \mathfrak{M}_{k+j-3,k-3}(U(N)) - d^{old}(N) \text{ if } \omega(N) \text{ is odd,} \\ \dim S_{k,j}^{new}(U'(N)) &= \dim S_{k,j}(U'(N)) - d^{old}(N) \qquad\quad \text{if } \omega(N) \text{ is even,} \end{aligned} \tag{6}$$

where

$$\begin{aligned} d^{old}(N) = & \sum_{\substack{M|N \\ \omega(M)=odd}} \dim S_{j+2}^{new}(\Gamma_0^{(1)}(M)) \times \dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)) \\ & + \delta_{j0} \sum_{\substack{M|N \\ \omega(M)=odd}} S_{2k-2}^{new,+}(\Gamma_0^{(1)}(N/M)) \\ & + \delta_{j0} \sum_{\substack{M|N \\ \omega(M)=even}} S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N/M)) + \delta_{j0}\delta_{k3}\delta_{\omega(N),odd}, \end{aligned} \tag{7}$$

where we put $\delta_{\omega(N),odd} = 1$ if $\omega(N)$ is odd and $= 0$ otherwise.

Here the definition and liftings are explained in the following conjecture.

Conjecture 4.5. *For each $M|N$, where N is squarefree, we have the following lifts.*

(1) *When $\omega(M)$ is odd, there should exist an injective lifting*:

$$S_{j+2}^{new}(\Gamma_0^{(1)}(M)) \times \dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)) \to \begin{cases} \mathfrak{M}_{k+j-3,k-3}(U(N)) & \text{if } \omega(N) \text{ is odd,} \\ S_{k,j}(U'(N)) & \text{if } \omega(N) \text{ is even.} \end{cases}$$

(2) *When $\omega(M)$ is odd, then there should exist an injective lifting*:

$$S_{2k-2}^{new,+}(\Gamma_0^{(1)}(N/M)) \to \begin{cases} \mathfrak{M}_{k-3,k-3}(U(N) & \text{if } \omega(N) \text{ is odd,} \\ S_k(U'(N)) & \text{if } \omega(N) \text{ is even.} \end{cases}$$

(3) *When $\omega(M)$ is even, there should exist an injective lifting*:

$$S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N/M)) \to \begin{cases} \mathfrak{M}_{k-3,k-3}(U(N)) & \text{if } \omega(N) \text{ is odd,} \\ S_k(U'(N)) & \text{if } \omega(N) \text{ is even.} \end{cases}$$

(4) *If we define new forms by the orthogonal complement of the space spanned by the images of all the above liftings in $\mathfrak{M}_{k+j-3,k-3}(U(N))$ or $S_{k,j}(U'(N))$ and also by the constant functions in $\mathfrak{M}_{0,0}(N)$ in case $(k,j) = (3,0)$ and $\omega(N)$ is odd, then*

$\dim \mathfrak{M}^{new}_{k+j-3,k-3}(U(N))$ *or* $\dim S^{new}_{k,j}(U'(N))$ *is given by* (6).

This fits dimensional relations and examples we give later. So we propose Conjecture 1.3 and 1.4.

By the way, if $f \in S^{new,-}_{2k-2}(\Gamma_0^{(1)}(M))$ is lifted through Jacobi forms to $\iota(f) \in S_{k,j}(K(N))$, then the spinor L function is written by

$$L(s,\iota(f),Sp) = \zeta(s-k+1)\zeta(s-k+2)L(s,f)$$

up to bad Euler factors.

4.4. Numerical examples of non-lift.

Since the description is fairly complicated, examples seem indispensable. In this subsection, we shortly give examples of non-lift in Conjecture 1.3 for $N=6$. The examples of liftings and Conjecture 1.4 will be explained more in detail in next section. The dimensions of $\dim S^{new,\pm}_{2k-2}(\Gamma_0^{(1)}(M))$ can be calculated by the trace formula of the Atkin–Lehner involution in [**Y73**]. We remark also that $\dim S_{2k-2}(SL_2(\mathbb{Z})) = \dim S^{new,(-1)^{k-1}}_{2k-2}(SL_2(\mathbb{Z}))$.

Then together with our new dimension formulas and [**Kit11**], we obtain the following table of dimensions.

Total dimension of $\dim S_k(K(N))$.

$level\backslash k$	3	4	5	6	7	8	9	10	11
$K(6)$	0	0	0	1	1	2	3	4	5
$K(3)$	0	0	0	1	0	1	1	2	1
$K(2)$	0	0	0	0	0	1	0	1	1
$K(1)$	0	0	0	0	0	0	0	1	0

Dimension of $S^{new,\pm}_{2k-2}(\Gamma_0(N))$. *The first one is for* $+$ *and the second for* $-$

$level\backslash 2k-2$	4	6	8	10	12	14	16	18	20
$\Gamma_0^{(1)}(6)$	1,0	1,0	1,0	1,0	2,1	1,0	2,1	2,1	2,1
$\Gamma_0^{(1)}(3)$	0,0	1,0	1,0	1,1	1,0	2,1	1,1	2,1	2,1
$\Gamma_0^{(1)}(2)$	0,0	0,0	1,0	1,0	0,0	1,1	1,0	1,0	1,1
$\Gamma_0^{(1)}(1)$	0,0	0,0	0,0	0,0	1,0	0,0	1,0	0,1	1,0

By our conjecture and the fact that $S_2(\Gamma_0^{(1)}(2)) = S_2(\Gamma_0^{(1)}(3)) = 0$, the dimensions of new forms of $K(6)$ or $U'(6)$ are given for $j=0$ and $N=6$ by

$$\begin{aligned}\dim S_k^{new}(K(6)) &= \dim S_k(K(6)) - 2\dim S_k(K(2)) - 2\dim S_k(K(3)) + 4\dim S_k(K(1)) \\ &+ \dim S^{new,-}_{2k-2}(\Gamma_0^{(1)}(2)) + \dim S^{new,-}_{2k-2}(\Gamma_0^{(1)}(3)) - \dim S^{new,-}_{2k-2}(SL_2(\mathbb{Z})) = \dim S^{new}_{k,0}(U'(6)) \\ &= \dim S_{k,0}(U'(6)) - \dim S^{new,+}_{2k-2}(\Gamma_0^{(1)}(2)) - \dim S^{new,+}_{2k-2}(\Gamma_0^{(1)}(3)) - \dim S^{new,-}_{2k-2}(SL_2(\mathbb{Z})),\end{aligned}$$

so we obtain the following table.

The dimension of new forms $\dim S_k^{new}(K(N))$.

$level \backslash k$	3	4	5	6	7	8	9	10	11
$K(6)$	0	0	0	0	1	0	2	2	3
$K(3)$	0	0	0	1	0	1	1	1	1
$K(2)$	0	0	0	0	0	1	0	0	1
$K(1)$	0	0	0	0	0	0	0	1	0

The dimension of new forms of $K(N)$ which are not Gritsenko lift.

$level \backslash k$	3	4	5	6	7	8	9	10	11
$K(6)$	0	0	0	0	0	0	1	1	2
$K(3)$	0	0	0	0	0	0	0	0	0
$K(2)$	0	0	0	0	0	0	0	0	0
$K(1)$	0	0	0	0	0	0	0	0	0

The dimension of $S_k(U^{'}(6))$ and $S_k^{new}(U(6))$

k	3	4	5	6	7	8	9	10	11
$S_k(U^{'}(6))$	0	1	2	2	2	3	4	6	6
$S_k^{new}(U(6))$	0	0	0	0	1	0	2	2	3

We see the case $k = 10$ more in detail. The space $S_{10}^{new}(K(3))$ consists of the lift from $S_{18}^{new,-}(\Gamma_0^{(1)}(3))$, and $S_{10}^{old}(K(3))$ consists of the lift form $S_{18}(SL_2(\mathbb{Z}))$, both one dimensional. $S_{10}^{new}(K(2)) = 0$ and $S_{10}^{old}(K(2))$ and $S_{10}(K(1))$ consists of the lift from $S_{18}(SL_2(\mathbb{Z}))$. The basis of $S_{10}(K(6))$ consists of a non-lift and the lift from $S_{18}^{new,-}(\Gamma_0^{(1)}(6))$, $S_{18}^{new,-}(\Gamma_0(3))$, and $S_{18}^{new,-}(SL_2(\mathbb{Z}))$. Here we note that the lift from $S_{18}^{new,-}(\Gamma_0^{(1)}(6))$ is regarded as a new form. On the other hand, the basis of $S_{10}(U^{'}(6))$ consists of a non-lift, one lift from $S_{18}^{new,-}(\Gamma_0^{(1)}(6))$ (which is a new form), two lifts from $S_{18}^{new,+}(\Gamma_0^{(1)}(2))$, one lift from $S_{18}^{new,+}(\Gamma_0^{(1)}(2))$, and one lift from $S_{18}^{new,-}(SL_2(\mathbb{Z}))$.

The following numerical examples up to $k = 11$, which is independent of any assumptions, support all these theoretical observations.

THEOREM 4.6. (i) *If $k \leq 8$, then the space $S_k(K(6)) = S_{k,0}(K(6))$ is generated by Gritsenko lifts from Jacobi cusp forms of weight k and index 6. The lifts from $S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N))$ $(N = 1, 2, 3, 6)$ to $S_k(K(6))$ are 2, 2, and 3 dimensional for $k = 9$, 10, and 11 respectively and the codimensions in $S_k(K(6))$ are 1, 1, and 2.*

(ii) *Assume $k \leq 11$. Then for each cusp eigenform g in $S_{2k-2}^{new,-}(SL(2;\mathbb{Z}))$, $S_{2k-2}^{new,+}(\Gamma_0^{(1)}(2))$, $S_{2k-2}^{new,+}(\Gamma_0^{(1)}(3))$ or $S_{2k-2}^{new,-}(\Gamma_0^{(1)}(6))$, there is a cusp form F in $S_{k,0}(U'(6))$ such that $H_5(t,F) = (t-5^{k-2})(t-5^{k-1})h_5(t,g)$, where $h_5(t,g)$ is the Hecke polynomial of g at 5. If $k \leq 8$, then there are no other cusp eigenforms of $U_k(U^{'}(6))$.*

If $k = 9$, 10, 11, then such cusp forms generate 3, 5, and 4-dimensional subspace of $S_k(U'(6))$ respectively. The codimension is 1 for $k = 9$ and 10 and 2 for $k = 11$.

(iii) *If we take the Hecke eigenforms $\phi_k \in S_k(K(6))$ for $k = 9$, 10 and $\phi_{11,a}$, $\phi_{11,b} \in S_{11}(K(6))$, and $\psi_k \in S_k(U'(6))$ for $k = 9$, 10, $\psi_{11,a}$, $\psi_{11,b} \in S_{11}(U^{'}(6))$ in a suitable order which are not obtained by lifting in the above sense, then eigenvalues of the Hecke operators $T_k(5)$ and $T_k(5^2)$ for $S_k(K(6))$ and $S_k(U^{'}(6))$ coincide with each other. The Hecke polynomials of these forms in $S_k(U^{'}(6))$ at 5 are given by*

$$\begin{aligned}
H_5(t,\psi_9) &= t^4 + 33540t^3 - 15293281250t^2 + 33540 \cdot 5^{15}t + 5^{30},\\
H_5(t,\psi_{10}) &= t^4 + 88980t^3 + 1170167968750t^2 + 88980 \cdot 5^{17}t + 5^{34},\\
H_5(t,\psi_{11,a}) &= t^4 - 5029620t^3 + 11396230468750t^2 - 5029620 \cdot 5^{19}t + 5^{38},\\
H_5(t,\psi_{11,b}) &= t^4 + 222420t^3 + 21376386718750t^2 + 222420 \cdot 5^{19}t + 5^{38}.
\end{aligned}$$

All absolute values of the zeros of these polynomials are $5^{(2k-3)/2}$. We have $H_5(t,\phi_9) = H_5(t,\psi_9)$, $H_5(t,\phi_{10}) = H_5(t,\psi_{10})$, $H_5(t,\phi_{11,a}) = H_5(t,\psi_{11,a})$, $H_5(t,\phi_{11,b}) = H(t,\psi_{11,b})$.

It is not easy at all to construct explicitly the non-lifted Hecke eigenforms and calculate the Hecke polynomials. We constructed $\psi \in S_{k,0}(U^{'}(6))$ for $k \leq 11$ and gave $H_5(t,g)$. But on the other hand, the results on non-lift paramodular forms in $S_{k,0}(K(6))$ for $9 \leq k \leq 11$ are due to Cris Poor and David Yuen. They kindly informed the authors of the results on ϕ and $H_5(t,\phi)$ obtained by their original technique in their book [**PSY**] in preparation.

5. Numerical examples of non-split group and liftings.

In the case when N is a prime, where the non-split group is the compact twist of $Sp(2,\mathbb{R})$, we have several numerical examples in [**Ibu84**] for $N = 2$ and similar calculation is possible for further N. In fact there are several related experimental results in this case, but we omit them here. But the case when N is a product of two distinct primes is completely different from the case N is a prime, since the group in question is a non-split $\mathbb{Q}$ form of $Sp(2,\mathbb{R})$. So in this section, we sketch the calculation in this case more in detail.

5.1. Hecke operators.

We review briefly Hecke operators on $S_{k,0}(U'(N))$ (cf. [**Sug84**]). We put $\Gamma = U'(N)$. For $m \in \mathbb{N}$ and $f \in S_{k,0}(\Gamma)$, we define

$$(T_k(m)f)(Z) := m^{2k-3} \sum_{\sigma \in \Gamma \backslash S_m} \det(CZ+D)^{-k} f(\sigma\langle Z\rangle),$$

where we put

$$S_m := \left\{ g = \begin{bmatrix} \alpha\,\beta \\ \gamma\,\delta \end{bmatrix} \,\middle|\, \begin{array}{l} \alpha,\delta \in \mathfrak{O},\ \beta \in \mathfrak{A}^{-1},\ \gamma \in \mathfrak{A}, \\ \alpha\overline{\beta} + \beta\overline{\alpha} = \gamma\overline{\delta} + \delta\overline{\gamma} = 0,\ \alpha\overline{\delta} + \beta\overline{\gamma} = m \end{array} \right\}$$

and $\sigma \in \Gamma\backslash S_m$ runs over a complete system of representatives of $\Gamma\backslash S_m$. We fix a prime number $p \nmid N$. For Hecke eigenform $f \in S_{k,0}(\Gamma)$, we put

$$T_k(p)f = \lambda(p)f, \qquad T_k(p^2)f = \lambda(p^2)f.$$

Then (for $j = 0$) we define

$$H_p(t,f) := t^4 - \lambda(p)t^3 + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})t^2 - \lambda(p)p^{2k-3}t + p^{4k-6}.$$

Then $H_p(p^{-s}, f)$ is the Euler p factor of the spinor L function of f. For any $X \subset B$, we write $X^0 = \{x \in X; Tr(x) = 0\}$. By easy consideration, we can give complete systems of representatives of $U'(N)\backslash S_p$ and $U'(N)\backslash S_{p^2}$ for a prime number $p \nmid N$ as follows. We put $\mathfrak{O}^{\times,1} = \{x \in \mathfrak{O}^\times \mid x\bar{x} = 1\}$.

$U'(N)\backslash S_p$

(1) $\begin{bmatrix} 1 & x \\ 0 & p \end{bmatrix}$, where x ranges over a complete system of representatives of $(\mathfrak{A}^{-1})^0/p(\mathfrak{A}^{-1})^0$.

(2) $\begin{bmatrix} a_i & \alpha_{i,j} \\ 0 & a_i \end{bmatrix}$, where a_i ranges over a complete system of representatives of $\mathfrak{O}^{\times,1}$-equivalent classes of elements of $\mathfrak{O}$ with norm p. For each a_i, $\alpha_{i,j}$ ranges over a complete system of representatives of $\{x \in \mathfrak{A}^{-1} \mid \mathrm{tr}(a_i^{-1}x) = 0\}/((a_i\mathfrak{A}^{-1})^0)$.

(3) $\begin{bmatrix} b_i & \beta_{i,j} \\ 0 & p^{-1}b_i \end{bmatrix}$, where b_i ranges over a complete system of representatives of $\mathfrak{O}^{\times,1}$-equivalent classes of elements of $\mathfrak{O}$ with norm p^2. For each b_i, $\beta_{i,j}$ ranges over a complete system of representatives of $\{x \in \mathfrak{A}^{-1} \mid \mathrm{tr}(b_i^{-1}x) = 0\}/(p^{-1}b_i(\mathfrak{A}^{-1})^0)$.

$U'(N)\backslash S_{p^2}$

(1) $\begin{bmatrix} 1 & x \\ 0 & p^2 \end{bmatrix}$, where x ranges over a complete system of representatives of $(\mathfrak{A}^{-1})^0/p^2(\mathfrak{A}^{-1})^0$.

(2) $\begin{bmatrix} a_i & \alpha_{i,j} \\ 0 & pa_i \end{bmatrix}$, where a_i ranges over a complete system of representatives of $\mathfrak{O}^{\times,1}$-equivalent classes of elements of $\mathfrak{O}$ with norm p. For each a_i, $\alpha_{i,j}$ ranges over a complete system of representatives of $\{x \in \mathfrak{A}^{-1} \mid \mathrm{tr}(a_i^{-1}x) = 0\}/(pa_i(\mathfrak{A}^{-1})^0)$.

(3) $\begin{bmatrix} b_i & \beta_{i,j} \\ 0 & b_i \end{bmatrix}$, where b_i ranges over a complete system of representatives of $\mathfrak{O}^{\times,1}$-equivalent classes of elements of $\mathfrak{O}$ with norm p^2. For each b_i, $\beta_{i,j}$ ranges over a complete system of representatives of $\{x \in \mathfrak{A}^{-1} \mid \mathrm{tr}(b_i^{-1}x) = 0\}/((b_i\mathfrak{A}^{-1})^0)$.

(4) $\begin{bmatrix} c_i & \gamma_{i,j} \\ 0 & p^{-1}c_i \end{bmatrix}$, where c_i ranges over a complete system of representatives of $\mathfrak{O}^{\times,1}$-equivalent classes of elements of $\mathfrak{O}$ with norm p^3. For each c_i, $\gamma_{i,j}$ ranges over a complete system of representatives of $\{x \in \mathfrak{A}^{-1} \mid \mathrm{tr}(c_i^{-1}x) = 0\}/(p^{-1}c_i(\mathfrak{A}^{-1})^0)$.

(5) $\begin{bmatrix} d_i & \delta_{i,j} \\ 0 & p^{-2}d_i \end{bmatrix}$, where d_i ranges over a complete system of representatives of $\mathfrak{O}^{\times,1}$-equivalent classes of elements of $\mathfrak{O}$ with norm p^4. For each d_i, $\delta_{i,j}$ ranges over a complete system of representatives of $\{x \in \mathfrak{A}^{-1} \mid \mathrm{tr}(d_i^{-1}x) = 0\}/(p^{-2}(d_i\mathfrak{A}^{-1})^0)$.

5.2. Discriminant 6.

Our lifting conjecture to $S_{k,0}(U'(6))$ from $S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(N))$ with $N = 1, 2, 3, 6$ seem to be a part of the lifting theory of Sugano or Oda, though there is no definite result on the image. Anyway we would like to see L functions more closely by experiment in order to say something more definite.

The second named author constructed the graded ring of Siegel modular forms $M_{k,0}(U'(6))$ explicitly. We can use the following theorem for our experiment.

THEOREM 5.1 ([**Kit12**]). *The graded ring of Siegel modular forms with respect to $U'(6)$ is given explicitly by*

$$\bigoplus_{k=0}^{\infty} M_{k,0}(U'(6)) = \mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{5b}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \\ \oplus \chi_{15}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{5b}\chi_{15}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6],$$

where we denote by E_k $(k = 2, 4, 6)$ the Eisenstein series of weight k which were defined in [**Hir99**]*, and denote by χ_{5a}, χ_{5b} and χ_{15} the Siegel cusp forms of weight 5, 5 and 15 respectively, which are defined in* [**Kit12**]*. The four modular forms E_2, E_4, χ_{5a} and E_6 are algebraically independent over $\mathbb{C}$. The fundamental relations among the generators are also given in* [**Kit12**].

For any $f(Z) \in M_{k,0}(U'(6))$, we have the following Fourier expansion

$$f(Z) = C(0) + \sum_{\eta \in L^*, \eta J > 0} C(\eta)\boldsymbol{e}[\mathrm{Tr}(\eta Z J)] \qquad (\boldsymbol{e}[z] = e^{2\pi i z}),$$

where

$$L^* = \left\{ \left. \begin{bmatrix} (5x+10z)\sqrt{6} & (x+5y)\sqrt{5} + x\sqrt{30} \\ (x+5y)\sqrt{5} - x\sqrt{30} & -(5x+10z)\sqrt{6} \end{bmatrix} \Big/ 60 \ \right| \ x, y, z \in \mathbb{Z} \right\}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and $\eta J > 0$ means ηJ is positive definite. See [**Kit12**, Section 2.4 and Section 3.1] for details. Note that $f(Z)$ is a cusp form if and only if $C(0) = 0$.

Now we consider Fourier coefficients of $(T_k(n)f)(Z)$ for $f(Z) \in S_{k,0}(U'(6))$. For any cusp form $f(Z) = \sum_{\eta \in L^*, \eta J > 0} C(\eta)\boldsymbol{e}[\mathrm{Tr}(\eta Z J)] \in S_{k,0}(U'(6))$, we put

$$(T_k(n)f)(Z) = \sum_{\eta \in L^*, \eta J > 0} C(n;\eta)\boldsymbol{e}[\mathrm{Tr}(\eta Z J)].$$

Calculating carefully by taking explicit representatives of $U'(6)\backslash S_5$ and $U'(6)\backslash S_{5^2}$, we obtain the following relations:

PROPOSITION 5.2.

(1) $C(5;[2,1,-1]) = C([10,5,-5])$,
(2) $C(5;[2,0,-1]) = C([10,0,-5]) + 2 \cdot 5^{k-2}C([2,0,-1])$,
(3) $C(5;[4,2,-2]) = C([20,10,-10])$,
(4) $C(5;[4,0,-2]) = C([20,0,-10]) + 2 \cdot 5^{k-2}C([4,0,-2])$,
(5) $C(5;[4,1,-2]) = C([20,5,-10]) + 2 \cdot 5^{k-2}C([4,1,-2])$,
(6) $C(5;[5,1,-2]) = C([25,5,-10]) + 2 \cdot 5^{k-2}C([7,-1,-3])$,
(7) $C(5^2;[2,1,-1]) = C([50,25,-25])$,
(8) $C(5^2;[2,0,-1]) = C([50,0,-25]) + 2 \cdot 5^{k-2}C([10,0,-5]) + 2 \cdot 5^{2k-4}C([2,0,-1])$,
(9) $C(5^2;[4,1,-2]) = C([100,25,-50]) + 2 \cdot 5^{k-2}(C[20,5,-10]) + 2 \cdot 5^{2k-4}C([4,1,-2])$.

In the next subsection, we calculate Hecke eigenvalues for $S_{k,0}(U'(6))$ with $4 \le k \le 11$ by using Proposition 5.2. A basis of each space is obtained by the ring generators of Theorem 5.1 and their Fourier coefficients can be calculated in the manner of [**Kit12**].

5.3. Results of numerical experiment.

As before we define

$$S^{new,\pm}_{2k-2}(\Gamma_0^{(1)}(N)) = \left\{ f \in S^{new}_{2k-2}(\Gamma_0^{(1)}(N)) \;\middle|\; u_N f = \mp(-1)^k f \right\},$$

where u_N is the Atkin–Lehner involution defined by the action of $\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$.

weight 4: $\dim S_{4,0}(U'(6)) = 1$. If we put

$$\phi_1 = {E_2}^2 - E_4,$$

then $\phi_1 \in S_{4,0}(U'(6))$, and we have

$$T_4(5)\phi_1 = 156\phi_1, \qquad T_4(5^2)\phi_1 = 16561\phi_1$$

by Proposition 5.2 (1), (7) and Table 1. We obtain

$$H_5(t,\phi_1) = (t-5^2)(t-5^3)(t^2-6t+5^5).$$

Table 1.

η	${E_2}^2$	E_4
$[0,0,0]$	1	1
$[2,1,-1]$	96	960/13
$[10,5,-5]$	245376	241920
$[50,25,-25]$	757762656	9846144960/13

On the other hand, the spaces of cusp forms of one variable are given as follows: $\dim S_6^{new,+}(\Gamma_0^{(1)}(2)) = \dim S_6^{new,-}(\Gamma_0^{(1)}(2)) = \dim S_6^{new,-}(\Gamma_0^{(1)}(3)) = \dim S_6^{new,-}(\Gamma_0^{(1)}(6)) = 0$, and $S_6^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_1$, $S_6^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_2$, where

$$f_1 = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + O(q^6),$$

and

$$f_2 = q + 4q^2 - 9q^3 + 16q^4 - 66q^5 + O(q^6),$$

respectively.

Hence we find $H_5(t,\phi_1) = (t-5^2)(t-5^3)h_5(t,f_1)$, where $h_5(t,f_1)$ is the Hecke polynomial of f_1 at 5.

weight 5: $\dim S_{5,0}(U'(6)) = 2$. If we put

$$\phi_1 = \chi_{5a}, \qquad \phi_2 = \chi_{5b},$$

then we see that ϕ_1 and ϕ_2 are Hecke eigenbasis of $S_{5,0}(U'(6))$ and

$$T_5(5)\phi_1 = 1140\phi_1, \qquad T_5(5^2)\phi_1 = 835225\phi_1,$$
$$T_5(5)\phi_2 = 540\phi_2, \qquad T_5(5^2)\phi_2 = 277225\phi_2$$

by Proposition 5.2 (1), (2), (7), (8) and Table 2. We obtain

$$H_5(t,\phi_1) = (t-5^3)(t-5^4)(t^2 - 390t + 5^7),$$
$$H_5(t,\phi_2) = (t-5^3)(t-5^4)(t^2 + 210t + 5^7).$$

Table 2.

η	χ_{5a}	χ_{5b}
$[0,0,0]$	0	0
$[2,1,-1]$	0	1
$[2,0,-1]$	1	0
$[10,5,-5]$	0	540
$[10,0,-5]$	890	0
$[50,25,-25]$	0	277225
$[50,0,-25]$	581475	0

On the other hand, the spaces of cusp forms of one variable are given as follows: $\dim S_8^{new,+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 + O(q^6).$$

$\dim S_8^{new,-}(\Gamma_0^{(1)}(2)) = 0$.
$\dim S_8^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_2$, where

$$f_2 = q + 6q^2 - 27q^3 - 92q^4 + 390q^5 + O(q^6).$$

$\dim S_8^{new,-}(\Gamma_0^{(1)}(3)) = 0$.
$\dim S_8^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_3$, where

$$f_3 = q + 8q^2 + 27q^3 + 64q^4 - 114q^5 + O(q^6).$$

$\dim S_8^{new,-}(\Gamma_0^{(1)}(6)) = 0$.

Hence we find $H_5(t,\phi_1) = (t-5^3)(t-5^4)h_5(t,f_2)$ and $H_5(t,\phi_2) = (t-5^3)(t-5^4)h_5(t,f_1)$.

weight 6: $\dim S_{6,0}(U'(6)) = 2$. If we put

$$\phi_1 = 225{E_2}^3 - 247E_2E_4 + 22E_6, \quad \phi_2 = 70{E_2}^3 - 39E_2E_4 - 31E_6,$$

then we see that ϕ_1 and ϕ_2 are Hecke eigenbasis of $S_{6,0}(U'(6))$ and

$$\begin{aligned} T_6(5)\phi_1 &= 4620\phi_1, & T_6(5^2)\phi_1 &= 13785025\phi_1, \\ T_6(5)\phi_2 &= 2220\phi_2, & T_6(5^2)\phi_2 &= 6369025\phi_2 \end{aligned}$$

by Proposition 5.2 (1), (2), (7) and Table 3. We obtain

$$\begin{aligned} H_5(t,\phi_1) &= (t-5^4)(t-5^5)(t^2 - 870t + 5^9), \\ H_5(t,\phi_2) &= (t-5^4)(t-5^5)(t^2 + 1530t + 5^9). \end{aligned}$$

Table 3.

η	${E_2}^3$	E_2E_4	E_6
$[0,0,0]$	1	1	1
$[2,1,-1]$	144	1584/13	2016/341
$[2,0,-1]$	216	3096/13	7560/341
$[10,5,-5]$	11848896	153119808/13	3945062016/341
$[10,0,-5]$	43651440	43758000	14784532560/341
$[50,25,-25]$	22596758341776	293755169840688/13	7705222882562016/341

On the other hand, the spaces of cusp forms of one variable are given as follows: $\dim S_{10}^{new,+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q + 16q^2 - 156q^3 + 256q^4 + 870q^5 + O(q^6).$$

$\dim S_{10}^{new,-}(\Gamma_0^{(1)}(2)) = 0$.
$\dim S_{10}^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_2$, where

$$f_2 = q + 18q^2 + 81q^3 - 188q^4 - 1530q^5 + O(q^6).$$

$\dim S_{10}^{new,-}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_3$, where

$$f_3 = q - 36q^2 - 81q^3 + 784q^4 - 1314q^5 + O(q^6).$$

$\dim S_{10}^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_4$, where

$$f_4 = q - 16q^2 + 81q^3 + 256q^4 + 2694q^5 + O(q^6).$$

$\dim S_{10}^{new,-}(\Gamma_0^{(1)}(6)) = 0$.

Hence we find $H_5(t,\phi_1) = (t-5^4)(t-5^5)h_5(t,f_1)$ and $H_5(t,\phi_2) = (t-5^4)(t-5^5)h_5(t,f_2)$.

weight 7: $\dim S_{7,0}(U'(6)) = 2$. If we put

$$\phi_1 = E_2\chi_{5a}, \qquad \phi_2 = E_2\chi_{5b},$$

then we see that ϕ_1 and ϕ_2 are Hecke eigenbasis of $S_{7,0}(U'(6))$ and

$$T_7(5)\phi_1 = 13380\phi_1, \qquad T_7(5^2)\phi_1 = 172290025\phi_1,$$
$$T_7(5)\phi_2 = 7020\phi_2, \qquad T_7(5^2)\phi_2 = 161796025\phi_2$$

by Proposition 5.2 (1), (2), (7), (8) and Table 4. We obtain

$$H_5(t,\phi_1) = (t-5^5)(t-5^6)(t^2+5370t+5^{11}),$$
$$H_5(t,\phi_2) = (t-5^5)(t-5^6)(t^2+11730t+5^{11}).$$

Table 4.

η	$E_2\chi_{5a}$	$E_2\chi_{5b}$
$[0,0,0]$	0	0
$[2,1,-1]$	0	1
$[2,0,-1]$	1	0
$[10,5,-5]$	0	7020
$[10,0,-5]$	7130	0
$[50,25,-25]$	0	161796025
$[50,0,-25]$	108196275	0

On the other hand, the spaces of cusp forms of one variable are given as follows:
$\dim S_{12}^{new,+}(\Gamma_0^{(1)}(2)) = 0$.
$\dim S_{12}^{new,-}(\Gamma_0^{(1)}(2)) = 0$.
$\dim S_{12}^{new,+}(\Gamma_0^{(1)}(3)) = 1$ and $S_{12}^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_1$, where

$$f_1 = q + 78q^2 - 243q^3 + 4036q^4 - 5370q^5 + O(q^6).$$

$\dim S_{12}^{new,-}(\Gamma_0^{(1)}(3)) = 0$.
$\dim S_{12}^{new,+}(\Gamma_0^{(1)}(6)) = 2$ and $S_{12}^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_2 \oplus \mathbb{C}f_3$, where

$$f_2 = q + 32q^2 + 243q^3 + 1024q^4 + 3630q^5 + O(q^6),$$
$$f_3 = q - 32q^2 - 243q^3 + 1024q^4 + 5766q^5 + O(q^6),$$

$\dim S_{12}^{new,-}(\Gamma_0^{(1)}(6)) = 1$ and $S_{12}^{new,-}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_4$, where

$$f_4 = q - 32q^2 + 243q^3 + 1024q^4 - 11730q^5 + O(q^6).$$

Hence we find $H_5(t, \phi_1) = (t - 5^5)(t - 5^6)h_5(t, f_1)$ and $H_5(t, \phi_2) = (t - 5^5)(t - 5^6)h_5(t, f_4)$.

weight 8: $\dim S_{8,0}(U'(6)) = 3$. If we put

$$\begin{aligned}
\phi_1 &= 110043{E_2}^4 - 218010{E_2}^2E_4 - 2728E_2E_6 + 110695{E_4}^2, \\
\phi_2 &= -4(69719 - 6427\sqrt{1969}){E_2}^4 + 13(59939 - 3733\sqrt{1969}){E_2}^2E_4 \\
&\quad + 682(13 + \sqrt{1969})E_2E_6 - 22139(23 - \sqrt{1969}){E_4}^2, \\
\phi_3 &= -4(69719 + 6427\sqrt{1969}){E_2}^4 + 13(59939 + 3733\sqrt{1969}){E_2}^2E_4 \\
&\quad + 682(13 - \sqrt{1969})E_2E_6 - 22139(23 + \sqrt{1969}){E_4}^2,
\end{aligned}$$

then we see that ϕ_1, ϕ_2, ϕ_3 are Hecke eigenbasis of $S_{8,0}(U'(6))$ and

$$\begin{aligned}
&T_6(5)\phi_1 = 36300\phi_1, && T_6(5^2)\phi_1 = 4018080625\phi_1, \\
&T_6(5)\phi_2 = (114108 + 384\sqrt{1969})\phi_2, && T_6(5^2)\phi_2 = (8716867153 + 51634944\sqrt{1969})\phi_2, \\
&T_6(5)\phi_3 = (114108 - 384\sqrt{1969})\phi_3, && T_6(5^2)\phi_3 = (8716867153 - 51634944\sqrt{1969})\phi_3
\end{aligned}$$

by Proposition 5.2 (1), (2), (3), (7) and Table 5 and 6. We obtain

$$\begin{aligned}
H_5(t, \phi_1) &= (t - 5^6)(t - 5^7)(t^2 + 57450t + 5^{13}), \\
H_5(t, \phi_2) &= (t - 5^6)(t - 5^7)(t^2 - (20358 + 384\sqrt{1969})t + 5^{13}), \\
H_5(t, \phi_3) &= (t - 5^6)(t - 5^7)(t^2 - (20358 - 384\sqrt{1969})t + 5^{13}).
\end{aligned}$$

On the other hand, the spaces of cusp forms of one variable are given as follows: $\dim S_{14}^{new,+}(\Gamma_0^{(1)}(2)) = 1$ and $S_{14}^{new,+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + O(q^6).$$

$\dim S_{14}^{new,-}(\Gamma_0^{(1)}(2)) = 1$ and $S_{14}^{new,-}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_2$, where

$$f_2 = q - 64q^2 - 1836q^3 + 4096q^4 + 3990q^5 + O(q^6).$$

$\dim S_{14}^{new,+}(\Gamma_0^{(1)}(3)) = 2$ and $S_{14}^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_3 \oplus \mathbb{C}f_4$, where

$$\begin{aligned} f_3 &= q + (-27 + 3\sqrt{1969})q^2 + 729q^3 \\ &\quad + (10258 - 162\sqrt{1969})q^4 + (20358 + 384\sqrt{1969})q^5 + O(q^6), \\ f_4 &= q + (-27 - 3\sqrt{1969})q^2 + 729q^3 \\ &\quad + (10258 + 162\sqrt{1969})q^4 + (20358 - 384\sqrt{1969})q^5 + O(q^6). \end{aligned}$$

Table 5.

η	${E_2}^4$	${E_2}^2 E_4$
$[0,0,0]$	1	1
$[2,1,-1]$	192	2208/13
$[2,0,-1]$	288	4032/13
$[4,2,-2]$	14592	162624/13
$[10,5,-5]$	130860288	1673186688/13
$[10,0,-5]$	780547392	10190996352/13
$[20,10,-10]$	975806198784	12683189263104/13
$[50,25,-25]$	144288528482774208	1875748634717301408/13
$[50,0,-25]$	942999141117375072	12258991765888909632/13

Table 6.

η	$E_2 E_6$	${E_2}^2$
$[0,0,0]$	1	1
$[2,1,-1]$	18384/341	1920/13
$[2,0,-1]$	32112/341	4320/13
$[4,2,-2]$	1228704/341	1845120/169
$[10,5,-5]$	41501838528/341	21377180160/169
$[10,0,-5]$	264658285152/341	133022208960/169
$[20,10,-10]$	332756404101504/341	164856944171520/169
$[50,25,-25]$	49202143925286410448/341	1875746380332904320/13
$[50,0,-25]$	321562470788343886032/341	159366930121797852960/169

$\dim S_{14}^{new,-}(\Gamma_0^{(1)}(3)) = 1$ and $S_{14}^{new,-}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_5$, where

$$f_5 = q - 12q^2 - 729q^3 - 8048q^3 - 30210q^5 + O(q^6).$$

$\dim S_{14}^{new,+}(\Gamma_0^{(1)}(6)) = 1$ and $S_{14}^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_6$, where

$$f_6 = q + 64q^2 - 729q^3 + 4096q^4 + 54654q^5 + O(q^6).$$

$\dim S_{14}^{new,-}(\Gamma_0^{(1)}(6)) = 0$.

Hence we find $H_5(t,\phi_1) = (t-5^6)(t-5^7)h_5(t,f_1)$ and $H_5(t,\phi_2) = (t-5^6)(t-5^7)h_5(t,f_3)$ and $H_5(t,\phi_3) = (t-5^6)(t-5^7)h_5(t,f_4)$.

weight 9: $\dim S_{9,0}(U'(6)) = 4$. If we put

$$\phi_1 = -13(E_4 - {E_2}^2)\chi_{5b} + 2{E_2}^2\chi_{5b}, \quad \phi_2 = -52(E_4 - {E_2}^2)\chi_{5a} + 3{E_2}^2\chi_{5a},$$
$$\phi_3 = 169(E_4 - {E_2}^2)\chi_{5a} + 16{E_2}^2\chi_{5a}, \quad \phi_4 = 65(E_4 - {E_2}^2)\chi_{5b} + 3{E_2}^2\chi_{5b},$$

then we see that ϕ_1, ϕ_2, ϕ_3, ϕ_4 are Hecke eigenbasis of $S_{9,0}(U'(6))$ and

$$\begin{aligned} T_9(5)\phi_1 &= 559260\phi_1, & T_9(5)\phi_2 &= 749460\phi_2,\\ T_9(5)\phi_3 &= 353940\phi_3, & T_9(5)\phi_4 &= -33540\phi_4,\\ T_9(5^2)\phi_1 &= 203206513225\phi_1, & T_9(5^2)\phi_2 &= 362968807225\phi_2,\\ T_9(5^2)\phi_3 &= 111952039225\phi_3, & T_9(5^2)\phi_4 &= 10314697225\phi_4 \end{aligned}$$

by Proposition 5.2 (1), (2), (3), (4), (7), (8) and Table 7. We obtain

$$\begin{aligned} H_5(t,\phi_1) &= (t-5^7)(t-5^8)(t^2 - 90510t + 5^{15}),\\ H_5(t,\phi_2) &= (t-5^7)(t-5^8)(t^2 - 280710t + 5^{15}),\\ H_5(t,\phi_3) &= (t-5^7)(t-5^8)(t^2 + 114810t + 5^{15}),\\ H_5(t,\phi_4) &= t^4 + 33540t^3 - 15293281250t^2 + 33540 \cdot 5^{15}t + 5^{30}. \end{aligned}$$

Table 7.

η	${E_2}^2\chi_{5a}$	${E_2}^2\chi_{5b}$	$E_4\chi_{5a}$	$E_4\chi_{5b}$
$[0,0,0]$	0	0	0	0
$[2,1,-1]$	0	1	0	1
$[2,0,-1]$	1	0	1	0
$[4,2,-2]$	0	112	0	1168/13
$[4,0,-2]$	150	0	2238/13	0
$[10,5,-5]$	0	422460	0	5218380/13
$[10,0,-5]$	347450	0	4332530/13	0
$[20,10,-10]$	0	112979520	0	1408351680/13
$[20,0,-10]$	-5390340	0	29991180/13	0
$[50,25,-25]$	0	158693017225	0	1973982231925/13
$[50,0,-25]$	140501129475	0	133718153475	0

On the other hand, the spaces of cusp forms of one variable are given as follows: $\dim S_{16}^{new,+}(\Gamma_0^{(1)}(2)) = 1$ and $S_{16}^{new,+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q - 128q^2 + 6252q^3 + 16384q^4 + 90510q^5 + O(q^6).$$

$\dim S_{16}^{new,-}(\Gamma_0^{(1)}(2)) = 0$.
$\dim S_{16}^{new,+}(\Gamma_0^{(1)}(3)) = 1$ and $S_{16}^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_2$, where

$$f_2 = q - 234q^2 - 2187q^3 + 21988q^4 + 280710q^5 + O(q^6).$$

$\dim S_{16}^{new,-}(\Gamma_0^{(1)}(3)) = 1$ and $S_{16}^{new,-}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_3$, where

$$f_3 = q - 72q^2 + 2187q^3 - 27584q^4 - 221490q^5 + O(q^6).$$

$\dim S_{16}^{new,+}(\Gamma_0^{(1)}(6)) = 2$ and $S_{16}^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_4 \oplus \mathbb{C}f_5$, where

$$f_4 = q + 128q^2 + 2187q^3 + 16384q^4 + 77646q^5 + O(q^6),$$
$$f_5 = q - 128q^2 - 2187q^3 + 16384q^4 - 314490q^5 + O(q^6).$$

$\dim S_{16}^{new,-}(\Gamma_0^{(1)}(6)) = 1$ and $S_{16}^{new,-}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_6$, where

$$f_6 = q + 128q^2 - 2187q^3 + 16384q^4 - 114810q^5 + O(q^6).$$

Hence we find $H_5(t,\phi_1) = (t-5^7)(t-5^8)h_5(t,f_1)$, $H_5(t,\phi_2) = (t-5^7)(t-5^8)h_5(t,f_2)$ and $H_5(t,\phi_3) = (t-5^7)(t-5^8)h_5(t,f_6)$, and $H_5(t,\phi_4)$ does not have such factorization. All absolute values of zeros of $H_5(t,\phi_4)$ are $5^{15/2}$.

weight 10: $\dim S_{10,0}(U'(6)) = 6$. If we put

$$\begin{aligned}
\phi_1 &= -1141885{E_2}^5 + 2578420{E_2}^3E_4 - 395560{E_2}^2E_6 \\
&\quad - 1413347E_2{E_4}^2 + 372372E_4E_6 + 6519398400{\chi_{5a}}^2, \\
\phi_2 &= \chi_{5a}\chi_{5b}, \\
\phi_3 &= -11881{E_2}^5 - 16523{E_2}^3E_4 + 67859{E_2}^2E_6 \\
&\quad + 27040E_2{E_4}^2 - 66495E_4E_6 + 358566912{\chi_{5a}}^2, \\
\phi_4 &= -1005445{E_2}^5 + 2058550{E_2}^3E_4 - 56265{E_2}^2E_6 \\
&\quad - 1045603E_2{E_4}^2 + 48763E_4E_6 - 814924800{\chi_{5a}}^2, \\
\phi_5 &= 5(8553171360955 - 88807370233\sqrt{14569}){E_2}^5 \\
&\quad + 65(-1258629706843 + 16378550233\sqrt{14569}){E_2}^3E_4 \\
&\quad + 1705(-1426727601 - 93536405\sqrt{14569}){E_2}^2E_6 \\
&\quad + 1352(29037896401 - 460871627\sqrt{14569})E_2{E_4}^2 \\
&\quad + 31031(71490083 + 5220879\sqrt{14569})E_4E_6 \\
&\quad + 814924800(-57532981 + 1170775\sqrt{14569}){\chi_{5a}}^2, \\
\phi_6 &= 5(8553171360955 + 88807370233\sqrt{14569}){E_2}^5 \\
&\quad + 65(-1258629706843 - 16378550233\sqrt{14569}){E_2}^3E_4 \\
&\quad + 1705(-1426727601 + 93536405\sqrt{14569}){E_2}^2E_6 \\
&\quad + 1352(29037896401 + 460871627\sqrt{14569})E_2{E_4}^2 \\
&\quad + 31031(71490083 - 5220879\sqrt{14569})E_4E_6 \\
&\quad + 814924800(-57532981 - 1170775\sqrt{14569}){\chi_{5a}}^2,
\end{aligned}$$

then we see that ϕ_i with $1 \le i \le 6$ are Hecke eigenbasis of $S_{10,0}(U'(6))$ and

$$\begin{aligned}
T_{10}(5)\phi_1 &= 3598860\phi_1, & T_{10}(5^2)\phi_1 &= 8331662440225\phi_1, \\
T_{10}(5)\phi_2 &= 2988900\phi_2, & T_{10}(5^2)\phi_2 &= 5742986100625\phi_2,
\end{aligned}$$

$$T_{10}(5)\phi_3 = 1317900\phi_3, \qquad T_{10}(5^2)\phi_3 = 2462729550625\phi_3,$$
$$T_{10}(5)\phi_4 = -88980\phi_4, \qquad T_{10}(5^2)\phi_4 = -1314838418975\phi_4,$$

$$T_{10}(5)\phi_5 = (2535180 + 10560\sqrt{14569})\phi_5,$$
$$T_{10}(5^2)\phi_5 = (5924648411425 + 28793001600\sqrt{14569})\phi_5,$$
$$T_{10}(5)\phi_6 = (2535180 - 10560\sqrt{14569})\phi_6,$$
$$T_{10}(5^2)\phi_6 = (5924648411425 - 28793001600\sqrt{14569})\phi_6$$

by Proposition 5.2 (1), (2), (3), (4), (5), (6), (7), (9) and Table 8, 9 and 10. We obtain

$$H_5(t,\phi_1) = (t-5^8)(t-5^9)(t^2 - 1255110t + 5^{17}),$$
$$H_5(t,\phi_2) = (t-5^8)(t-5^9)(t^2 - 645150t + 5^{17}),$$
$$H_5(t,\phi_3) = (t-5^8)(t-5^9)(t^2 + 1025850t + 5^{17}),$$
$$H_5(t,\phi_4) = t^4 + 88980t^3 + 1170167968750t^2 + 88980 \cdot 5^{17}t + 5^{34},$$
$$H_5(t,\phi_5) = (t-5^8)(t-5^9)(t^2 - (191430 + 10560\sqrt{14569})t + 5^{17}),$$
$$H_5(t,\phi_6) = (t-5^8)(t-5^9)(t^2 - (191430 - 10560\sqrt{14569})t + 5^{17}).$$

On the other hand, the spaces of cusp forms of one variable are given as follows: $\dim S_{18}^{new,+}(\Gamma_0^{(1)}(2)) = 1$ and $S_{18}^{new,+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q + 256q^2 + 6084q^3 + 65536q^4 + 1255110q^5 + O(q^6).$$

$\dim S_{18}^{new,-}(\Gamma_0^{(1)}(2)) = 0$.

Table 8.

η	${E_2}^5$	${E_2}^3E_4$
$[0,0,0]$	1	1
$[2,1,-1]$	240	2832/13
$[2,0,-1]$	360	4968/13
$[4,2,-2]$	24000	271104/13
$[4,0,-2]$	52920	748440/13
$[4,1,-2]$	69840	888624/13
$[5,1,-2]$	151200	2003616/13
$[7,-1,-3]$	151200	2003616/13
$[10,5,-5]$	793067328	9910081728/13
$[10,0,-5]$	5406473232	70447499280/13
$[20,10,-10]$	51731773500672	672439405059072/13
$[20,0,-10]$	595736929851696	7744645747810800/13
$[20,5,-10]$	2559451070543904	33273204034445280/13
$[25,5,-10]$	18685218962737728	242906427734299200/13
$[50,25,-25]$	299811787249828271856	299811726806080374096
$[100,25,-50]$	1952991598376967264494285040	25388890779536535712114175760/13

Table 9.

η	${E_2}^2E_6$	$E_2{E_4}^2$
$[0,0,0]$	1	1
$[2,1,-1]$	$34752/341$	$2544/13$
$[2,0,-1]$	$56664/341$	$5256/13$
$[4,2,-2]$	$2176608/341$	$3075648/169$
$[4,0,-2]$	$6881976/341$	$10598904/169$
$[4,1,-2]$	$11605824/341$	$11135376/169$
$[5,1,-2]$	$31968864/341$	$26872992/169$
$[7,-1,-3]$	$31968864/341$	$26872992/169$
$[10,5,-5]$	$190716173568/341$	$123830489664/169$
$[10,0,-5]$	$152074473552/31$	$918620482320/169$
$[20,10,-10]$	$1601383983449472/31$	$8740853223641856/169$
$[20,0,-10]$	$203046243577628976/341$	$100681219283171760/169$
$[20,5,-10]$	$872843492365121664/341$	$432555923946540960/169$
$[25,5,-10]$	$637142423323305772 8/341$	$3157766502908212800/169$
$[50,25,-25]$	$10223565248711099047 9296/341$	$389755170450982823524 8/13$
$[100,25,-50]$	$6659701351820954338872875990 40/341$	$2538889078013933211338443416 0/13$

Table 10.

η	E_4E_6	${\chi_{5a}}^2$	$\chi_{5a}\chi_{5b}$
$[0,0,0]$	1	0	0
$[2,1,-1]$	$353568/4433$	0	0
$[2,0,-1]$	$834840/4433$	0	0
$[4,2,-2]$	$27911712/4433$	0	0
$[4,0,-2]$	$91053720/4433$	1	0
$[4,1,-2]$	$149770080/4433$	0	1
$[5,1,-2]$	$417610080/4433$	-2	0
$[7,-1,-3]$	$417610080/4433$	-2	0
$[10,5,-5]$	$2347962277248/4433$	192	0
$[10,0,-5]$	$21815524301040/4433$	1812	0
$[20,10,-10]$	$228989195274897792/4433$	14688	0
$[20,0,-10]$	$2639604828357339120/4433$	1468746	0
$[20,5,-10]$	$11347079657888387520/4433$	1146464	2207650
$[25,5,-10]$	$82828080448639358400/4433$	-5733092	0
$[50,25,-25]$	$13290632296118463915727 68/4433$	488501760	0
$[100,25,-50]$	$8657611757585643477529978140960/4433$	2137471976320	3713083756875

$\dim S_{18}^{new,+}(\Gamma_0^{(1)}(3)) = 2$ and $S_{18}^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_2 \oplus \mathbb{C}f_3$, where

$$\begin{aligned} f_2 &= q + (297 - 3\sqrt{14569})q^2 + 6561q^3 + (88258 - 1782\sqrt{14569})q^4 \\ &\quad + (191430 + 10560\sqrt{14569})q^5 + O(q^6), \\ f_3 &= q + (297 + 3\sqrt{14569})q^2 + 6561q^3 + (88258 + 1782\sqrt{14569})q^4 \\ &\quad + (191430 - 10560\sqrt{14569})q^5 + O(q^6). \end{aligned}$$

$\dim S_{18}^{new,-}(\Gamma_0^{(1)}(3)) = 1$ and $S_{18}^{new,-}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_4$, where

$$f_4 = q + 204q^2 - 6561q^3 - 89456q^4 - 163554q^5 + O(q^6).$$

$\dim S_{18}^{new,+}(\Gamma_0^{(1)}(6)) = 2$ and $S_{18}^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_5 \oplus \mathbb{C}f_6$, where

$$f_5 = q - 256q^2 + 6561q^3 + 65536q^4 - 72186q^5 + O(q^6),$$
$$f_6 = q + 256q^2 - 6561q^3 + 65536q^4 - 199650q^5 + O(q^6).$$

$\dim S_{18}^{new,-}(\Gamma_0^{(1)}(6)) = 1$ and $S_{18}^{new,-}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_7$, where

$$f_7 = q - 256q^2 - 6561q^3 + 65536q^4 + 645150q^5 + O(q^6).$$

$\dim S_{18}(SL_2(\mathbb{Z})) = 1$ and $S_{18}(SL_2(\mathbb{Z})) = \mathbb{C}f_8$, where

$$f_8 = q - 528q^2 - 4284q^3 + 147712q^4 - 1025850q^5 + O(q^6).$$

Hence we find $H_5(t,\phi_1) = (t-5^8)(t-5^9)h_5(t,f_1)$, $H_5(t,\phi_2) = (t-5^8)(t-5^9)h_5(t,f_7)$, $H_5(t,\phi_3) = (t-5^8)(t-5^9)h_5(t,f_8)$, $H_5(t,\phi_5) = (t-5^8)(t-5^9)h_5(t,f_2)$ and $H_5(t,\phi_6) = (t-5^8)(t-5^9)h_5(t,f_3)$, and $H_5(t,\phi_4)$ does not have such factorization. All absolute values of zeros of $H_5(t,\phi_4)$ are $5^{17/2}$.

weight 11: $\dim S_{11,0}(U'(6)) = 6$. If we put

$$\begin{aligned}
\phi_1 &= (-168651{E_2}^3 + 209937E_2E_4 - 40238E_6)\chi_{5b},\\
\phi_2 &= (284240{E_2}^3 - 307983E_2E_4 + 27280E_6)\chi_{5b},\\
\phi_3 &= (38070{E_2}^3 - 38727E_2E_4 + 1705E_6)\chi_{5a},\\
\phi_4 &= (-5160{E_2}^3 + 3848E_2E_4 + 1705E_6)\chi_{5b},\\
\phi_5 &= (15(648516161263 + 489095155\sqrt{87481}){E_2}^3\\
&\quad + 13(925073695727 + 448963853\sqrt{87481})E_2E_4\\
&\quad + 3410(-470385337 - 3317995\sqrt{87481})E_6)\chi_{5a},\\
\phi_6 &= (15(648516161263 - 489095155\sqrt{87481}){E_2}^3\\
&\quad + 13(925073695727 - 448963853\sqrt{87481})E_2E_4\\
&\quad + 3410(-470385337 + 3317995\sqrt{87481})E_6)\chi_{5a},
\end{aligned}$$

then we see that ϕ_i with $1 \le i \le 6$ are Hecke eigenbasis of $S_{11,0}(U'(6))$ and

$$\begin{aligned}
T_{11}(5)\phi_1 &= 18265500\phi_1, & T_{11}(5^2)\phi_1 &= 214947093765625\phi_1,\\
T_{11}(5)\phi_2 &= 5869260\phi_2, & T_{11}(5^2)\phi_2 &= 61035253963225\phi_2,\\
T_{11}(5)\phi_3 &= 5029620\phi_3, & T_{11}(5^2)\phi_3 &= 10086149610025\phi_3,\\
T_{11}(5)\phi_4 &= -222420\phi_4, & T_{11}(5^2)\phi_4 &= -25141613327975\phi_4,
\end{aligned}$$

$$T_{11}(5)\phi_5 = (14726820 + 12480\sqrt{87481})\phi_5,$$
$$T_{11}(5^2)\phi_5 = (149477240554800 + 221331427200\sqrt{87481})\phi_5,$$
$$T_{11}(5)\phi_6 = (14726820 - 12480\sqrt{87481})\phi_6,$$
$$T_{11}(5^2)\phi_6 = (149477240554800 - 221331427200\sqrt{87481})\phi_6$$

by Proposition 5.2 (1), (2), (3), (4), (5), (6), (8), (9) and Table 11 and 12. We obtain

$$H_5(t,\phi_1) = (t-5^9)(t-5^{10})(t^2 - 6546750t + 5^{19}),$$
$$H_5(t,\phi_2) = (t-5^9)(t-5^{10})(t^2 + 5849490t + 5^{19}),$$
$$H_5(t,\phi_3) = t^4 - 5029620t^3 + 11396230468750t^2 - 5029620\cdot 5^{19}t + 5^{38},$$
$$H_5(t,\phi_4) = t^4 + 222420t^3 + 21376386718750t^2 + 222420\cdot 5^{19}t + 5^{38},$$
$$H_5(t,\phi_5) = (t-5^9)(t-5^{10})(t^2 - (3008070 + 12480\sqrt{87481})t + 5^{19}),$$
$$H_5(t,\phi_6) = (t-5^9)(t-5^{10})(t^2 - (3008070 - 12480\sqrt{87481})t + 5^{19}).$$

Table 11.

η	${E_2}^3\chi_{5a}$	${E_2}^3\chi_{5b}$	$E_2E_4\chi_{5a}$
$[0,0,0]$	0	0	0
$[2,1,-1]$	0	1	0
$[2,0,-1]$	1	0	1
$[4,2,-2]$	0	160	0
$[4,0,-2]$	222	0	3174/13
$[4,1,-2]$	128	189	1376/13
$[5,1,-2]$	0	0	0
$[7,-1,-3]$	0	0	0
$[10,5,-5]$	0	4343436	0
$[10,0,-5]$	8611226	0	113980370/13
$[20,10,-10]$	0	4639849344	0
$[20,0,-10]$	8279773068	0	109778560860/13
$[20,5,-10]$	899383552	21129170490	10846037440/13
$[25,5,-10]$	0	0	0
$[50,25,-25]$	0	36668746923865	0
$[50,0,-25]$	83674652292435	0	1110487815845175/13

On the other hand, the spaces of cusp forms of one variable are given as follows: $\dim S_{20}^{new,+}(\Gamma_0^{(1)}(2)) = 1$ and $S_{20}^{new,+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q - 512q^2 - 13092q^3 + 262144q^4 + 6546750q^5 + O(q^6).$$

$\dim S_{20}^{new,-}(\Gamma_0^{(1)}(2)) = 1$ and $S_{20}^{new,-}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_2$, where

$$f_2 = q + 512q^2 - 53028q^3 + 262144q^4 - 5556930q^5 + O(q^6).$$

Table 12.

η	$E_2E_4\chi_{5b}$	$E_6\chi_{5a}$	$E_6\chi_{5b}$
$[0,0,0]$	0	0	0
$[2,1,-1]$	1	0	1
$[2,0,-1]$	0	1	0
$[4,2,-2]$	$1792/13$	0	$7472/341$
$[4,0,-2]$	0	$9606/341$	0
$[4,1,-2]$	$2745/13$	$-3440/341$	$-1647/341$
$[5,1,-2]$	0	0	0
$[7,-1,-3]$	0	0	0
$[10,5,-5]$	$55264860/13$	0	3499308
$[10,0,-5]$	0	$2579088034/341$	0
$[20,10,-10]$	$59153495040/13$	0	$117991449792/31$
$[20,0,-10]$	0	$195376036692/31$	0
$[20,5,-10]$	$273821857650/13$	$-215836328032/341$	$5736213328770/341$
$[25,5,-10]$	0	0	0
$[50,25,-25]$	$459989594885125/13$	0	$8634632000650445/341$
$[50,0,-25]$	0	$24125014802966415/341$	0

$\dim S_{20}^{new,+}(\Gamma_0^{(1)}(3)) = 2$ and $S_{20}^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_3 \oplus \mathbb{C}f_4$, where

$$\begin{aligned} f_3 &= q + (351 + 3\sqrt{87481})q^2 - 19683q^3 + (386242 + 2106\sqrt{87481})q^4 \\ &\quad + (3008070 + 12480\sqrt{87481})q^5 + O(q^6), \\ f_4 &= q + (351 - 3\sqrt{87481})q^2 - 19683q^3 + (386242 - 2106\sqrt{87481})q^4 \\ &\quad + (3008070 - 12480\sqrt{87481})q^5 + O(q^6). \end{aligned}$$

$\dim S_{20}^{new,-}(\Gamma_0^{(1)}(3)) = 1$ and $S_{20}^{new,-}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_5$, where

$$f_5 = q - 1104q^2 + 19683q^3 + 694528q^4 + 3516270q^5 + O(q^6).$$

$\dim S_{20}^{new,+}(\Gamma_0^{(1)}(6)) = 2$ and $S_{20}^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_6 \oplus \mathbb{C}f_7$, where

$$\begin{aligned} f_6 &= q - 512q^2 - 19683q^3 + 262144q^4 - 3732474q^5 + O(q^6), \\ f_7 &= q + 512q^2 + 19683q^3 + 262144q^4 + 1953390q^5 + O(q^6). \end{aligned}$$

$\dim S_{20}^{new,-}(\Gamma_0^{(1)}(6)) = 1$ and $S_{20}^{new,-}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_8$, where

$$f_8 = q - 512q^2 + 19683q^3 + 262144q^4 - 5849490q^5 + O(q^6).$$

Hence we find $H_5(t,\phi_1) = (t-5^9)(t-5^{10})h_5(t,f_1)$, $H_5(t,\phi_2) = (t-5^9)(t-5^{10})h_5(t,f_8)$, $H_5(t,\phi_5) = (t-5^9)(t-5^{10})h_5(t,f_3)$ and $H_5(t,\phi_6) = (t-5^9)(t-5^{10})h_5(t,f_4)$, and $H_5(t,\phi_3)$ and $H_5(t,\phi_4)$ do not have such factorization. All absolute values of zeros of $H_5(t,\phi_3)$ and $H_5(t,\phi_4)$ are $5^{19/2}$.

6. Preliminaries to proving Main Theorem 1.1.

We describe a brief outline of the method for proving Main Theorem 1.1 (Theorem 3.1). The details of the proof are described in Section 7 and 8. We use the Selberg trace formula. Roughly speaking, we have to calculate the sum of contributions of all $K(N)$-conjugacy classes of $K(N)$. The method for evaluating each contribution depends on the types of conjugacy classes. We divide $\Gamma = K(N)$ into disjoint union of three subsets $\Gamma^{(e)}$, $\Gamma^{(p)}$ and $\Gamma^{(h)}$ as follows:

(i) $\Gamma^{(e)}$ consists of torsion elements of Γ.

(ii) $\Gamma^{(p)}$ consists of non-semi-simple elements of Γ whose semi-simple factors belong to $\Gamma^{(e)}$.

(iii) $\Gamma^{(h)}$ consists of the other elements of Γ than the above two types.

We denote the contribution of each set by $I(\Gamma^{(e)})_{k,j}$, $I(\Gamma^{(p)})_{k,j}$ and $I(\Gamma^{(h)})_{k,j}$. Then it is known that $I(\Gamma^{(h)})_{k,j} = 0$ and

$$\dim S_{k,j}(\Gamma) = I(\Gamma^{(e)})_{k,j} + I(\Gamma^{(p)})_{k,j}.$$

The methods for evaluating $I(\Gamma^{(e)})_{k,j}$ and $I(\Gamma^{(p)})_{k,j}$ are quite different. We evaluate $I(\Gamma^{(e)})_{k,j}$ in Section 7 and $I(\Gamma^{(p)})_{k,j}$ in Section 8.

6.1. Semi-simple elements.

We explain briefly how to evaluate $I(\Gamma^{(e)})_{k,j}$ explicitly. This method was developed in [**Has80**], [**HI80**], [**HI82**], [**HI83**], [**Has83**], [**Has84**], [**Ibu85**].

We define an element $\alpha(\mu,\nu)$ of $Sp(2;\mathbb{R})$ by

$$\alpha(\mu,\nu) = \begin{bmatrix} \cos\mu & 0 & \sin\mu & 0 \\ 0 & \cos\nu & 0 & \sin\nu \\ -\sin\mu & 0 & \cos\mu & 0 \\ 0 & -\sin\nu & 0 & \cos\nu \end{bmatrix}.$$

We define a function $J_0(\alpha(\mu,\nu))$ and a subgroup $C_0(\alpha(\mu,\nu); Sp(2;\mathbb{R}))$ of $Sp(2;\mathbb{R})$ as in [**Wak12**, (a), (b-1)–(b-5) on pages 206, 207].

Let γ be an element of $\Gamma^{(e)}$. Then there exists $g \in Sp(2;\mathbb{R})$ such that $g^{-1}\gamma g = \alpha(\mu,\nu)$ for some μ and ν. We define

$$J_0(\gamma) = J_0(\alpha(\mu,\nu)) \text{ and } C_0(\gamma; Sp(2;\mathbb{R})) = gC_0(\alpha(\mu,\nu); Sp(2;\mathbb{R}))g^{-1},$$

and

$$J_0'(\gamma) = \begin{cases} J_0(\gamma) & \text{if } -1_4 \notin C_0(\gamma; Sp(2;\mathbb{R})), \\ \frac{1}{2}J_0(\gamma) & \text{if } -1_4 \in C_0(\gamma; Sp(2;\mathbb{R})). \end{cases}$$

We put $G := GSp(2;\mathbb{Q})^+ = \left\{ g \in GL(4,\mathbb{Q}) \;\middle|\; g \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix} {}^t g = \lambda(g) \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}, \lambda(g) > 0 \right\}$

and

$$R_p := M(4;\mathbb{Z}_p) \text{ if } p \nmid N \text{ and } R_p := \begin{bmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \text{ if } p \mid N.$$

We define $c_{k,j} = 2^{-6} \cdot \pi^{-3} \cdot (k-2)(j+k-1)(j+2k-3)$.

Theorem 6.1. *The contribution $I(\Gamma^{(e)})_{k,j}$ can be obtained by*

$$I(\Gamma^{(e)})_{k,j} = c_{k,j} \sum_{\{g\}_G} J_0'(g) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p).$$

See [**Has83**, Theorem 2–4] and [**Wak12**, Theorem B.1] for details. In this section, we explain only rough procedure.

The first sum of Theorem 6.1 is taken over G-conjugacy classes $\{g\}_G$ of torsion elements of G which satisfies $\{g\}_G \cap \Gamma \neq \emptyset$. Each $H_i(k,j,N)$ in Theorem 3.1 means the partial sum taken over all G-conjugacy classes whose characteristic polynomials are of the form $f_i(\pm x)$.

$$\begin{aligned}
f_1(x) &= (x-1)^4, & f_1(-x) &= (x+1)^4, \\
f_2(x) &= (x-1)^2(x+1)^2, & & \\
f_3(x) &= (x-1)^2(x^2+1), & f_3(-x) &= (x+1)^2(x^2+1), \\
f_4(x) &= (x-1)^2(x^2+x+1), & f_4(-x) &= (x+1)^2(x^2-x+1), \\
f_5(x) &= (x-1)^2(x^2-x+1), & f_5(-x) &= (x+1)^2(x^2+x+1), \\
f_6(x) &= (x^2+1)^2, & & \\
f_7(x) &= (x^2+x+1)^2, & f_7(-x) &= (x^2-x+1)^2, \\
f_8(x) &= (x^2+1)(x^2+x+1), & f_8(-x) &= (x^2+1)(x^2-x+1), \\
f_9(x) &= (x^2+x+1)(x^2-x+1), & & \\
f_{10}(x) &= x^4+x^3+x^2+x+1, & f_{10}(-x) &= x^4-x^3+x^2-x+1, \\
f_{11}(x) &= x^4+1, & & \\
f_{12}(x) &= x^4-x^2+1. & &
\end{aligned}$$

The second sum in Theorem 6.1 for each $\{g\}_G$ is taken over G-genus of $\mathbb{Z}$-orders in $Z(g)$, the commutator algebra of g in $M(4;\mathbb{Q})$. The third product runs over all prime numbers. We give $M_G(\Lambda)$ and $c_p(g, R_p, \Lambda_p)$ explicitly separately for each characteristic polynomial in Section 7.

The advantage of Theorem 6.1 is that all necessary data are obtained by local calculations. All local data $c_p(g, R_p, \Lambda_p)$ have already been obtained in [**HI80**] if $p \nmid N$ and in [**Ibu85**] if $p \mid N$. So what we have to do is to put the local data together carefully by considering a gap between global and local conjugacy, which is summarized as follows. For each $f(x) = f_i(x)$, let $G[f]$ be the set of torsion elements of G whose characteristic polynomials are $f(\pm x)$, and consider the decomposition of $G[f]$ by G-conjugation.

Let G_A be the adelization of G, and similarly consider the decomposition of $G_A[f]$ by G_A-conjugation. It was proved by Asai [**Asa76**] that the following natural map ϕ is injective.

$$\phi : G[f]/{\sim_G} \longrightarrow G_A[f]/{\sim_{G_A}}.$$

However, it is not necessarily surjective.

THEOREM 6.2 (cf. [**HI80**], [**Has83**]). (i) *If* $f(x) = f_1(\pm x)$, $f_2(x)$, $f_3(\pm x)$, $f_4(\pm x)$, $f_5(\pm x)$, $f_8(\pm x)$ *or* $f_{10}(\pm x)$, *then the map* ϕ *is surjective. Thus we have only to put the local data together in all the combinations of local conjugacy.*

(ii) *If* $f(x) = f_6(x)$ *or* $f_7(\pm x)$, *then the map* ϕ *is not surjective. We put* $F = \mathbb{Q}(\sqrt{a^2 - 4b})$ *for* $f(x) = (x^2 + ax + b)^2$. *Then there is a correspondence between* $\{g\}_G \in G[f]/{\sim_G}$ *and the set of isomorphic classes of quaternion algebras* $Z_0(g)$ *over* $\mathbb{Q}$ *which is contained in* $Z(g)$. *This correspondence is one-to-one or two-to-one according as* $Z_0(g)$ *is indefinite or definite. If* $Z_0(g)$ *is indefinite, then* g *is* $Sp(2;\mathbb{R})$*-conjugate to* $\alpha(\mu, -\mu)$. *If* $Z_0(g)$ *is definite, then* g *is* $Sp(2;\mathbb{R})$*-conjugate to* $\alpha(\mu, \mu)$ *and* $\{g\}_G$ *and* $\{g^{-1}\}_G$ *correspond to* $Z_0(g)$. *In addition, a* G*-conjugacy class* $\{g\}_G$ *appears in the first sum of Theorem* 6.1 *if and only if* $D(Z_0(g)) \mid d_F$, *where* $D(Z_0(g))$ *and* d_F *are the discriminants of* $Z_0(g)$ *and* F.

(iii) *If* $f(x) = f_9(x)$, $f_{11}(x)$ *or* $f_{12}(x)$, *then the map* ϕ *is not surjective.*

REMARK. In the case (iii) above, for each $\{g\}_G \in G[g]/{\sim_G}$, the characteristic polynomial of g^2 is $f_6(x)$ or $f_7(\pm x)$. Hence we can sometimes use $Z_0(g^2)$ in (ii) for the case (iii) to judge if some G_A conjugacy class is in the image of ϕ.

6.2. Non-semi-simple elements.

We explain briefly how to evaluate $I(\Gamma^{(p)})_{k,j}$. Main tools were contained in [**Has83**] and [**Wak12**], and for prime levels the explicit results are known in [**Ibu85**] and [**Ibu07a**]. Let γ be an element of $\Gamma^{(p)}$. Then there exists $g \in Sp(2;\mathbb{R})$ such that $g^{-1}\gamma g$ is one of the following types.

Unipotent : $f(x) = f_1(\pm x)$

$$(1)\ \pm \begin{bmatrix} 1&0&1&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \qquad (2)\ \pm \begin{bmatrix} 1&0&0&1\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \qquad (3)\ \pm \begin{bmatrix} 1&0&0&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix},$$

$\boldsymbol{\delta}$-parabolic : $f(x) = f_2(x)$

$$(4)\ \begin{bmatrix} 1&0&s_1&0\\ 0&-1&0&s_2\\ 0&0&1&0\\ 0&0&0&-1 \end{bmatrix};\ (s_1, s_2) = (\pm 1, \pm 1), \qquad (5)\ \begin{bmatrix} 1&0&s&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{bmatrix};\ s = \pm 1,$$

Elliptic/parabolic : $f(x) = f_3(\pm x),\ f_4(\pm x),\ f_5(\pm x)$

$$(6)\ \pm\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \lambda \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \lambda = \pm 1, \qquad (7)\ \pm\begin{bmatrix} -1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 1 & 0 & \lambda \\ -\sqrt{3}/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \lambda = \pm 1,$$

$$(8)\ \pm\begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 1 & 0 & \lambda \\ -\sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \lambda = \pm 1,$$

Paraelliptic : $f(x) = f_6(x),\ f_7(\pm x)$

$$(9)\ \begin{bmatrix} 0 & 1 & 0 & s \\ -1 & 0 & -s & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix};\ s = \pm 1, \qquad (10)\ \pm\begin{bmatrix} 1/2 & \sqrt{3}/2 & s/2 & \sqrt{3}s/2 \\ -\sqrt{3}/2 & 1/2 & -\sqrt{3}s/2 & s/2 \\ 0 & 0 & 1/2 & \sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix};\ s = \pm 1$$

We denote by $I_i(k, j, N)$ $(i = 1, 2, \ldots, 10)$ the contribution of all elements of type (1)–(10) respectively.

Let $C(\gamma; Sp(2;\mathbb{R}))$ be the centralizer of γ in $Sp(2;\mathbb{R})$. We define a function $J_0(\gamma; s)$ and a subgroup $C_0(\gamma; Sp(2;\mathbb{R}))$ of $C(\gamma; Sp(2;\mathbb{R}))$ as in [**Wak12**, (e) and (f) on page 207–211].

Definition 6.3. For each $\gamma \in \Gamma$, we set

$$\begin{aligned}[]
[\gamma]_\Gamma = \{\gamma' \in \Gamma \mid \gamma_s = \gamma'_s,\ & C_0(\gamma'; Sp(2;\mathbb{R})) = C_0(\gamma; Sp(2;\mathbb{R})), \\
& C(\gamma'; Sp(2;\mathbb{R})) \simeq C(\gamma; Sp(2;\mathbb{R}))\},
\end{aligned}$$

where γ_s (resp. γ'_s) is the semisimple factor of the Jordan decomposition of γ(resp. γ'). We call the set $[\gamma]_\Gamma$ the family represented by γ. We use the notation $C_0(F; Sp(2;\mathbb{R}))$ for a family F. We define $C_0(F;\Gamma) = C_0(F; Sp(2;\mathbb{R})) \cap \Gamma$.

For a subgroup C of $Sp(2;\mathbb{R})$, we define $\overline{C} = \big(\{\pm 1_4\} \cdot C\big)/\{\pm 1_4\}$. We define $c_{k,j} = 2^{-6} \cdot \pi^{-3} \cdot (k-2)(j+k-1)(j+2k-3)$.

Theorem 6.4. *The contribution $I(\Gamma^{(p)})_{k,j}$ can be obtained by*

$$\begin{aligned}
I(\Gamma^{(p)})_{k,j} = {} & \frac{c_{k,j}}{2} \sum_{F} \mathrm{vol}(C_0(F;\Gamma)\backslash C_0(F; Sp(2;\mathbb{R}))) \cdot \lim_{s\to+0} \sum_{\gamma \in F/\sim} \frac{J_0(\gamma; s)}{[\overline{C}(\gamma;\Gamma) : \overline{C_0}(\gamma;\Gamma)]} \\
& + \frac{c_{k,j}}{2} \sum_{F} \frac{\mathrm{vol}(C_0(F;\Gamma)\backslash C_0(F; Sp(2;\mathbb{R})))}{[\overline{C}(F;\Gamma) : \overline{C_0}(F;\Gamma)]} \cdot \lim_{s\to+0} \sum_{\gamma \in F} J_0(\gamma; s).
\end{aligned}$$

See [**Has83**] and [**Wak12**] for details. The first sum in Theorem 6.4 is taken over a complete system of representatives of Γ-conjugacy classes of families of $Sp(2;\mathbb{R})$-conjugacy types (1), (2) and (3) above, and the notation $F/\sim$ means a complete system of representatives of Γ-conjugacy classes of elements of each family F. The second sum is taken over a complete system of representatives of Γ-conjugacy classes of families of $Sp(2;\mathbb{R})$-conjugacy types (4)–(10) above. Here $\gamma \in F$ is taken over all elements of each family F. Explicit formulas for the limits in each sum are given in [**Wak12**, page 207–212]. We calculate the first and the second sums of Theorem 6.4 separately for each $Sp(2;\mathbb{R})$-conjugacy types (1)–(10) in Section 8. In this section, we summarize necessary lemmas which will be used repeatedly.

In order to use Theorem 6.4, we should classify Γ-conjugacy classes of families which are extended over in the first and the second sum. It requires elaborate works, but the following proposition on the classification of equivalence classes of cusps gives a lot of help to us. The one-dimensional cusps of the Satake compactification of $\Gamma\backslash\mathfrak{H}_2$ corresponds bijectively with $\Gamma\backslash Sp(2;\mathbb{Q})/P_{2,1}(\mathbb{Q})$, and zero-dimensional cusps with $\Gamma\backslash Sp(2;\mathbb{Q})/P_{2,0}(\mathbb{Q})$, where

$$P_{2,0}(\mathbb{Q}) = \left\{ \begin{bmatrix} A & B \\ 0_2 & D \end{bmatrix} \in Sp(2;\mathbb{Q}) \right\}, \qquad P_{2,1}(\mathbb{Q}) = \left\{ \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \in Sp(2;\mathbb{Q}) \right\}.$$

Poor and Yuen [**PY13**] gave representatives of cusps for general N (See also [**Ibu93**] for the prime level case). We quote it in the case of squarefree N in the following proposition.

Proposition 6.5 (Poor and Yuen [**PY13**]). *Let N be a squarefree positive integer. Then we have*

$$\begin{aligned} Sp(2;\mathbb{Q}) &= K(N)P_{2,0}(\mathbb{Q}) \\ &= \bigsqcup_{m|N, m>0} K(N) \begin{bmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m & 1 \end{bmatrix} P_{2,1}(\mathbb{Q}), \end{aligned}$$

Corollary 6.6. *Let g be an element of $\Gamma^{(p)}$. Then g is $K(N)$-conjugate to an element in $P_{2,0}(\mathbb{Q})$ or $x_m P_{2,1}(\mathbb{Q}) x_m^{-1}$ for some* $x_m = \begin{bmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m & 1 \end{bmatrix}$.

Let G be a group and H a subgroup of G. There are many cases where we should consider the splitting of a G-conjugacy class into H-conjugacy classes, that is, classify H-conjugacy classes of $\{\gamma\}_G \cap H$ for each G-conjugacy class $\{\gamma\}_G$. Let $C(\gamma; G)$ be the centralizer of γ in G and $M(\gamma; G/H) = \{x \in G; x^{-1}\gamma x \in H\}$.

Proposition 6.7. *We have a bijective map*

$$\big(\{\gamma\}_G \cap H\big)/\sim_H \xrightarrow{\sim} C(\gamma;G)\backslash M(\gamma;G/H)/H,$$

which sends the H-conjugacy class $\{x^{-1}\gamma x\}_H$ to the double coset $C(\gamma;G)xH$.

7. Proof of Main Theorem 1.1: semi-simple elements.

In this section, we give a proof to obtain the contribution $I(\Gamma^{(e)})_{k,j}$ in detail. Notations are the same as in Section 6.

7.1. The contribution $H_1(k,j,N)$.

We consider the contribution $H_1(k,j,N)$ of $\pm 1_4$, namely the partial sum of Theorem 6.1 for $\{1_4\}_G$ and $\{-1_4\}_G$. As is well-known, it corresponds to the volume of the fundamental domain. If we put

$$B(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap Sp(2;\mathbb{Q}).$$

then we have

$$[Sp(2;\mathbb{Z}) : B(N)] = \prod_{p|N}(p^2+1)(p+1)^2 \quad \text{and} \quad [K(N) : B(N)] = \prod_{p|N}(p+1)^2.$$

This can be proved, for example, by Bruhat–Tits theory. Now we have $H_1(k,j,N) = H_1^{Sp(2;\mathbb{Z})}(k,j) \times \prod_{p|N}(p^2+1)$, where $H_1^{Sp(2;\mathbb{Z})}(k,j) = 2^{-7}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3)$. (cf. [**Wak12**, Theorem 7.1])

7.2. The contribution $H_2(k,j,N)$.

We consider the contribution of the elliptic elements whose characteristic polynomial is $f_2(x) = (x-1)^2(x+1)^2$. The set of such elements of $G = GSp(2;\mathbb{Q})^+$ consists of a single G-conjugacy class $\{g\}_G$, and the contribution $H_2(k,j,N)$ is given by

$$H_2(k,j,N) = c_{k,j} J_0'(g) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p). \tag{8}$$

We have

$$C_0(g;Sp(2;\mathbb{R})) = \left\{ \begin{bmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{bmatrix} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in SL(2;\mathbb{R}) \right\} \ni -1_4$$

and

$$J_0'(g) = \frac{1}{2} J_0(g) = {c_{k,j}}^{-1} 2^{-7} \pi^{-4} (-1)^k (j+k-1)(k-2)$$

by [**Wak12**, (b-5)]. We can use the local data $c_p(g, R_p, \Lambda_p)$ from [**HI80**, Proposition 13] for prime numbers $p \nmid N$. On the other hand, the local data for $p \mid N$ need to be calculated newly since in [**Ibu85**] the global conjugacy classes are given directly and local data have not been used.

Proposition 7.1. *For a prime number p with $p \mid N$, we have*

$$c_p(g, R_p, \Lambda_p) = \begin{cases} 2 \text{ if } \Lambda_p \sim Z(g)_p \cap R_p, \\ 1 \text{ if } \Lambda_p \sim Z(g)_p \cap x R_p x^{-1} \text{ and } p = 2, \\ 0 \text{ otherwise,} \end{cases}$$

where $x = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, *and where $\sim$ means conjugation by $Z(g)_p^\times \cap G_p$.*

Proof. We define $M_p(g, R_p) = \{x \in G_p \mid x^{-1} g x \in R_p\}$. By direct calculations, a complete system of representatives of $(Z(g)^\times \cap G)_p \backslash M_p(g, R_p) / (R_p^\times \cap G_p)$ is given by $\{x_1, x_2\}$ if $p \neq 2$, and $\{x_1, x_2, x_3\}$ if $p = 2$, where

$$x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

We can check $Z(g)_p \cap x_1 R_p x_1^{-1} \sim Z(g)_p \cap x_2 R_p x_2^{-1}$ and $Z(g)_p \cap x_1 R_p x_1^{-1} \not\sim Z(g)_p \cap x_3 R_p x_3^{-1}$ if $p = 2$. □

Putting all local data together, we have $\prod_p c_p(g, R_p, \Lambda_p)$ and $M_G(\Lambda)$ as follows:

$$\prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{\omega(N)} & \text{if } \Lambda \sim L_1, \\ 2^{\omega(N)} & \text{if } \Lambda \sim L_2 \text{ and } 2 \nmid N, \\ 2^{\omega(N)-1} & \text{if } \Lambda \sim L_3 \text{ and } 2 \mid N, \\ 0 & \text{otherwise,} \end{cases}$$

where L_1, L_2 and L_3 are certain $\mathbb{Z}$-orders in $Z(g)$ such that $M_G(L_1) = \pi^4/9$, $M_G(L_2) = 2\pi^4/3$ and $M_G(L_3) = \pi^4$. Here $\Lambda \sim L$ means that two $\mathbb{Z}$-orders Λ and L belong to the same G-genus, that is, they are locally conjugate by $Z(g)_p^\times \cap G_p$ for all p. Hence the sum of (8) runs over these three $\mathbb{Z}$-orders, and we obtain $H_2(k, j, N)$.

7.3. The contribution $H_3(k, j, N)$, $H_4(k, j, N)$ and $H_5(k, j, N)$.

We consider the contributions of the elliptic elements with characteristic polynomials

$$
\begin{aligned}
f_3(x) &= (x-1)^2(x^2+1), & f_3(-x) &= (x+1)^2(x^2+1),\\
f_4(x) &= (x-1)^2(x^2+x+1), & f_4(-x) &= (x+1)^2(x^2-x+1),\\
f_5(x) &= (x-1)^2(x^2-x+1), & f_5(-x) &= (x+1)^2(x^2+x+1).
\end{aligned}
$$

For each i, the set of such elements of $G = GSp(2;\mathbb{Q})^+$ with $f_i(\pm x)$ consists of four G-conjugacy classes represented by g, g^{-1}, $-g$ and $-g^{-1}$ for some g, and the contribution $H_i(k,j,N)$ is given by

$$
H_i(k,j,N) = 2\cdot c_{k,j}\cdot \left(J_0'(g)+J_0'(g^{-1})\right)\sum_{\Lambda} M_G(\Lambda)\prod_p c_p(g,R_p,\Lambda_p). \tag{9}
$$

For each i, g is $Sp(2;\mathbb{R})$-conjugate to $\alpha(\theta,0)$ where $\theta=\pi/2$, $2\pi/3$, $\pi/3$ respectively. We have

$$
C_0(g;Sp(2;\mathbb{R})) = \left\{ \begin{bmatrix} 1&0&0&0\\ 0&a&0&b\\ 0&0&1&0\\ 0&c&0&d \end{bmatrix} ; \begin{bmatrix} a&b\\ c&d \end{bmatrix} \in SL(2;\mathbb{R}) \right\} \not\ni -1_4
$$

and

$$
\begin{aligned}
J_0'(g)+J_0'(g^{-1}) &= J_0(g)+J_0(g^{-1})\\
&= c_{k,j}^{-1}\cdot \frac{(j+k-1)\sin(k-2)\theta-(k-2)\sin(j+k-1)\theta}{2^4\pi^2\sin\theta(1-\cos\theta)}
\end{aligned}
$$

by [**Wak12**, (b-4)]. We quote necessary local data from [**HI80**, Proposition 14] and [**Ibu85**, Proposition 2.3] below.

$$
H_3(k,j,N):\quad \prod_p c_p(g,R_p,\Lambda_p) = \begin{cases} 2^{\omega(N)} & \text{if } \Lambda\sim L_1,\\ 2^{\omega(N)-1} & \text{if } \Lambda\sim L_2 \text{ and } 2\mid N,\\ 0 & \text{otherwise,} \end{cases}
$$

$$
H_4(k,j,N):\quad \prod_p c_p(g,R_p,\Lambda_p) = \begin{cases} 2^{\omega(N)} & \text{if } \Lambda\sim L_3,\\ 2^{\omega(N)-1} & \text{if } \Lambda\sim L_4 \text{ and } 3\mid N,\\ 0 & \text{otherwise,} \end{cases}
$$

$$
H_5(k,j,N):\quad \prod_p c_p(g,R_p,\Lambda_p) = \begin{cases} 2^{\omega(N)} & \text{if } \Lambda\sim L_5,\\ 0 & \text{otherwise,} \end{cases}
$$

where L_1, L_2, L_3, L_4 and L_5 are certain $\mathbb{Z}$-orders in $Z(g)$ such that $M_G(L_1)=\pi^2/12$, $M_G(L_2)=\pi^2/4$, $M_G(L_3)=\pi^2/18$, $M_G(L_4)=\pi^2/3$ and $M_G(L_5)=\pi^2/18$. Hence we obtain $H_3(k,j,N)$, $H_4(k,j,N)$ and $H_5(k,j,N)$ by (9).

7.4. The contribution $H_6(k,j,N)$ and $H_7(k,j,N)$.

We consider the contributions of elliptic elements whose characteristic polynomials are $f_6(x)=(x^2+1)^2$ and $f_7(\pm x)=(x^2\pm x+1)^2$. The set of such elements of $G =$

$GSp(2;\mathbb{Q})^+$ consists of infinitely many G-conjugacy classes, but only finite numbers of G-conjugacy classes appear in the first sum of Theorem 6.1.

First we consider $H_6(k,j,N)$. In this case, three G-conjugacy classes $\{g_1\}_G$, $\{g_2\}_G$ and $\{g_2{}^{-1}\}_G$ appear, where

$$g_1=\alpha\left(\frac{\pi}{2},-\frac{\pi}{2}\right)=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix},\qquad g_2=\alpha\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{bmatrix},$$

and the contribution $H_6(k,j,N)$ is given by

$$\begin{aligned}H_6(k,j,N)&=c_{k,j}\cdot J_0'(g_1)\sum_{\Lambda}M_G(\Lambda)\prod_p c_p(g_1,R_p,\Lambda_p)\\&\quad+c_{k,j}\cdot\Big(J_0'(g_2)+J_0'(g_2{}^{-1})\Big)\sum_{\Lambda}M_G(\Lambda)\prod_p c_p(g_2,R_p,\Lambda_p).\end{aligned}\tag{10}$$

We have

$$\begin{aligned}C_0(g_1;Sp(2;\mathbb{R}))&=\left\{\begin{bmatrix}a&0&b&0\\0&a&0&b\\c&0&d&0\\0&c&0&d\end{bmatrix};\begin{bmatrix}a&b\\c&d\end{bmatrix}\in SL(2;\mathbb{R})\right\}\ni-1_4,\\C_0(g_2;Sp(2;\mathbb{R}))&=\{1_4\}\not\ni-1_4,\end{aligned}$$

and

$$\begin{aligned}&J_0'(g_1)=2^{-1}J_0(g_1)=c_{k,j}{}^{-1}2^{-5}\pi^{-2}(j+2k-3)(-1)^{j/2},\\&J_0'(g_2)+J_0'(g_2{}^{-1})=J_0(g_2)+J_0(g_2^{-1})=c_{k,j}{}^{-1}2^{-2}(j+1)(-1)^{k+j/2}\end{aligned}$$

by [**Wak12**, (b-3), (b-2)]. The remaining parts of (10) depend on the conditions of N. We collect necessary local data as follows from [**HI80**, Proposition 15, 16] and [**Ibu85**, Proposition 2.4, 2.5].

(i) If $2\mid N$ and $\sharp\{p\mid N;(-1/p)=-1\}=$ even, we have

$$\prod_p c_p(g_1,R_p,\Lambda_p)=\begin{cases}1\text{ if }\Lambda\sim L_1,\\1\text{ if }\Lambda\sim L_2,\\0\text{ otherwise},\end{cases}\qquad\prod_p c_p(g_2,R_p,\Lambda_p)=\begin{cases}1\text{ if }\Lambda\sim L_3,\\1\text{ if }\Lambda\sim L_4,\\0\text{ otherwise},\end{cases}$$

and $M_G(L_i)=a_i\prod_{p\mid N}(p+(-1/p))$, where $a_i=\pi^2/8$ if $i=1$, $\pi^2/4$ if $i=2$, $1/32$ if $i=3$, and $1/192$ if $i=4$.

(ii) If $2\mid N$ and $\sharp\{p\mid N;(-1/p)=-1\}=$ odd, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_1, \\ 1 \text{ if } \Lambda \sim L_2, \\ 0 \text{ otherwise}, \end{cases} \qquad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_3, \\ 1 \text{ if } \Lambda \sim L_4, \\ 0 \text{ otherwise}, \end{cases}$$

and $M_G(L_i) = a_i \prod_{p|N}(p + (-1/p))$, where $a_i = \pi^2/4$ if $i = 1$, $\pi^2/24$ if $i = 2$, $1/64$ if $i = 3$, and $1/32$ if $i = 4$.

(iii) If $2 \nmid N$ and $\sharp \{p \mid N; (-1/p) = -1\} =$ even, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_1, \\ 1 \text{ if } \Lambda \sim L_2, \\ 0 \text{ otherwise}, \end{cases} \qquad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_3, \\ 0 \text{ otherwise}, \end{cases}$$

and $M_G(L_i) = a_i \prod_{p|N}(p + (-1/p))$, where $a_i = \pi^2/4$ if $i = 1$, $\pi^2/6$ if $i = 2$, and $1/32$ if $i = 3$.

(iv) If $2 \nmid N$ and $\sharp \{p \mid N; (-1/p) = -1\} =$ odd, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_1, \\ 0 \text{ otherwise}, \end{cases} \qquad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_2, \\ 1 \text{ if } \Lambda \sim L_3, \\ 0 \text{ otherwise}, \end{cases}$$

and $M_G(L_i) = a_i \prod_{p|N}(p + (-1/p))$, where $a_i = \pi^2/4$ if $i = 1$, $1/32$ if $i = 2$, and $1/48$ if $i = 3$.

Summing up all the cases, we obtain $H_6(k, j, N)$ by (10) as follows:

$$H_6(k, j, N) = \left(\alpha \cdot (j + 2k - 3)(-1)^{j/2} + \beta \cdot (j + 1)(-1)^{k+j/2}\right) \cdot \prod_{p|N} \left(p + \left(\frac{-1}{p}\right)\right),$$

where

$$\begin{aligned}
&\alpha = 2^{-8} \cdot 3, && \beta = 2^{-8} \cdot 3^{-1} \cdot 7, && \text{if } 2 \mid N \text{ and } \sharp\{p \mid N\ ;\ \left(\frac{-1}{p}\right) = -1\} = \text{even}, \\
&\alpha = 2^{-8} \cdot 3^{-1} \cdot 7, && \beta = 2^{-8} \cdot 3, && \text{if } 2 \mid N \text{ and } \sharp\{p \mid N\ ;\ \left(\frac{-1}{p}\right) = -1\} = \text{odd}, \\
&\alpha = 2^{-7} \cdot 3^{-1} \cdot 5, && \beta = 2^{-7}, && \text{if } 2 \nmid N \text{ and } \sharp\{p \mid N\ ;\ \left(\frac{-1}{p}\right) = -1\} = \text{even}, \\
&\alpha = 2^{-7}, && \beta = 2^{-7} \cdot 3^{-1} \cdot 5, && \text{if } 2 \nmid N \text{ and } \sharp\{p \mid N\ ;\ \left(\frac{-1}{p}\right) = -1\} = \text{odd}.
\end{aligned}$$

By careful calculation, we can rewrite $H_6(k, j, N)$ as the simple formula in Theorem 3.1.

Next we consider $H_7(k, j, N)$. This is calculated in the same way as $H_6(k, j, N)$. In this case, six G-conjugacy classes $\{g_1\}_G$, $\{g_2\}_G$, $\{{g_2}^{-1}\}_G$, $\{-g_1\}_G$, $\{-g_2\}_G$, $\{-{g_2}^{-1}\}_G$ appear in the first sum of Theorem 6.1, where g_1 and g_2 are some elements of G such that $g_1 \sim \alpha(2\pi/3, -2\pi/3)$, $g_2 \sim \alpha(2\pi/3, 2\pi/3)$ by $Sp(2; \mathbb{R})$-conjugation. The contribution is given by

$$H_7(k,j,N) = 2 \cdot c_{k,j} \cdot J_0'(g_1) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g_1, R_p, \Lambda_p)$$
$$+ 2 \cdot c_{k,j} \cdot \left(J_0'(g_2) + J_0'({g_2}^{-1})\right) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g_2, R_p, \Lambda_p). \qquad (11)$$

We have the same definitions $C_0(g_1; Sp(2;\mathbb{R}))$ and $C_0(g_2; Sp(2;\mathbb{R}))$ as in $H_6(k,j,N)$, and

$$J_0'(g_1) = 2^{-1} J_0(g_1) = {c_{k,j}}^{-1} 2^{-3} 3^{-1} \pi^{-2} (j+2k-3)[1,-1,0;3]_j,$$
$$J_0'(g_2) + J_0'({g_2}^{-1}) = J_0(g_2) + J_0(g_2^{-1}) = {c_{k,j}}^{-1} 3^{-1}(j+1)[0,1,-1;3]_{j+2k}.$$

The local data are given as follows:

(i) If $3 \mid N$ and $\sharp\{p \mid N; (-3/p) = -1\}$ = even, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_1, \\ 1 \text{ if } \Lambda \sim L_2, \\ 0 \text{ otherwise,} \end{cases} \qquad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_3, \\ 0 \text{ otherwise,} \end{cases}$$

and $M_G(L_i) = a_i \prod_{p|N}(p + (-1/p))$, where $a_i = \pi^2/27$ if $i=1$, $4\pi^2/27$ if $i=2$, and $1/54$ if $i=3$.

(ii) If $3 \mid N$ and $\sharp\{p \mid N; (-3/p) = -1\}$ = odd, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_1, \\ 0 \text{ otherwise,} \end{cases} \qquad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_2, \\ 1 \text{ if } \Lambda \sim L_3, \\ 0 \text{ otherwise,} \end{cases}$$

and $M_G(L_i) = a_i \prod_{p|N}(p + (-1/p))$, where $a_i = 4\pi^2/27$ if $i=1$, $1/216$ if $i=2$, and $1/54$ if $i=3$.

(iii) If $3 \nmid N$ and $\sharp\{p \mid N; (-3/p) = -1\}$ = even, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_1, \\ 0 \text{ otherwise,} \end{cases} \qquad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_2, \\ 0 \text{ otherwise,} \end{cases}$$

and $M_G(L_i) = a_i \prod_{p|N}(p + (-1/p))$, where $a_i = 2\pi^2/9$ if $i=1$, and $1/72$ if $i=2$.

(iv) If $3 \nmid N$ and $\sharp\{p \mid N; (-3/p) = -1\}$ = odd, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_1, \\ 0 \text{ otherwise,} \end{cases} \qquad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 \text{ if } \Lambda \sim L_2, \\ 0 \text{ otherwise,} \end{cases}$$

and $M_G(L_i) = a_i \prod_{p|N}(p + (-1/p))$, where $a_i = \pi^2/9$ if $i=1$, and $1/36$ if $i=2$.

We can obtain $H_7(k,j,N)$ by (11) by the same way as $H_6(k,j,N)$.

7.5. The contribution $H_8(k, j, N)$.

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_8(x) = (x^2+1)(x^2+x+1)$ or $f_8(-x) = (x^2+1)(x^2-x+1)$. The set of such elements of $G = GSp(2;\mathbb{Q})^+$ consists of eight G-conjugacy classes. Let g_1 be an element of G such that g_1 is $Sp(2;\mathbb{R})$-conjugate to $\alpha(\pi/2, 2\pi/3)$, and h an element of $Sp(2;\mathbb{R})$ such that $g_1 = h^{-1}\alpha(\pi/2, 2\pi/3)h$. Then the G-conjugacy classes are represented by $g_1 = h^{-1}\alpha(\pi/2, 2\pi/3)h$, $g_2 = h^{-1}\alpha(\pi/2, -2\pi/3)h$, $g_3 = h^{-1}\alpha(-\pi/2, 2\pi/3)h$, $g_4 = h^{-1}\alpha(-\pi/2, -2\pi/3)h$, and $-g_1$, $-g_2$, $-g_3$, $-g_4$. The contribution $H_8(k, j, N)$ is given by

$$H_8(k,j,N) = 2 \cdot c_{k,j} \cdot \left(J_0'(g_1) + J_0'(g_2) + J_0'(g_3) + J_0'(g_4)\right) \cdot \sum_{L_G(\Lambda)} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p) \tag{12}$$

by Theorem 6.1. We have

$$C_0(g_1; Sp(2;\mathbb{R})) = C_0(g_2; Sp(2;\mathbb{R})) = C_0(g_3; Sp(2;\mathbb{R})) = C_0(g_4; Sp(2;\mathbb{R})) = \{1_4\} \not\ni -1_4$$

and

$$J_0'(g_1) + J_0'(g_2) + J_0'(g_3) + J_0'(g_4) = J_0(g_1) + J_0(g_2) + J_0(g_3) + J_0(g_4) = c_{k,j}^{-1} \cdot C_8(k,j)$$

by [**Wak12**, (b-1)], where $C_8(k,j)$ is given in subsection 2.4. The local data are given by

$$\prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{\sharp N} & \text{if } \Lambda \sim L, \\ 0 & \text{otherwise,} \end{cases}$$

where L is a certain $\mathbb{Z}$-order in $Z(g)$ such that $M_G(L) = 1/24$. Hence we obtain $H_8(k, j, N)$ by (12).

7.6. The contribution $H_9(k, j, N)$.

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_9(x) = (x^2+x+1)(x^2-x+1)$. The set of such elements of $G = GSp(2;\mathbb{Q})^+$ consists of infinitely many G-conjugacy classes, which are $Sp(2;\mathbb{R})$-conjugate to $g_1 = \alpha(\pi/3, 2\pi/3)$, g_1^{-1}, $g_2 = \alpha(\pi/3, -2\pi/3)$ and g_2^{-1}, but exactly four G-conjugacy classes out of them appear in the first sum of Theorem 6.1 since they are the only G-conjugacy classes which have nonempty intersections with $K(N)$. Since they are $Sp(2;\mathbb{R})$-conjugate to g_1, g_1^{-1}, g_2 and g_2^{-1}, the contribution $H_9(k, j, N)$ is given by

$$H_9(k,j,N) = c_{k,j} \left\{J_0'(g_1) + J_0'({g_1}^{-1}) + J_0'(g_2) + J_0'({g_2}^{-1})\right\} \prod_{\Lambda} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p). \tag{13}$$

We have $C_0(g_1; Sp(2;\mathbb{R})) = C_0(g_2; Sp(2;\mathbb{R})) = \{1_4\}$ and

$$J_0'(g_1) + J_0'({g_1}^{-1}) + J_0'(g_2) + J_0'({g_2}^{-1}) = J_0(g_1) + J_0({g_1}^{-1}) + J_0(g_2) + J_0({g_2}^{-1})$$

$$= c_{k,j}^{-1} \cdot C_9(k,j)$$

by [**Wak12**, (b-1)] where $C_9(k,j)$ is given in subsection 2.4. We can see from [**HI80**, Proposition 18] and [**Ibu85**, Proposition 2.8] that

$$\prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{\omega(N)} & \text{if } \Lambda \sim L_1, \\ 2^{\omega(N)} & \text{if } \Lambda \sim L_2 \text{ and } 2 \nmid N, \\ 0 & \text{otherwise,} \end{cases}$$

where L_1 and L_2 are certain $\mathbb{Z}$-orders in $Z(g)$ such that $M_G(L_1) = 1/36$ and $M_G(L_2) = 1/12$. Thus we can obtain $H_9(k,j,N)$ by (13).

7.7. The contribution $H_{10}(k,j,N)$.

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_{10}(x) = x^4 + x^3 + x^2 + x + 1$ or $f_{10}(-x) = x^4 - x^3 + x^2 - x + 1$. The set of such elements of $G = GSp(2;\mathbb{Q})^+$ consists of infinitely many G-conjugacy classes, which are $Sp(2;\mathbb{R})$-conjugate to $g = \alpha(2\pi/5, -4\pi/5)$, g^2, g^3 and g^4, but only four G-conjugacy classes out of them can appear in the first sum of Theorem 6.1 since they can have nonempty intersections with $K(N)$. Since they are $Sp(2;\mathbb{R})$-conjugate to g, g^2, g^3 and g^4, the contribution $H_{10}(k,j,N)$ is given by

$$H_{10}(k,j,N) = c_{k,j} \left\{ J_0'(g) + J_0'(g^2) + J_0'(g^3) + J_0'(g^4) \right\} \prod_{\Lambda} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p). \tag{14}$$

We have $C_0(g; Sp(2;\mathbb{R})) = \{1_4\}$ and

$$\begin{aligned} J_0'(g) + J_0'(g^2) + J_0'(g^3) + J_0'(g^4) &= J_0(g) + J_0(g^2) + J_0(g^3) + J_0(g^4) \\ &= c_{k,j}^{-1} \cdot C_{10}(k,j) \end{aligned}$$

by [**Wak12**, (b-1)] where $C_{10}(k,j)$ is given in subsection 2.4. We can see from [**HI80**, Proposition 18] that

$$\prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{\sharp N(\pm 1;5)} & \text{if } \Lambda \sim L \text{ and } \{p \mid N; p \equiv 2, 3 \bmod 5\} = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where L is a certain $\mathbb{Z}$-orders in $Z(g)$ such that $M_G(L) = 1/10$. Thus we can obtain $H_{10}(k,j,N)$ by (14).

7.8. The contribution $H_{11}(k,j,N)$.

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_{11}(x) = x^4 + 1$. The set of such elements of $G = GSp(2;\mathbb{Q})^+$ consists of infinitely many G-conjugacy classes, which are $Sp(2;\mathbb{R})$-conjugate to $\gamma_1 = \alpha(\pi/4, 3\pi/4)$, γ_1^{-1}, $\gamma_2 = \alpha(\pi/4, -3\pi/4)$ and γ_2^{-1}, but only four G-conjugacy classes out of them can appear in the first sum of Theorem 6.1 since they can have nonempty intersections with $K(N)$. They are $Sp(2;\mathbb{R})$-conjugate to γ_1, γ_1^{-1}, γ_2 and γ_2^{-1}. We need to take notice of

Theorem 6.2 (iii) and the Remark after that. There is a single G_p-conjugacy class $\{g_p\}_{G_p}$ with $Z_0(g_p^2)$ = split if $p \equiv 1, 3, 5 \bmod 8$, and there are two G_p-conjugacy classes $\{g_p\}_{G_p}$ and $\{h_p\}_{G_p}$ with $Z_0(g_p^2)$ = split and $Z_0(h_p^2)$ = division if $p = 2$ or $p \equiv 7 \bmod 8$. For a given combination of local conjugacy classes $\{g_v\}_{G_v}$ for each place v, the necessary and sufficient condition for existence of the corresponding G-conjugacy is

(i) the number of prime numbers such that $Z_0(g_p^2)$ is division is finite, and

(ii) if the number of prime numbers such that $Z_0(g_p^2)$ is division is even (resp. odd), then g_∞ is $Sp(2;\mathbb{R})$-conjugate to g_1 or g_1^{-1} (resp. g_2 or g_2^{-1}).

As a result, for a fixed conjugacy class $g_1^{\pm 1}$ or $g_2^{\pm 1}$ at ∞, we can take any combination of local conjugacy classes except for $p = 2$ and choose one of two G_2-conjugacy classes to make the above condition (ii) hold. For each G_p-conjugacy classes, $c_p(g_p, R_p, \mathcal{O}_p)$ are given as follows.

(i) if $p \equiv 1 \bmod 8$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N. \end{cases}$$

(ii) if $p \equiv 3, 5 \bmod 8$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N. \end{cases}$$

(iii) if $p \equiv 7 \bmod 8$,

$$c_p(g_p, R_p, \mathcal{O}_p) + c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N. \end{cases}$$

(iv) if $p = 2$,

$$c_p(g_p, R_p, \mathcal{O}_p) = c_p(h_p, R_p, \mathcal{O}_p) = 1.$$

It follows that $H_{11}(k, j, N) = 0$ if $\{p \mid N; p \equiv 3, 5 \bmod 8\} \neq \emptyset$. Otherwise,

$$\begin{aligned} H_{11}(k,j,N) = 2^{-3} \cdot c_{k,j}^{-1} \cdot \big(J_0'(\gamma_1) + J_0'(\gamma_1^{-1}) + J_0'(\gamma_2) + J_0'(\gamma_2^{-1})\big) \\ \cdot \prod_{\substack{p \nmid N \\ p \neq 2}} c_p(g_p, R_p, \mathcal{O}_p) \cdot \prod_{\substack{p \nmid N \\ p \equiv 1(8)}} c_p(g_p, R_p, \mathcal{O}_p) \\ \cdot \prod_{\substack{p \nmid N \\ p \equiv 7(8)}} \Big(c_p(g_p, R_p, \mathcal{O}_p) + c_p(h_p, R_p, \mathcal{O}_p)\Big) \\ \cdot \big\{c_2(g_2, R_2, \mathcal{O}_2) \text{ or } c_2(h_2, R_2, \mathcal{O}_2)\big\} \\ = 2^{-3} \cdot c_{k,j}^{-1} \cdot \big(J_0'(\gamma_1) + J_0'(\gamma_1^{-1}) + J_0'(\gamma_2) + J_0'(\gamma_2^{-1})\big) \cdot 2^{\sharp\{p \mid N; p \equiv \pm 1(8)\}}. \end{aligned}$$

We have $C_0(\gamma_1; Sp(2;\mathbb{R})) = C_0(\gamma_2; Sp(2;\mathbb{R})) = \{1_4\}$ and

$$\begin{aligned} J_0'(\gamma_1) + J_0'(\gamma_1^{-1}) + J_0'(\gamma_2) + J_0'(\gamma_2^{-1}) &= J_0(\gamma_1) + J_0(\gamma_1^{-1}) + J_0(\gamma_2) + J_0(\gamma_2^{-1}) \\ &= c_{k,j} \cdot C_{11}(k,j). \end{aligned}$$

Thus we obtain $H_{11}(k,j,N)$.

7.9. The contribution $H_{12}(k,j,N)$.

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_{12}(x) = x^4 - x^2 + 1$. The set of such elements of $G = GSp(2;\mathbb{Q})^+$ consists of infinitely many G-conjugacy classes, which are $Sp(2;\mathbb{R})$-conjugate to $\gamma_1 = \alpha(\pi/6, 5\pi/6)$, $\gamma_2 = \alpha(\pi/6, -5\pi/6)$, γ_1^{-1} and γ_2^{-1}, but only four G-conjugacy classes out of them can appear in the first sum of Theorem 6.1 since they can have nonempty intersections with $K(N)$. They are $Sp(2;\mathbb{R})$-conjugate to γ_1, γ_2, γ_1^{-1} and γ_2^{-1}. We need to take notice of Theorem 6.2 (iii) and use Remark after that. There is a single G_p-conjugacy class $\{g_p\}_{G_p}$ if $p \equiv 1, 5, 7 \bmod 12$, and there are two G_p-conjugacy classes $\{g_p\}_{G_p}$ and $\{h_p\}_{G_p}$ otherwise. For each $g_p \in G_p$, necessary local data are given as follows.

(i) if $p \equiv 1 \bmod 12$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \qquad Z_0(g_p^2) = \text{split}.$$

(ii) if $p \equiv 5, 7 \bmod 12$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \qquad Z_0(g_p^2) = \text{split}.$$

(iii) if $p \equiv 11 \bmod 12$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \qquad Z_0(g_p^2) = \text{split},$$

$$c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \qquad Z_0(h_p^2) = \text{division}.$$

(iv) if $p = 2$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \qquad Z_0(g_p^2) = \text{split},$$

$$c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 1 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \qquad Z_0(h_p^2) = \text{division}.$$

(v) if $p = 3$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \qquad Z_0(g_p^2) = \text{division},$$

$$c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 1 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \qquad Z_0(h_p^2) = \text{split}.$$

It follows that $H_{12}(k, j, N) = 0$ if $\{p \mid N; p \equiv 5, 7 \bmod 12\} \neq \emptyset$. Otherwise, $H_{12}(k, j, N)$ is given as follows:

(I) if "$2 \mid N$ and $3 \mid N$" or "$2 \nmid N$ and $3 \nmid N$",

(i) if $\sharp\{p \mid N; p \equiv 11 \bmod 12\} = $ even,

$$H_{12}(k, j, N) = 2^{-2}3^{-1}\big(J_0'(\gamma_2) + J_0'(\gamma_2^{-1})\big) \cdot 2^{\sharp\{p|N; p \equiv \pm 1 (12)\}}.$$

(ii) if $\sharp\{p \mid N; p \equiv 11 \bmod 12\} = $ odd,

$$H_{12}(k, j, N) = 2^{-2}3^{-1}\big(J_0'(\gamma_1) + J_0'(\gamma_1^{-1})\big) \cdot 2^{\sharp\{p|N; p \equiv \pm 1 (12)\}}.$$

(II) if "$2 \nmid N$ and $3 \mid N$" or "$2 \mid N$ and $3 \nmid N$",

(i) if $\sharp\{p \mid N; p \equiv 11 \bmod 12\} = $ even,

$$H_{12}(k, j, N) = 2^{-2}3^{-1}\big(J_0'(\gamma_1) + J_0'(\gamma_1^{-1})\big) \cdot 2^{\sharp\{p|N; p \equiv \pm 1 (12)\}}.$$

(ii) if $\sharp\{p \mid N; p \equiv 11 \bmod 12\} = $ odd,

$$H_{12}(k, j, N) = 2^{-2}3^{-1}\big(J_0'(\gamma_2) + J_0'(\gamma_2^{-1})\big) \cdot 2^{\sharp\{p|N; p \equiv \pm 1 (12)\}}.$$

We have $C_0(\gamma_1; Sp(2; \mathbb{R})) = C_0(\gamma_2; Sp(2; \mathbb{R})) = \{1_4\}$ and

$$\begin{aligned} J_0'(\gamma_1) + J_0'(\gamma_1^{-1}) &= J_0(\gamma_1) + J_0(\gamma_1^{-1}) = [1, -1, 0; 3]_j, \\ J_0'(\gamma_2) + J_0'(\gamma_2^{-1}) &= J_0(\gamma_2) + J_0(\gamma_2^{-1}) = [0, -1, 1; 3]_{j+2k}. \end{aligned}$$

Thus we obtain $H_{12}(k, j, N)$. By careful calculation, we can rewrite $H_{12}(k, j, N)$ as in Theorem 3.1.

8. Proof of Main Theorem 1.1: non-semi-simple elements.

8.1. Unipotent elements.

In this subsection, we evaluate the contributions $I_1(k, j, N)$, $I_2(k, j, N)$, and $I_3(k, j, N)$. Let γ be a unipotent element of $K(N)$. We write $SM(2; \mathbb{R}) = \{X \in M(2; \mathbb{R});\ {}^tX = X\}$. Then by virtue of [**Mor74**], γ has a contribution to the dimension formula only when γ is $Sp(2; \mathbb{Q})$-conjugate to an element of the form:

$$\begin{bmatrix} 1_2 & S \\ 0_2 & 1_2 \end{bmatrix}, \quad S \in SM(2; \mathbb{R}).$$

Since $K(N)$ has a single equivalence class of 0-dimensional cusps (Proposition 6.5), γ is $K(N)$-conjugate to an element of the above form. We divide it into the following four cases:
(i) S: definite.
(ii) S: indefinite, $\det S \neq 0$, and $-\det S \in (\mathbb{Q}^\times)^2$.
(iii) S: indefinite, $\det S \neq 0$, and $-\det S \notin (\mathbb{Q}^\times)^2$.
(iv) $\det S = 0$.

As for (iii), it is known that such γ has no contribution to the dimension formula ([**Wak12**, Section 4.13]). We denote by $I_1(k,j,N)$, $I_2(k,j,N)$, $I_3(k,j,N)$ the contribution of all elements where $\pm\gamma$ correspond to (i), (ii) and (iv) respectively.

Case 1. We assume that S is definite. We can evaluate the contribution $I_1(k,j,N)$ as follows. (cf. [**Wak12**, (e-2) on page 207]) There is a single $K(N)$-conjugacy class of family, which is given by

$$F = \left\{ \begin{bmatrix} 1_2 & T \\ 0_2 & 1_2 \end{bmatrix} \in K(N) \;\middle|\; T : \text{definite} \right\}.$$

We define

$$C_0(F; Sp(2;\mathbb{R})) = \left\{ \begin{bmatrix} 1_2 & X \\ 0_2 & 1_2 \end{bmatrix} \;\middle|\; X \in SM(2;\mathbb{R}) \right\},$$

and $C_0(F;K(N)) = C_0(F;Sp(2;\mathbb{R})) \cap K(N)$.

We see

$$\begin{aligned}
&\left\{ \begin{bmatrix} A & 0_2 \\ 0_2 & {}^tA^{-1} \end{bmatrix} \;\middle|\; A \in GL(2;\mathbb{Q}) \right\} \cap K(N) \\
&= \left\{ \begin{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} & 0_2 \\ 0_2 & \dfrac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{bmatrix} \;\middle|\; a,c,d \in \mathbb{Z}, b \in N\mathbb{Z}, ad-bc = \pm 1 \right\} \\
&\simeq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2;\mathbb{Z}) \;\middle|\; c \in N\mathbb{Z} \right\} =: G\Gamma_0(N).
\end{aligned}$$

We define a lattice L in $SM(2;\mathbb{R})$ by

$$L = \left\{ \begin{bmatrix} a & b \\ b & c/N \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{Z} \right\}.$$

Then, by Theorem 6.4 and the formula in [**Wak12**, (e-2) on page 207], we have

$$I_1(k,j,N)=\mathrm{vol}(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R})))\cdot\frac{j+1}{2^2\pi}\cdot\frac{1}{[G\Gamma_0(N):\Gamma_0(N)]}\cdot\frac{\mathrm{vol}(\Gamma_0(N)\backslash\mathfrak{H}_1)}{\mathrm{vol}(L)},\tag{15}$$

where $\mathrm{vol}(L)=\int_{L\backslash SM(2;\mathbb{R})}dX$. It is easy to see that

$$\mathrm{vol}(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R})))=\mathrm{vol}(L)=\frac{1}{N}.$$

Since we have $[G\Gamma_0(N):\Gamma_0(N)]=2$ and $\mathrm{vol}(\Gamma_0(N)\backslash\mathfrak{H}_1)=[SL(2;\mathbb{Z}):\Gamma_0(N)]\cdot\mathrm{vol}(SL(2;\mathbb{Z})\backslash\mathfrak{H}_1)=\prod_{p|N}(p+1)\cdot(\pi/3)$, we obtain by (15)

$$I_1(k,j,N)=2^{-3}3^{-1}(j+1)\prod_{p|N}(p+1).$$

Case 2. We assume that S is indefinite, $\det S\neq 0$, and $-\det S\in(\mathbb{Q}^\times)^2$. We can evaluate the contribution $I_2(k,j,N)$ as follows (cf. [**Wak12**, (e-3) on page 207]). There is a single $K(N)$-conjugacy class of family, which is given by

$$F=\left\{\begin{bmatrix}1_2 & T\\ 0_2 & 1_2\end{bmatrix}\in K(N)\;\middle|\;T:\text{indefinite and }-\det S\in(\mathbb{Q}^\times)^2\right\}.$$

We define

$$C_0(F;Sp(2;\mathbb{R}))=\left\{\begin{bmatrix}1_2 & X\\ 0_2 & 1_2\end{bmatrix}\;\middle|\;X\in SM(2;\mathbb{R})\right\}.$$

and $C_0(F;K(N))=C_0(F;Sp(2;\mathbb{R}))\cap K(N)$. Let $\beta_1,\dots,\beta_t$ be the set of non-equivalent cusps of $\Gamma_0(N)$ in $\overline{\mathfrak{H}_1}$, and V_j be the element of $SL(2;\mathbb{Q})$ which satisfies $V_j\langle\beta_j\rangle=\infty$. Then $\left\{\begin{bmatrix}a & b\\ c & 0\end{bmatrix}\;\middle|\;a,b,c\in\mathbb{R}\right\}\cap V_jL\,{}^tV_j$ has a unique basis of the form:

$$\begin{bmatrix}t_1^{(j)} & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}t_2^{(j)} & d_j\\ d_j & 0\end{bmatrix};\;d_j>0,\;t_1^{(j)}>|t_2^{(j)}|\geq 0.$$

Let B_j be the parabolic subgroup of $\Gamma_0(N)$ which stabilizes β_j, and we define c_j by

$$B_j={V_j}^{-1}\left\{\pm\begin{bmatrix}1 & nt_1^{(j)}c_j/2d_j\\ 0 & 1\end{bmatrix}\;\middle|\;n\in\mathbb{Z}\right\}V_j.$$

Then, by Theorem 6.4 and the formula in [**Wak12**, (e-3) on page 207], we have

$$I_2(k,j,N)=-\mathrm{vol}(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R})))\cdot\frac{(j+1)}{2^4\cdot 3}\cdot\frac{1}{[G\Gamma_0(N):\Gamma_0(N)]}\cdot\sum_{j=1}^{t}\frac{c_j}{{d_j}^3}.\tag{16}$$

In our case, a complete system of representatives of non-equivalent cusp of $\Gamma_0(N)$ is given by $\{1/n \mid n \text{ is a positive divisor of } N\}$. We put $\beta_n = 1/n$, then we can take $V_n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$. We can easily obtain

$$\left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\} \cap V_j L\, {}^tV_j = \mathbb{Z} \begin{bmatrix} 1/n & 0 \\ 0 & 0 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$V_n B_n\, {V_n}^{-1} = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \;\middle|\; b \in \frac{N}{n}\mathbb{Z} \right\},$$

hence we get $c_n = 2N$ and $d_n = 1$. Hence we obtain by (16)

$$I_2(k,j,N) = -2^{\omega(N)-4} \cdot 3^{-1} \cdot (j+1),$$

where $\omega(N)$ is the number of prime divisors of N.

Case 3. We assume that $\det S = 0$. We can evaluate the contribution $I_3(k,j,N)$ as follows (cf. [**Wak12**, (e-4) on page 208]). In this case, the elements in question are $Sp(2;\mathbb{Q})$ conjugate to $\left(\begin{smallmatrix} 1_2 & S \\ 0 & 1_2 \end{smallmatrix}\right)$ with $S = \left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right)$. Using Proposition 6.5 for $P_{2,1}(\mathbb{Q})$, we see that each unipotent element of $K(N)$ of this type is $K(N)$-conjugate to an element of the form:

$$\begin{bmatrix} 1 & 0 & nm & n \\ 0 & 1 & n & n/m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

for a positive divisor m of N and non-zero integer n. We can take a complete system of representatives of $K(N)$-conjugacy classes of families as

$$\begin{aligned} F_m &= \left\{ \begin{bmatrix} 1 & 0 & nm & n \\ 0 & 1 & n & n/m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; n \in \mathbb{Z} - \{0\} \right\} \\ &= g_m \cdot \left\{ \begin{bmatrix} 1 & 0 & n & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; n \in \mathbb{Z} - \{0\} \right\} \cdot {g_m}^{-1}, \end{aligned}$$

where

$$g_m = \begin{bmatrix} \sqrt{m} & 0 & 0 & 0 \\ 1/\sqrt{m} & 1/\sqrt{m} & 0 & 0 \\ 0 & 0 & 1/\sqrt{m} & -1/\sqrt{m} \\ 0 & 0 & 0 & \sqrt{m} \end{bmatrix}.$$

So there are $2^{\omega(N)}$ families since $2^{\omega(N)}$ is the number of divisors of N. For each family F_m, we define $C_0(F_m; Sp(2;\mathbb{R}))$ as in [**Wak12**, (e-4) on page 208]. Then we have

$$\mathrm{vol}(C_0(F_m; K(N))\backslash C_0(F_m; Sp(2;\mathbb{R}))) = \frac{\pi^2}{3} N.$$

Hence, by Theorem 6.4 and the formula in [**Wak12**, (e-4) on page 209], we obtain

$$\begin{aligned} I_3(k,j,N) &= -2^{\omega(N)} \cdot \mathrm{vol}\big(C_0(F_m; K(N))\backslash C_0(F_m; Sp(2;\mathbb{R}))\big) \cdot \frac{(j+1)(j+2k-3)}{2^5 \cdot 3 \cdot \pi^2} \\ &= -2^{\omega(N)-5} 3^{-2} N(j+1)(j+2k-3). \end{aligned}$$

8.2. δ-parabolic elements.

In this subsection, we evaluate $I_4(k,j,N)$ and $I_5(k,j,N)$. Let γ be a δ-parabolic element of $K(N)$, that is, γ is $Sp(2;\mathbb{R})$-conjugate to an element of the form:

$$\delta(s_1,s_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (s_1,s_2) = (\pm 1, \pm 1) \text{ or } (\pm 1, 0). \tag{17}$$

We denote by $I_4(k,j,N)$ the contribution for γ with $(s_1,s_2) = (\pm 1, \pm 1)$ and by $I_5(k,j,N)$ for those with $(s_1,s_2) = (\pm 1, 0)$.

We can see from Corollary 6.6 that such elements are $K(N)$-conjugate to elements of the form:

$$\begin{bmatrix} A & B \\ 0_2 & {}^tA^{-1} \end{bmatrix} \quad (A \cdot {}^tB = B \cdot {}^tA).$$

Since $A \in M(2;\mathbb{Z})$ is of order two and $\mathrm{Tr}A = 0$, it is well-known that A is $GL(2;\mathbb{Z})$-conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Thus γ is $\iota(GL(2;\mathbb{Z}))$-conjugate to

$$X_1(S) = \iota\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \begin{bmatrix} 1_2 & S \\ 0_2 & 1_2 \end{bmatrix}, \quad \text{or} \quad X_2(S) = \iota\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\right) \begin{bmatrix} 1_2 & S \\ 0_2 & 1_2 \end{bmatrix}$$

where ι is a map defined by $\iota(A) = \begin{bmatrix} A & 0_2 \\ 0_2 & {}^tA^{-1} \end{bmatrix}$ and S is a symmetric matrix.

We define $\Gamma' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2;\mathbb{Z}) \,\middle|\, b \in N\mathbb{Z} \right\}$, then $\iota(GL(2;\mathbb{Z})) \cap K(N) = \iota(\Gamma')$. We try to determine how $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}_{GL(2;\mathbb{Z})} \cap \Gamma'$ and $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}_{GL(2;\mathbb{Z})} \cap \Gamma'$ split into Γ'-conjugacy classes. By Proposition 6.7, we can achieve this purpose by determining complete systems of representatives of the double coset decompositions:

$$C\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; GL(2;\mathbb{Z})\right) \Big\backslash M\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; GL(2;\mathbb{Z})/\Gamma'\right) \Big/ \Gamma', \tag{18}$$

$$C\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}; GL(2;\mathbb{Z})\right) \Big\backslash M\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}; GL(2;\mathbb{Z})/\Gamma'\right) \Big/ \Gamma'. \tag{19}$$

We consider the cases $(s_1, s_2) = (\pm 1, \pm 1)$ and $(\pm 1, 0)$ in (17) separately.

Case 1. We consider the case of $(s_1, s_2) = (\pm 1, \pm 1)$ and $2 \nmid N$. For each positive divisor d of N, we fix elements x_d and y_d of $GL(2;\mathbb{Z})$ which satisfies

$$x_d \equiv \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p \text{ if } p \mid d, \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bmod p \text{ if } p \nmid d, \end{cases} \qquad y_d \equiv \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p \text{ if } p \mid d, \\ \begin{bmatrix} 1 & 0 \\ p-2 & 1 \end{bmatrix} \bmod p \text{ if } p \nmid d. \end{cases}$$

Then, by straightforward calculation, we see that complete systems of representatives of (18) and (19) are given by x_d $(d|N,\ d>0)$ and y_d $(d|N,\ d>0)$ respectively. Hence a complete system of representatives of $\iota(\Gamma')$-conjugacy classes is given by $\iota(x_d)^{-1}X_1(S)\iota(x_d)$ and $\iota(y_d)^{-1}X_2(S)\iota(y_d)$ $(d|N,\ d>0)$. Since $\iota(\Gamma') \subset K(N)$, we need to examine which $\iota(\Gamma')$-conjugacy classes are the same as $K(N)$-conjugacy classes. By careful calculation, we see that a complete system of representatives of $K(N)$-conjugacy classes of elements in question is given by

$$\iota(x_d)^{-1}X_1\left(\begin{bmatrix} m/d & 0 \\ 0 & n \end{bmatrix}\right)\iota(x_d), \qquad \iota(x_d)^{-1}X_1\left(\begin{bmatrix} m/d & 1 \\ 1 & n \end{bmatrix}\right)\iota(x_d),$$
$$\iota(y_d)^{-1}X_2\left(\begin{bmatrix} m+n/d & -m \\ -m & 2m \end{bmatrix}\right)\iota(y_d), \quad \iota(y_d)^{-1}X_2\left(\begin{bmatrix} m+n/d & -m \\ -m & 2m-1 \end{bmatrix}\right)\iota(y_d), \quad (d \mid N, d>0).$$

Finally, we determine $K(N)$-conjugacy classes of families and evaluate each contribution. For each positive divisor d of N, we define

$$g_{1,d} = \begin{bmatrix} \sqrt{d} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad g_{2,d} = \begin{bmatrix} \sqrt{d} & 0 & 0 & \sqrt{d}/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1/\sqrt{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$g_{3,d}=\begin{bmatrix}\sqrt{2d} & \sqrt{d/2} & 0 & 0\\ 0 & -1/\sqrt{2} & 0 & 0\\ 0 & 0 & 1/\sqrt{2d} & 0\\ 0 & 0 & 1/\sqrt{2} & -\sqrt{2}\end{bmatrix}, \qquad g_{4,d}=\begin{bmatrix}\sqrt{2d} & \sqrt{d/2} & \sqrt{2d}/8 & -\sqrt{2d}/4\\ 0 & -1/\sqrt{2} & \sqrt{2}/8 & 0\\ 0 & 0 & 1/\sqrt{2d} & 0\\ 0 & 0 & 1/\sqrt{2} & -\sqrt{2}\end{bmatrix}.$$

Then we obtain the following result by careful calculation.

Proposition 8.1. *A complete system of representatives of $K(N)$-conjugacy classes of families is given as*

$$F_{1,d}=\iota(x_d)^{-1}g_{1,d}^{-1}\cdot\left\{\delta(m,n)\,\middle|\,\begin{matrix}m,n\in\mathbb{Z}\\ m,n\neq 0\end{matrix}\right\}\cdot g_{1,d}\ \iota(x_d),$$

$$F_{2,d}=\iota(x_d)^{-1}g_{2,d}^{-1}\cdot\left\{\delta(m,n)\,\middle|\,\begin{matrix}m,n\in\mathbb{Z}\\ m,n\neq 0\end{matrix}\right\}\cdot g_{2,d}\ \iota(x_d),$$

$$F_{3,d}=\iota(y_d)^{-1}g_{3,d}^{-1}\cdot\left\{\delta(Nm+2n,m)\,\middle|\,\begin{matrix}m,n\in\mathbb{Z}\\ m\neq 0,\ Nm+2n\neq 0\end{matrix}\right\}\cdot g_{3,d}\ \iota(y_d),$$

$$F_{4,d}=\iota(y_d)^{-1}g_{4,d}^{-1}\cdot\left\{\delta(Nm+2n-N/2,m-1/2)\,\middle|\,\begin{matrix}m,n\in\mathbb{Z}\\ Nm+2n-N/2\neq 0\end{matrix}\right\}\cdot g_{4,d}\ \iota(y_d)$$

for positive divisors d of N, where ι is the map defined by $\iota(A)=\begin{bmatrix}A & 0_2\\ 0_2 & {}^tA^{-1}\end{bmatrix}$.

For each family F, we define $C_0(F;Sp(2;\mathbb{R}))$ as in [**Wak12**, (f-2) on page 210].

Proposition 8.2. *Necessary data for each family are as follows.*

(i) *If $F=F_{1,d}$ or $F_{2,d}$,*

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big)=1,\quad [C(F;K(N)):C_0(F;K(N))]=2$$

and $\lim_{s\to+0}\sum_{\gamma'\in F}J_0(\gamma';s)=c_{k,j}^{-1}\cdot 2^{-3}\cdot(-1)^k$, *hence the contribution of F is* $2^{-5}\cdot(-1)^k$.

(ii) *If $F=F_{3,d}$,*

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big)=2,\quad [C(F;K(N)):C_0(F;K(N))]=2$$

and $\lim_{s\to+0}\sum_{\gamma'\in F}J_0(\gamma';s)=c_{k,j}^{-1}\cdot 2^{-4}\cdot(-1)^k$, *hence the contribution of F is* $2^{-5}\cdot(-1)^k$.

(iii) *If $F=F_{4,d}$,*

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big)=2,\quad [C(F;K(N)):C_0(F;K(N))]=2$$

and $\lim_{s\to+0}\sum_{\gamma'\in F} J_0(\gamma';s) = c_{k,j}^{-1}\cdot(1-(-1/N))\cdot 2^{-4}\cdot(-1)^k$, *hence the contribution of* F *is* $(1-(-1/N))\cdot 2^{-5}\cdot(-1)^k$.

Proof. Direct calculations and [**Wak12**, (f-2) on page 210]. In (iii), $(-1/N)$ appears since

$$\begin{aligned}\lim_{s\to+0}\sum_{\gamma'\in F} J_0(\gamma';s) &= c_{k,j}^{-1}\cdot 2^{-5}\cdot(-1)^k\cdot\sum_{t=0}^{1}\left(1+i\cdot\cot\frac{(t+1/2)\pi}{2}\right)\left(1-i\cdot\cot\frac{-N(t+1/2)\pi}{2}\right)\\ &= c_{k,j}^{-1}\cdot 2^{-5}\cdot(-1)^k\cdot\begin{cases}0 \text{ if } N\equiv 1 \bmod 4,\\ 4 \text{ if } N\equiv 3 \bmod 4.\end{cases}\end{aligned}$$ $\square$

It follows that the total contribution of all families is

$$I_4(k,j,N) = \left(4-\left(\frac{-1}{N}\right)\right)\cdot 2^{\omega(N)-5}\cdot(-1)^k.$$

Case 2. We consider the case of $(s_1,s_2)=(\pm1,\pm1)$ and $2\mid N$. For each positive divisor d of $N/2$, we fix elements $x_d^{(1)}$, $x_d^{(2)}$, $x_d^{(3)}$ and y_d of $GL(2;\mathbb{Z})$ which satisfies

$$x_d^{(1)} \equiv \begin{cases}\begin{bmatrix}1&0\\0&1\end{bmatrix} \bmod p \text{ if } p\mid d \text{ or } p=2,\\ \begin{bmatrix}0&1\\-1&0\end{bmatrix} \bmod p \text{ if } p\nmid d,\end{cases} \qquad x_d^{(2)} \equiv \begin{cases}\begin{bmatrix}1&0\\0&1\end{bmatrix} \bmod p \text{ if } p\mid d,\\ \begin{bmatrix}0&1\\-1&0\end{bmatrix} \bmod p \text{ if } p\nmid d \text{ or } p=2,\end{cases}$$

$$x_d^{(3)} \equiv \begin{cases}\begin{bmatrix}1&0\\0&1\end{bmatrix} \bmod p \text{ if } p\mid d,\\ \begin{bmatrix}0&1\\-1&0\end{bmatrix} \bmod p \text{ if } p\nmid d,\\ \begin{bmatrix}1&0\\1&1\end{bmatrix} \bmod p \text{ if } p=2,\end{cases} \qquad y_d \equiv \begin{cases}\begin{bmatrix}1&0\\0&1\end{bmatrix} \bmod p \text{ if } p\mid d \text{ or } p=2,\\ \begin{bmatrix}1&0\\p-2&1\end{bmatrix} \bmod p \text{ if } p\nmid d.\end{cases}$$

By similar argument as in Case 1, we obtain the following result.

Proposition 8.3. *A complete system of representatives of* $K(N)$*-conjugacy classes of families is given as*

$$F_{1,d}^{(i)} = \iota(x_d^{(i)})^{-1} g_{1,d}^{-1}\cdot\left\{\delta(m,n)\,\middle|\,\begin{matrix}m,n\in\mathbb{Z}\\ m,n\neq 0\end{matrix}\right\}\cdot g_{1,d}\ \iota(x_d^{(i)}),$$

$$F_{2,d}^{(i)} = \iota(x_d^{(i)})^{-1} g_{2,d}^{-1}\cdot\left\{\delta(m,n)\,\middle|\,\begin{matrix}m,n\in\mathbb{Z}\\ m,n\neq 0\end{matrix}\right\}\cdot g_{2,d}\ \iota(x_d^{(i)}),$$

$$F^{(i)}_{3,d}=\iota(x^{(i)}_d)^{-1}g_{3,d}^{-1}\cdot\left\{\delta(Nm+2n,m)\,\middle|\,\begin{array}{l}m,n\in\mathbb{Z}\\ m\neq 0,\ Nm+2n\neq 0\end{array}\right\}\cdot g_{3,d}\ \iota(x^{(i)}_d),$$

$$F_{4,d}=\iota(y_d)^{-1}g_{4,d}^{-1}\cdot\left\{\delta(Nm+2n-N/2,m-1/2)\,\middle|\,\begin{array}{l}m,n\in\mathbb{Z}\\ Nm+2n-N/2\neq 0\end{array}\right\}\cdot g_{4,d}\ \iota(y_d)$$

for positive divisors d of N and $i=1,2,3$, where $g_{1,d}$, $g_{2,d}$, $g_{3,d}$, $g_{4,d}$ are the same as in Case 1.

For each family F, we define $C_0(F;Sp(2;\mathbb{R}))$ as in [**Wak12**, (f-2) on page 210].

PROPOSITION 8.4. *Necessary data for each family are as follows*:

(i) *If $F=F^{(i)}_{1,d}$, $F^{(i)}_{2,d}$, or $F^{(i)}_{3,d}$, then*

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big)=1,\quad [C(F;K(N)):C_0(F;K(N))]=2$$

and $\lim_{s\to+0}\sum_{\gamma'\in F}J_0(\gamma';s)=c_{k,j}^{-1}\cdot 2^{-3}\cdot(-1)^k$, *hence the contribution of F is* $2^{-5}\cdot(-1)^k$.

(ii) *If $F=F_{4,d}$, then*

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big)=2,\quad [C(F;K(N)):C_0(F;K(N))]=2$$

and $\lim_{s\to+0}\sum_{\gamma'\in F}J_0(\gamma';s)=c_{k,j}^{-1}\cdot 2^{-4}\cdot(-1)^k$, *hence the contribution of F is* $2^{-5}\cdot(-1)^k$.

PROOF. Direct calculations and [**Wak12**, (f-2) on page 210]. □

It follows that the total contribution of all families is

$$I_4(k,j,N)=4\cdot 2^{\omega(N)-5}\cdot(-1)^k.$$

Case 3. We consider the case $(s_1,s_2)=(\pm1,0)$. For each positive integer d of N, we define

$$g_{1,d}=\begin{bmatrix}\sqrt{d}&0&0&0\\0&1&0&0\\0&0&1/\sqrt{d}&0\\0&0&0&1\end{bmatrix},\qquad g_{2,d}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix},$$

$$g_{3,d}=\begin{bmatrix}\sqrt{d}&0&0&-\sqrt{d}/2\\0&1&-1/2&0\\0&0&1/\sqrt{d}&0\\0&0&0&1\end{bmatrix},\qquad g_{4,d}=\begin{bmatrix}0&1&-1/2&0\\-1&0&0&1/2\\0&0&0&1\\0&0&-1&0\end{bmatrix}.$$

Then we obtain the following result.

Proposition 8.5. *A complete system of representatives of $K(N)$-conjugacy classes of families is given as*

$$F_{1,d} = g_{1,d}^{-1} \cdot \left\{ \delta(m,0) \,\middle|\, \begin{matrix} m \in \mathbb{Z} \\ m \neq 0 \end{matrix} \right\} \cdot g_{1,d}, \qquad F_{2,d} = g_{2,d}^{-1} \cdot \left\{ \delta(m,0) \,\middle|\, \begin{matrix} m \in \mathbb{Z} \\ m \neq 0 \end{matrix} \right\} \cdot g_{2,d},$$

$$F_{3,d} = g_{3,d}^{-1} \cdot \left\{ \delta(m,0) \,\middle|\, \begin{matrix} m \in \mathbb{Z} \\ m \neq 0 \end{matrix} \right\} \cdot g_{3,d}, \qquad F_{4,d} = g_{4,d}^{-1} \cdot \left\{ \delta(m,0) \,\middle|\, \begin{matrix} m \in \mathbb{Z} \\ m \neq 0 \end{matrix} \right\} \cdot g_{4,d},$$

$$F_{5,d} = -F_{1,d}, \qquad F_{6,d} = -F_{2,d},$$

$$F_{7,d} = -F_{3,d}, \qquad F_{8,d} = -F_{4,d}$$

for positive divisors d of N.

For each family F, we define $C_0(F; Sp(2;\mathbb{R}))$ as in [**Wak12**, (f-2) on page 210].

Proposition 8.6. *Necessary data for each family are as follows*:

(i) *If $F = F_{1,d}$, $F_{2,d}$, $F_{5,d}$ or $F_{6,d}$, then*

$$\mathrm{vol}\Big(C_0(F; K(N)) \backslash C_0(F; Sp(2;\mathbb{R}))\Big) = \frac{\pi^2}{3}, \quad [C(F; K(N)) : C_0(F; K(N))] = 1,$$
$$\lim_{s \to +0} \sum_{\gamma' \in F} J_0(\gamma'; s) = -c_{k,j}^{-1} \cdot 2^{-6} \cdot \pi^{-2} \cdot (-1)^k \cdot (2k+j-3),$$

hence the contribution of F is $-2^{-7} \cdot 3^{-1} \cdot (-1)^k \cdot (2k+j-3)$.

(ii) *If $F = F_{3,d}$, $F_{4,d}$, $F_{7,d}$ or $F_{8,d}$, then*

$$\mathrm{vol}\Big(C_0(F; K(N)) \backslash C_0(F; Sp(2;\mathbb{R}))\Big) = \pi^2, \quad [C(F; K(N)) : C_0(F; K(N))] = 1,$$
$$\lim_{s \to +0} \sum_{\gamma' \in F} J_0(\gamma'; s) = -c_{k,j}^{-1} \cdot 2^{-6} \cdot \pi^{-2} \cdot (-1)^k \cdot (2k+j-3),$$

hence the contribution of F is $-2^{-7} \cdot (-1)^k \cdot (2k+j-3)$.

Proof. Direct calculations and [**Wak12**, (f-1) on page 209]. □

It follows that the total contribution of all families is

$$I_5(k,j,N) = -2^{\omega(N)-4} \cdot 3^{-1} \cdot (-1)^k \cdot (2k+j-3).$$

8.3. Elliptic/parabolic elements.

In this subsection, we evaluate $I_6(k,j,N)$, $I_7(k,j,N)$ and $I_8(k,j,N)$. Let γ be an elliptic/parabolic element of $K(N)$, that is, γ is $Sp(2;\mathbb{R})$-conjugate to an element of the form:

$$\hat{\beta}(\theta,\lambda)=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&\lambda\\0&0&1&0\\0&0&0&1\end{bmatrix},\qquad \theta=\frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{\pi}{3},\qquad \lambda\neq 0. \tag{20}$$

Let m be a positive divisor of N and $m'=N/m$. Since N is squarefree, m and m' are mutually prime. We fix $\alpha,\beta\in\mathbb{Z}$ such that $m\alpha+m'\beta=-1$. We define matrices of $Sp(2;\mathbb{R})$ as follows:

$$g_m=\begin{bmatrix}-\beta\sqrt{m'} & \sqrt{m} & 0 & 0\\ \alpha/\sqrt{m'} & 1/\sqrt{m} & 0 & 0\\ 0 & 0 & \sqrt{m'} & -\alpha\sqrt{m}\\ 0 & 0 & -m\sqrt{m'} & -\beta m'\sqrt{m}\end{bmatrix},\quad T_1=1_4,\quad T_2=\begin{bmatrix}1 & 0 & 0 & 1/2\\ -1/2 & -1 & -1/2 & 0\\ 0 & 0 & 1 & -1/2\\ 0 & 0 & 0 & -1\end{bmatrix},$$

$$T_3=\begin{bmatrix}2c & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ c & 0 & \sqrt{3}c & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad T_4=\begin{bmatrix}2c & 0 & 0 & 1/3\\ c & 1 & c/\sqrt{3} & 0\\ c & 0 & \sqrt{3}c & -1/3\\ 0 & 0 & 0 & 1\end{bmatrix},\quad (c^2=\sqrt{3}/6).$$

Case 1: We consider elements of $K(N)$ whose characteristic polynomial is $(x-1)^2(x^2+1)$ and double the contribution. We denote by $I_6(k,j,N)$ the total of such contribution.

PROPOSITION 8.7. *A complete system of representatives of $K(N)$-conjugacy classes of families are given as follows:*

$$\begin{aligned}
F_{1,m}&=g_mT_1\left\{\hat{\beta}(\pi/2,n)\mid n\in\mathbb{Z},n\neq 0\right\}T_1^{-1}g_m^{-1},\\
F_{2,m}&=g_mT_1\left\{\hat{\beta}(-\pi/2,n)\mid n\in\mathbb{Z},n\neq 0\right\}T_1^{-1}g_m^{-1},\\
F_{3,m}&=g_mT_2\left\{\hat{\beta}(\pi/2,n-1/2)\mid n\in\mathbb{Z}\right\}T_2^{-1}g_m^{-1},\\
F_{4,m}&=g_mT_2\left\{\hat{\beta}(-\pi/2,n+1/2)\mid n\in\mathbb{Z}\right\}T_2^{-1}g_m^{-1},
\end{aligned}$$

for positive divisors m of N.

For each family F, we define $C_0(F;Sp(2;\mathbb{R}))$ as in [**Wak12**, (f-4) on page 212].

PROPOSITION 8.8. *Necessary data for each family are as follows.*

(i) *If $F=F_{1,m}$ or $F_{3,m}$,*

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big)=1,\quad [C(F;K(N)):C_0(F;K(N))]=4$$

and $\lim_{s\to+0}\sum_{\gamma'\in F}J_0(\gamma';s)=-c_{k,j}^{-1}\cdot 2^{-3}\cdot[i+(-1)^{j/2},-1+i(-1)^{j/2},-i-$

$(-1)^{j/2}, 1-i(-1)^{j/2}; 4]_k$, *hence the contribution of* F *is* $2^{-6} \cdot [i+(-1)^{j/2}, -1+i(-1)^{j/2}, -i-(-1)^{j/2}, 1-i(-1)^{j/2}; 4]_k$.

(ii) *If* $F = F_{2,m}$ *or* $F_{4,m}$,

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big) = 1, \quad [C(F;K(N)) : C_0(F;K(N))] = 4$$

and $\lim_{s\to+0}\sum_{\gamma'\in F} J_0(\gamma';s) = -c_{k,j}^{-1} \cdot 2^{-3} \cdot [-i+(-1)^{j/2}, -1-i(-1)^{j/2}, i-(-1)^{j/2}, 1+i(-1)^{j/2}; 4]_k$, *hence the contribution of* F *is* $2^{-6} \cdot [-i+(-1)^{j/2}, -1-i(-1)^{j/2}, i-(-1)^{j/2}, 1+i(-1)^{j/2}; 4]_k$.

PROOF. Direct calculations and [**Wak12**, (f-4) on page 212]. □

It follows that the total contribution of all families is

$$I_6(k,j,N) = -2^{\omega(N)-3} \cdot [(-1)^{j/2}, -1, -(-1)^{j/2}, 1; 4]_k.$$

Case 2: We consider elements of $K(N)$ whose characteristic polynomial is $(x-1)^2(x^2+x+1)$ and double the contribution. We denote by $I_7(k,j,N)$ the total of such contribution.

PROPOSITION 8.9. *A complete system of representatives of* $K(N)$*-conjugacy classes of families are given as follows*:

$$\begin{aligned}
F_{1,m} &= g_m T_3 \left\{\hat{\beta}(2\pi/3, n) \mid n \in \mathbb{Z}, n \neq 0\right\} T_3^{-1} g_m^{-1}, \\
F_{2,m} &= g_m T_3 \left\{\hat{\beta}(-2\pi/3, n) \mid n \in \mathbb{Z}, n \neq 0\right\} T_3^{-1} g_m^{-1}, \\
F_{3,m} &= g_m T_4 \left\{\hat{\beta}(2\pi/3, n+2N/3) \mid n \in \mathbb{Z}\right\} T_4^{-1} g_m^{-1}, \\
F_{4,m} &= g_m T_4 \left\{\hat{\beta}(-2\pi/3, n+N/3) \mid n \in \mathbb{Z}\right\} T_4^{-1} g_m^{-1},
\end{aligned}$$

for positive divisors m *of* N.

For each family F, we define $C_0(F;Sp(2;\mathbb{R}))$ as in [**Wak12**, (f-4) on page 212].

PROPOSITION 8.10. *Necessary data for each family are as follows.*

(i) *If* $F = F_{1,m}$ *or* $F_{2,m}$,

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big) = 1, \quad [C(F;K(N)) : C_0(F;K(N))] = 6,$$

and $\sum_{i=1}^{2}\lim_{s\to+0}\sum_{\gamma'\in F_{i,m}} J_0(\gamma';s) = -c_{k,j}^{-1} \cdot 2^{-1} \cdot 3^{-1} \cdot \big([1,-1,0;3]_k + [1,0,-1;3]_{j+k}\big)$, *hence the sum of the contributions of* $F_{1,m}$ *and* $F_{2,m}$ *is* $-2^{-3} \cdot 3^{-2} \cdot \big([1,-1,0;3]_k + [1,0,-1;3]_{j+k}\big)$.

(ii) *If* $F = F_{3,m}$ *or* $F_{4,m}$,

$$\mathrm{vol}\Big(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R}))\Big)=1,\quad [C(F;K(N)):C_0(F;K(N))]=3$$

and $\sum_{i=3}^{4}\lim_{s\to+0}\sum_{\gamma'\in F_{i,m}}J_0(\gamma';s)$

$$=-c_{k,j}^{-1}\cdot\begin{cases}2^{-1}\cdot 3^{-1}\cdot\big([1,-1,0;3]_k+[1,0,-1;3]_{j+k}\big) & \text{if } N\equiv 0 \bmod 3,\\ 3^{-2}\cdot\big([1,-2,1;3]_k+[2,-1,-1;3]_{j+k}\big) & \text{if } N\equiv 1 \bmod 3,\\ 3^{-2}\cdot\big([2,-1,-1;3]_k+[1,1,-2;3]_{j+k}\big) & \text{if } N\equiv 2 \bmod 3,\end{cases}$$

hence the sum of the contributions of $F_{3,m}$ and $F_{4,m}$ is

$$\begin{cases}-2^{-2}\cdot 3^{-2}\cdot\big([1,-1,0;3]_k+[1,0,-1;3]_{j+k}\big) & \text{if } N\equiv 0 \bmod 3,\\ -2^{-1}\cdot 3^{-3}\cdot\big([1,-2,1;3]_k+[2,-1,-1;3]_{j+k}\big) & \text{if } N\equiv 1 \bmod 3,\\ -2^{-1}\cdot 3^{-3}\cdot\big([2,-1,-1;3]_k+[1,1,-2;3]_{j+k}\big) & \text{if } N\equiv 2 \bmod 3.\end{cases}$$

PROOF. Direct calculations and [**Wak12**, (f-4) on page 212]. □

It follows that the total contribution of all families is

$$I_7(k,j,N)=-2^{\omega(N)-2}\cdot 3^{-2}\cdot\begin{cases}[3,-3,0;3]_k+[3,0,-3;3]_{j+k} & \cdots \text{ if } N\equiv 0 \bmod 3,\\ [5,-1,4;3]_k+[1,-4,-5;3]_{j+k} & \cdots \text{ if } N\equiv 1 \bmod 3,\\ [1,-5,-4;3]_k+[5,4,-1;3]_{j+k} & \cdots \text{ if } N\equiv 2 \bmod 3.\end{cases}$$

Case 3: We consider elements of $K(N)$ whose characteristic polynomial is $(x-1)^2(x^2-x+1)$ and double the contribution. We denote by $I_8(k,j,N)$ the total of such contribution.

PROPOSITION 8.11. *A complete system of representatives of $K(N)$-conjugacy classes of families are given as follows:*

$$F_{1,m}=g_mT_3\left\{\hat\beta(\pi/3,n)\mid n\in\mathbb{Z},n\neq 0\right\}T_3^{-1}g_m^{-1},$$
$$F_{2,m}=g_mT_3\left\{\hat\beta(-\pi/3,n)\mid n\in\mathbb{Z},n\neq 0\right\}T_3^{-1}g_m^{-1},$$

for positive divisors m of N.

For each family F, we define $C_0(F;Sp(2;\mathbb{R}))$ as in [**Wak12**, page 212].

PROPOSITION 8.12. *Necessary data for each family are as follows:*

$$\mathrm{vol}\Big(C_0(F_{1,m};K(N))\backslash C_0(F_{1,m};Sp(2;\mathbb{R}))\Big)=\mathrm{vol}\Big(C_0(F_{2,m};K(N))\backslash C_0(F_{2,m};Sp(2;\mathbb{R}))\Big)=1,$$
$$[C(F_{1,m};K(N)):C_0(F_{1,m};K(N))]=[C(F_{2,m};K(N)):C_0(F_{2,m};K(N))]=6,$$

and $\sum_{i=1}^{2}\lim_{s\to+0}\sum_{\gamma'\in F_{i,m}}J_0(\gamma';s)=-2^{-1}\cdot\big([-1,-1,0,1,1,0;6]_k+[1,0,-1,-1,0,1;6]_{j+k}\big)$, *hence the sum of the contributions of $F_{1,m}$ and $F_{2,m}$ is* $-2^{-3}\cdot 3^{-1}\cdot\big([-1,-1,0,1,1,0;6]_k+$

$[1,0,-1,-1,0,1;6]_{j+k})$.

Proof. Direct calculations and [**Wak12**, (f-4) on page 212]. □

It follows that the total contribution of all families is

$$I_8(k,j,N) = -2^{\omega(N)-2} \cdot 3^{-1} \cdot \big([-1,-1,0,1,1,0;6]_k + [1,0,-1,-1,0,1;6]_{j+k}\big).$$

8.4. Paraelliptic elements.

In this subsection, we prove $I_9(k,j,N)$ and $I_{10}(k,j,N)$ in Theorem 3.1. Let γ be a paraelliptic element of $K(N)$, that is, γ is $Sp(2;\mathbb{R})$-conjugate to an element of the form:

$$\gamma(\theta,s) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \theta = \frac{\pi}{2} \text{ or } \pm\frac{2\pi}{3}, s = \pm 1. \tag{21}$$

We consider the cases $\theta = \pi/2$ and $\theta = \pm 2\pi/3$ in (21) separately.

Case 1. We consider the case $\theta = \pi/2$. By similar calculation as in subsection 8.2 by using Corollary 6.6 and Proposition 6.7, we obtain the following result. We define

$$x_d = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \qquad \begin{pmatrix} d = 0,1,\dots,N-1 \\ d^2+1 \in N\mathbb{Z} \end{pmatrix}, \qquad g_1 = 1_4,$$

$$g_2 = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad g_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad g_4 = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proposition 8.13. *A complete system of representatives of $K(N)$-conjugacy classes of families is given by*

$$\begin{aligned} F_1 &= \iota(x_d)^{-1}{g_1}^{-1} \cdot \{\gamma(\pi/2,n) \mid n \in \mathbb{Z}, n \neq 0\} \cdot g_1\iota(x_d)^{-1}, \\ F_2 &= \iota(x_d)^{-1}{g_2}^{-1} \cdot \{\gamma(\pi/2,n) \mid n \in \mathbb{Z}, n \neq 0\} \cdot g_2\iota(x_d)^{-1}, \\ F_3 &= \iota(x_d)^{-1}{g_3}^{-1} \cdot \{\gamma(\pi/2,n+1/2) \mid n \in \mathbb{Z}\} \cdot g_3\iota(x_d)^{-1}, \\ F_4 &= \iota(x_d)^{-1}{g_4}^{-1} \cdot \{\gamma(\pi/2,n+1/2) \mid n \in \mathbb{Z}\} \cdot g_4\iota(x_d)^{-1}, \end{aligned}$$

where $d \in \{0,1,\dots,N-1\}$ such that $d^2+1 \in N\mathbb{Z}$.

For each family F, we define $C_0(F;Sp(2;\mathbb{R}))$ as in [**Wak12**, page 211].

Proposition 8.14. *For every family F, we have*

$$\frac{\mathrm{vol}(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R})))}{[\overline{C}(F;K(N)):\overline{C_0}(F;K(N))]}=\frac{1}{2},$$

and $\lim_{s\to+0}\sum_{\gamma'\in F}J_0(\gamma';s)=-c_{k,j}^{-1}\cdot 2^{-3}\cdot(-1)^{j/2}$, *hence the contribution of* F *is* $-2^{-5}\cdot(-1)^{j/2}$.

Proof. Direct calculations and [**Wak12**, (f-3) on page 211]. □

It follows that the total contribution of all families is

$$I_9(k,j,N)=-2^{-3}\cdot(-1)^{j/2}\cdot\prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right).$$

Case 2. We consider the case $\theta=\pm 2\pi/3$. We need only to double the contribution of the case $\theta=2\pi/3$ to obtain $I_{10}(k,j,N)$. We define

$$x_d=\begin{bmatrix}1&0\\d&1\end{bmatrix}\qquad\begin{pmatrix}d=0,1,\dots,N-1\\ d^2+d+1\in N\mathbb{Z}\end{pmatrix},\qquad g_1=\begin{bmatrix}1/\sqrt{2}&0&0&0\\1/\sqrt{6}&-\sqrt{2/3}&0&0\\0&0&\sqrt{2}&1/\sqrt{2}\\0&0&0&-\sqrt{3/2}\end{bmatrix},$$

$$g_2=\begin{bmatrix}1/\sqrt{2}&0&0&-\sqrt{2}/12\\1/\sqrt{6}&-\sqrt{2/3}&\sqrt{6}/18&7\sqrt{6}/36\\0&0&\sqrt{2}&1/\sqrt{2}\\0&0&0&-\sqrt{3/2}\end{bmatrix},\qquad g_3=\begin{bmatrix}1/\sqrt{2}&0&0&-\sqrt{2}/6\\1/\sqrt{6}&-\sqrt{2/3}&\sqrt{6}/9&7\sqrt{6}/18\\0&0&\sqrt{2}&1/\sqrt{2}\\0&0&0&-\sqrt{3/2}\end{bmatrix}.$$

Proposition 8.15. *A complete system of representatives of* $K(N)$*-conjugacy classes of families is given as*

$$\begin{aligned}F_{1,d}&=\iota(x_d)^{-1}{g_1}^{-1}\cdot\{\gamma(2\pi/3,n)\mid n\in\mathbb{Z},n\neq 0\}\cdot g_1\iota(x_d),\\F_{2,d}&=\iota(x_d)^{-1}{g_2}^{-1}\cdot\{\gamma(2\pi/3,n+1/3)\mid n\in\mathbb{Z}\}\cdot g_2\iota(x_d),\\F_{3,d}&=\iota(x_d)^{-1}{g_3}^{-1}\cdot\{\gamma(2\pi/3,n+2/3)\mid n\in\mathbb{Z}\}\cdot g_3\iota(x_d)\end{aligned}$$

where $d\in\{0,1,\dots,N-1\}$ *such that* $d^2+d+1\in N\mathbb{Z}$.

For each family F, we define $C_0(F;Sp(2;\mathbb{R}))$ as in [**Wak12**, page 211].

Proposition 8.16. *For every family* F*, we have*

$$\frac{\mathrm{vol}(C_0(F;K(N))\backslash C_0(F;Sp(2;\mathbb{R})))}{[\overline{C}(F;K(N)):\overline{C_0}(F;K(N))]}=\frac{1}{3},$$

and $\lim_{s\to+0}\sum_{\gamma'\in F}J_0(\gamma';s)=-c_{k,j}^{-1}\cdot 2^{-1}\cdot 3^{-1}\cdot[1,-1,0;3]_j$, *hence the contribution of* F *is* $-2^{-2}\cdot 3^{-2}\cdot[1,-1,0;3]_j$.

Proof. Direct calculations and [**Wak12**, (f-3) on page 211]. □

It follows that the total contribution of all families is

$$I_{10}(k,j,N) = -2^{-1}\cdot 3^{-1}\cdot [1,-1,0;3]_j \cdot \prod_{p|N}\left(1+\left(\frac{-3}{p}\right)\right).$$

9. Appendix: tables of dimensions.

9.1. Scalar valued case.

We give a table of dimensions of $S_{k,0}(K(N))$ by Main Theorem 1.1 (Theorem 3.1) and its remark. Concerning Theorem 4.6 (i), we also give a table of dimensions of $J_{k,m}^{cusp}$, the space of Jacobi cusp forms of weight k and index m, by a well-known formula. As is well-known, Gritsenko [**Gri95**] proved a lifting theory from $J_{k,N}^{cusp}$ to $S_{k,0}(K(N))$.

$\dim S_{k,0}(K(N))$

$N\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	3
2	0	0	0	0	0	1	0	1	1	2	0	2	1	4	1	4	2	7
3	0	0	0	1	0	1	1	2	1	4	1	4	3	6	3	10	4	11
5	0	0	1	1	1	2	2	4	4	6	5	9	8	13	12	18	16	25
6	0	0	0	1	1	2	3	4	5	10	6	11	15	20	17	30	27	40
7	0	1	1	2	2	4	4	7	7	11	11	16	16	24	24	33	33	45
10	0	1	1	2	4	6	7	12	15	21	23	32	38	52	55	72	81	103
11	0	1	2	3	3	6	7	12	14	20	22	32	36	48	54	69	76	97
13	1	2	3	5	7	10	13	19	23	31	37	48	56	72	82	102	115	140
14	0	1	2	3	5	10	12	19	26	36	42	58	70	92	105	132	152	189
15	0	1	2	5	5	10	14	21	26	40	44	62	74	96	109	144	156	197

$\dim J_{k,m}^{cusp}$

$m\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	2
2	0	0	0	0	0	1	0	1	1	2	0	2	1	3	1	3	1	4
3	0	0	0	1	0	1	1	2	1	3	1	3	2	4	2	5	2	5
5	0	0	1	1	1	2	2	3	3	4	3	5	4	6	5	7	5	8
6	0	0	0	1	1	2	2	3	3	5	3	5	5	7	5	8	6	9
7	0	1	1	2	2	3	3	5	4	6	5	7	6	9	7	10	8	11
10	0	1	1	2	3	4	4	6	6	8	7	9	9	12	10	13	12	15
11	0	1	2	3	3	5	5	7	7	9	8	11	10	13	12	15	13	17
13	1	2	3	4	5	6	7	9	9	11	11	13	13	16	15	18	17	20
14	0	1	2	3	4	6	6	8	9	11	10	13	13	16	15	18	17	21
15	0	1	2	4	4	6	7	9	9	12	11	14	14	17	16	20	18	22

9.2. Vector valued case.

We give a table of dimensions of $S_{k,j}(K(N))$ with $j \geq 2$ by Main Theorem 1.1 (Theorem 3.1). The theorem is valid for $k \geq 5$, but we believe that the formula is still applicable for $k = 3$ and 4. The values below for $k = 3$ and 4 are conjectural in most cases.

$\dim S_{k,2}(K(N))$

$N \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	2	0	3
2	0	0	0	0	0	0	0	1	1	1	1	4	4	6	4	9	7	12
3	0	0	0	0	0	1	1	2	2	4	4	8	8	13	11	18	16	26
5	0	0	0	0	1	2	3	5	7	10	13	19	22	31	33	44	48	63
6	0	0	0	1	2	4	6	11	15	20	25	38	46	59	65	87	96	121
7	0	0	1	1	3	5	8	12	16	22	29	39	47	61	70	88	100	124
10	0	0	1	3	7	12	19	30	42	56	73	99	123	154	181	226	262	316
11	0	1	2	4	9	14	21	31	43	57	75	97	119	151	178	217	254	304
13	0	1	3	5	12	19	29	43	59	79	104	134	166	208	248	301	354	422
14	0	1	4	8	17	27	42	63	87	115	151	197	245	304	363	443	520	618
15	0	1	4	8	17	29	44	65	89	121	157	205	253	318	377	461	538	646

$\dim S_{k,4}(K(N))$

$N \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	0	1	0	1	0	2	1	3	1	4	2	6
2	0	0	0	0	1	1	1	3	3	5	4	8	9	13	11	17	18	26
3	0	0	0	1	0	2	3	5	5	10	8	16	17	23	24	35	33	49
5	0	1	1	3	4	7	10	16	18	27	31	43	50	65	72	92	102	127
6	0	0	1	4	6	12	17	25	34	49	54	78	95	117	135	170	191	239
7	0	1	1	5	7	12	18	28	34	49	59	79	95	120	138	172	196	237
10	0	3	5	13	22	35	50	74	96	131	161	209	256	317	370	450	524	623
11	1	4	6	15	22	35	51	73	93	127	157	201	245	302	355	430	498	589
13	1	5	9	20	31	49	71	101	131	176	220	280	342	420	497	598	696	820
14	1	6	12	27	45	70	101	145	191	255	319	407	500	613	725	872	1020	1201
15	0	5	11	28	42	71	104	148	194	264	326	422	515	632	750	907	1049	1246

$\dim S_{k,6}(K(N))$

$N\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	1	0	1	1	2	1	3	2	5	3	7	4	9
2	0	0	0	0	1	2	2	4	6	7	7	13	14	19	19	28	28	37
3	0	0	1	2	2	5	7	9	12	18	18	27	31	40	43	59	59	77
5	0	0	3	4	7	13	17	24	33	43	52	69	81	102	118	145	162	197
6	0	0	3	8	13	22	33	45	62	82	97	130	159	192	223	279	312	372
7	1	3	7	11	18	28	38	52	69	89	109	139	166	204	238	286	327	387
10	0	4	12	23	42	63	91	127	170	218	274	347	422	513	605	725	840	982
11	3	5	18	27	44	68	94	126	170	216	270	338	409	494	587	695	804	941
13	4	10	25	40	64	96	134	180	239	305	381	475	575	694	823	973	1129	1316
14	2	10	29	51	86	131	185	253	339	433	543	682	829	1001	1188	1410	1639	1910
15	2	10	31	55	88	137	195	263	351	455	565	710	863	1043	1236	1472	1701	1990

9.3. Paramodular new forms.

We give tables of dimensions of $S_{k,j}^{new}(K(N))$ for $N \geq 2$ given in Section 4 by Proposition 4.4. (For $N = 1$, we have $S_{k,j}^{new}(K(1)) = S_{k,j}(K(1))$ by definition.) Since the space of new forms contain the lifts, we denote by $\mathcal{S}_k^*(K(N))$ the orthogonal complement in $S_k^{new}(K(N))$ of the space spanned by the lifts from $S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N))$ and we also give a table of $\dim \mathcal{S}_k^*(K(N))$.

$\dim S_{k,0}^{new}(K(N))$

$N\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0	0	0	0	1	0	0	1	1	0	1	1	2	1	2	2	3
3	0	0	0	1	0	1	1	1	1	3	1	3	3	4	3	8	4	7
5	0	0	1	1	1	2	2	3	4	5	5	8	8	11	12	16	16	21
6	0	0	0	0	1	0	2	2	3	4	5	5	10	9	12	12	18	19
7	0	1	1	2	2	4	4	6	7	10	11	15	16	22	24	31	33	41
10	0	1	0	1	3	3	5	7	9	12	16	18	25	29	35	40	51	57
11	0	1	2	3	3	6	7	11	14	19	22	31	36	46	54	67	76	93
13	1	2	3	5	7	10	13	18	23	30	37	47	56	70	82	100	115	136
14	0	0	1	1	3	4	7	10	15	19	25	32	43	50	63	73	91	106
15	0	1	1	3	4	7	11	15	20	28	36	45	58	70	86	102	123	144

$\dim \mathcal{S}_k^*(K(N))$

$N\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1
3	0	0	0	0	0	0	0	0	0	1	0	1	1	2	1	5	2	4
5	0	0	0	0	0	0	0	1	1	2	2	4	4	7	7	11	11	15
6	0	0	0	0	0	0	1	1	2	3	3	4	8	7	10	10	15	17
7	0	0	0	0	0	1	1	2	3	5	6	9	10	15	17	23	25	32
10	0	0	0	0	1	2	3	4	7	9	12	15	21	24	31	35	45	52
11	0	0	0	0	0	1	2	5	7	11	14	21	26	35	42	54	63	78
13	0	0	0	1	2	4	6	10	14	20	26	35	43	56	67	84	98	118
14	0	0	0	0	1	2	4	7	11	15	20	27	37	44	56	66	83	98
15	0	0	0	1	1	4	7	10	15	22	29	38	50	61	77	92	112	133

$\dim S_{k,2}^{new}(K(N))$

$N\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0	0	0	0	0	0	1	1	1	1	2	4	2	4	5	7	6
3	0	0	0	0	0	1	1	2	2	4	4	6	8	9	11	14	16	20
5	0	0	0	0	1	2	3	5	7	10	13	17	22	27	33	40	48	57
6	0	0	0	1	2	2	4	5	9	10	15	18	22	29	35	41	50	57
7	0	0	1	1	3	5	8	12	16	22	29	37	47	57	70	84	100	118
10	0	0	1	3	5	8	13	18	26	34	45	57	71	88	107	128	152	178
11	0	1	2	4	9	14	21	31	43	57	75	95	119	147	178	213	254	298
13	0	1	3	5	12	19	29	43	59	79	104	132	166	204	248	297	354	416
14	0	1	2	6	11	17	26	37	53	69	91	115	143	178	215	257	306	358
15	0	1	4	8	15	23	36	51	71	93	123	155	193	238	289	345	410	480

$\dim S_{k,4}^{new}(K(N))$

$N\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0	0	0	1	1	1	1	3	3	4	4	7	7	9	9	14	14
3	0	0	0	1	0	2	3	3	5	8	8	12	15	17	22	27	29	37
5	0	1	1	3	4	7	10	14	18	25	31	39	48	59	70	84	98	115
6	0	0	1	2	4	6	9	13	18	23	30	38	47	57	69	82	97	113
7	0	1	1	5	7	12	18	26	34	47	59	75	93	114	136	164	192	225
10	0	1	3	7	12	19	28	40	54	71	91	115	142	173	208	248	292	341
11	1	4	6	15	22	35	51	71	93	125	157	197	243	296	353	422	494	577
13	1	5	9	20	31	49	71	99	131	174	220	276	340	414	495	590	692	808
14	1	4	10	17	29	44	63	87	117	151	193	241	296	359	431	510	600	699
15	0	3	9	20	34	53	78	110	148	194	248	312	385	468	562	669	787	918

$\dim S_{k,6}^{new}(K(N))$

$N\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0	0	0	1	0	2	2	4	3	5	7	10	9	13	14	20	19
3	0	0	1	2	2	3	7	7	10	14	16	21	27	30	37	45	51	59
5	0	0	3	4	7	11	17	22	31	39	50	63	77	92	112	131	154	179
6	0	0	1	4	7	12	15	23	30	40	51	62	77	94	111	133	154	180
7	1	3	7	11	18	26	38	50	67	85	107	133	162	194	232	272	319	369
10	0	4	6	15	26	37	53	75	96	126	160	195	240	291	343	407	476	550
11	3	5	18	27	44	66	94	124	168	212	268	332	405	484	581	681	796	923
13	4	10	25	40	64	94	134	178	237	301	379	469	571	684	817	959	1121	1298
14	0	4	15	29	48	75	105	145	193	249	315	390	477	575	686	810	945	1098
15	2	10	23	43	70	105	147	201	265	341	429	530	647	779	926	1092	1275	1478

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Tomoyoshi Ibukiyama
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka 560-0043, Japan
E-mail: ibukiyam@math.sci.osaka-u.ac.jp

Hidetaka Kitayama
Department of Mathematics
Faculty of Education
Wakayama University
Sakaedani 930
Wakayama-City 640-8510, Japan
E-mail: hkitayam@center.wakayama-u.ac.jp