J. Math. Soc. Japan
Vol. 68, No. 1 (2016) pp. 169–191
doi: 10.2969/jmsj/06810169

Weighted L^p-boundedness of convolution type integral operators associated with bilinear estimates in the Sobolev spaces

By Kazumasa Fujiwara and Tohru Ozawa

(Received Mar. 3, 2014)

Abstract. We study the boundedness of integral operators of convolution type in the Lebesgue spaces with weights. As a byproduct, we give a simple proof of the fact that the standard Sobolev space $H^s(\mathbb{R}^n)$ forms an algebra for $s > n/2$. Moreover, an optimality criterion is presented in the framework of weighted L^p-boundedness.

1. Introduction.

We study the boundedness of integral operators of convolution type in the Lebesgue space with weights. A special attention will be made on an optimality criterion with respect to the growth rate of weights.

To illustrate the problem, we revisit the standard property that the Sobolev space $H^s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$ forms an algebra for $s > n/2$ from the point of view from the weighted $L^2(\mathbb{R}^n)$-boundedness of convolution. The corresponding bilinear estimate in the Sobolev space takes the form

$$\|uv\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s} \tag{1.1}$$

with $s > n/2$, where

$$\|u\|_{H^s} = \|(1-\Delta)^{s/2}u\|_{L^2} = \|(1+|\xi|^2)^{s/2}\hat{u}\|_{L^2},$$
$$\hat{u}(\xi) = \mathfrak{F}u(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp(-ix\cdot\xi)u(x)dx,$$

and Δ is the Laplacian in $\mathbb{R}^n$. The bilinear estimate of this type was may be traced back at least to the paper by Saut and Temam [**15**]. There are many papers on further refinements and improvements on this subject as well as various applications to nonlinear partial differential equations. (see for instance [**2**]–[**18**] and references therein.)

One of the purpose in this paper is to give a simple and elementary proof of (1.1), which avoids paradifferential technique for instance.

In the Fourier representation, multiplication of functions is realized by convolution of the corresponding Fourier transformed functions:

2010 *Mathematics Subject Classification.* Primary 26D15; Secondary 46E35, 42B.
Key Words and Phrases. pointwise multiplication, Sobolev spaces.

$$\mathfrak{F}(uv)(\xi) = (2\pi)^{n/2}(\hat{u} * \hat{v})(\xi) = (2\pi)^{n/2} \int_{\mathbb{R}^n} \hat{u}(\xi - \eta)\hat{v}(\eta) d\eta$$

and the estimate (1.1) is equivalent to the bilinear estimate of the form

$$\|\omega(\hat{u} * \hat{v})\|_{L^2} \leq C\|\omega\hat{u}\|_{L^2}\|\omega\hat{v}\|_{L^2}$$

with $\omega(\xi) = (1 + |\xi|^2)^{s/2}$, which is also rewritten as

$$\left\|\omega\left(\left(\frac{\hat{u}}{\omega}\right) * \left(\frac{\hat{v}}{\omega}\right)\right)\right\|_{L^2} \leq C\|\hat{u}\|_{L^2}\|\hat{v}\|_{L^2}. \tag{1.2}$$

By a duality argument, (1.2) is equivalent to the trilinear estimate of the form

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \omega(\xi) \frac{1}{\omega(\xi - \eta)} \frac{1}{\omega(\eta)} \hat{u}(\xi - \eta)\hat{v}(\eta)\hat{w}(\xi)\, d\eta\, d\xi \right| \\ &\quad \leq C\|\hat{u}\|_{L^2}\|\hat{v}\|_{L^2}\|\hat{w}\|_{L^2}. \end{aligned} \tag{1.3}$$

By a simple change of variables, (1.3) is equivalent to

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \omega(\xi + \eta) \frac{1}{\omega(\xi)} \frac{1}{\omega(\eta)} \hat{u}(\xi)\hat{v}(\eta)\hat{w}(\xi + \eta)\, d\eta\, d\xi \right| \\ &\quad \leq C\|\hat{u}\|_{L^2}\|\hat{v}\|_{L^2}\|\hat{w}\|_{L^2}. \end{aligned} \tag{1.4}$$

This gives a motivation to study the boundedness of the integrals of the form

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w_0(x + y)w_1(x)w_2(y)f(x + y)g(x)h(y)\, dx\, dy \tag{1.5}$$

with weight functions w_0, w_1, w_2, where w_1 and w_2 are supposedly the inverse weight of w_0.

The following theorem is basic in this direction.

THEOREM 1.1. *Let $2 \leq p \leq \infty$ and let w_0, w_1, w_2 be nonnegative, continuous functions on $[0, \infty)$ satisfying*

$$M_1 \equiv \sup_{r>0} w_0^{\#}(2r)w_2(r)\|w_1(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{1.6}$$

$$M_2 \equiv \sup_{r>0} w_0^{\#}(2r)w_1(r)\|w_2(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{1.7}$$

where

$$w_0^{\#}(r) = \sup_{0\leq\rho\leq r} w_0(\rho),$$

$$B(r) = \{x \in \mathbb{R}^n; |x| \leq r\}.$$

Then, the trilinear estimate

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} & w_0(|x+y|)\ w_1(|x|)\ w_2(|y|)\ |f(x+y)g(x)h(y)|\,dx\,dy \\ & \le (M_1+M_2)\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}} \qquad (1.8)\end{aligned}$$

holds for all $f \in L^p(\mathbb{R}^n)$, g, $h \in L^{p'}(\mathbb{R}^n)$, where p' is the dual exponent defined by $1/p+1/p'=1$.

Proof. For $f \in L^p$ we define the translation by $y \in \mathbb{R}^n$ by $(\tau_y f)(x) = f(x+y)$. For $S \subset \mathbb{R}^n$ we denote by χ_S its characteristic function. Then, by the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned}\iint_{|x|\le|y|} & w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy \\ & \le \iint w_0^{\#}(2|y|)\chi_{B(|y|)}(x)w_1(|x|)w_2(|y|)|\tau_y f(x)g(x)h(y)|\,dx\,dy \\ & \le \int w_0^{\#}(2|y|)\|\chi_{B(|y|)}w_1(|\cdot|)\|_{L^p}\|\tau_y f\cdot g\|_{L^{p'}}w_2(|y|)|h(y)|\,dy \\ & \le M_1\|\|\tau_y f\cdot g\|_{L^{p'}}\|_{L^p_y}\|h\|_{L^{p'}} \\ & = M_1\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}},\end{aligned}$$

where L^p_y is the L^p norm for the variable y. Similarly,

$$\begin{aligned}\iint_{|x|\ge|y|} & w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy \\ & \le \iint w_0^{\#}(2|x|)\chi_{B(|x|)}(y)w_1(|x|)w_2(|y|)|\tau_x f(y)g(x)h(y)|\,dx\,dy \\ & \le \int w_0^{\#}(2|x|)\|\chi_{B(|x|)}w_2(|\cdot|)\|_{L^p}\|\tau_x f\cdot h\|_{L^{p'}}w_1(|x|)|g(x)|\,dx \\ & \le M_2\|\|\tau_x f\cdot h\|_{L^{p'}}\|_{L^p}\|g\|_{L^{p'}} \\ & = M_2\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}}.\end{aligned}$$

Summing those inequalities, we have (1.8). □

Corollary 1.1. *Let $2 \le p \le \infty$ and let w_0, w_1, w_2 be nonnegative, continuous functions on $[0,\infty)$ satisfying*

$$M_1' = \sup_{r>0} w_0(2r)w_2(r)\|w_1(|\cdot|)\|_{L^p(B(r))} < \infty, \qquad (1.9)$$

$$M_2' = \sup_{r>0} w_0(2r)w_1(r)\|w_2(|\cdot|)\|_{L^p(B(r))} < \infty, \qquad (1.10)$$

and the estimate

$$w_0(r) \leq C' w_0(R) \tag{1.11}$$

for any r *and* R *with* $0 \leq r \leq R$ *with* $C' \geq 1$ *independent of* r *and* R. *Then, the trilinear estimate*

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w_0(|x+y|)\ w_1(|x|)\ w_2(|y|)\ |f(y+x)g(x)h(y)|\,dx\,dy \\ &\quad \leq C'(M_1' + M_2')\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}}\end{aligned}$$

holds for all $f \in L^p(\mathbb{R}^n)$, g, $h \in L^{p'}(\mathbb{R}^n)$.

Proof. By (1.11), we have $w_0^{\#}(2r) \leq C' w_0(2r)$ for any $r \geq 0$. Then, the corollary follows from Theorem 1.1 □

The bilinear estimate (1.1) follows by choosing $p = 2$, $w_0(r) = (1+r^2)^{s/2}$, $w_1(r) = w_2(r) = (1+r^2)^{-s/2}$ with $s > n/2$, which ensures the required square integrability. A natural question then arises in connection with minimal growth rate at infinity in space for w_0, $1/w_1$, $1/w_2$. Weight functions of the form $w(r) = (1+r^2)^{n/2}(1+\log(1+r))^s$ with $s > 1/2$ may be the first candidate with $w_0 = w$, $w_1 = w_2 = 1/w$. This is not optimal since $w(r) = (1+r^2)^{n/2}(1+\log(1+r))^{1/2}\big(1+\log(1+\log(1+r))\big)^s$ with $s > 1/2$ has a slower growth with keeping the required square integrability.

To describe emerging extra logarithmic factors in such an iteration procedure, it is convenient to introduce the following set $\mathcal{F}$ consisting of positive, continuous functions w on $[0,\infty)$ satisfying $1/w \in L^1_{\mathrm{loc}}(0,\infty)$ and the following assumptions (A1) and (A2):

(A1) For any $a \in \mathbb{R}$, there exists $C_a \geq 1$ such that for any r and R with $0 \leq r \leq R$, w satisfies the inequality

$$w(r)\bigg(\int_0^r \frac{1}{w(\rho)}d\rho + 1\bigg)^a \leq C_a w(R)\bigg(\int_0^R \frac{1}{w(\rho)}d\rho + 1\bigg)^a.$$

(A2) There exists $C > 0$ such that the inequality

$$w(2r) \leq C w(r)$$

holds for all $r > 0$.

Example 1. The function w defined by $w(r) = 1 + r$ belongs to $\mathcal{F}$ with $C_a = 1$ for $a \geq -1$, $C_a = e^{a+1}(-a)^{-a}$ for $a < -1$, and $C = 2$.

Example 2. The function w defined by $w(r) = (1+r)^s$ with $s > 1$ belongs to $\mathcal{F}$ with $C_a = 1$ for $a \geq -s$,

$$C_a = (-a)^{-a}(a+s-as)^{(as-a-s)/(s-1)} s^{(2s+a-as)/(s-1)}$$

for $a < -s$, and $C = 2^s$.

EXAMPLE 3. The function w defined by $w(r) = (1+r^2)^{s/2}$ for $s \geq 1$ belong to $\mathcal{F}$ with $C_a = 1$ for $a \geq 0$,

$$C_a = s^a(-a)^{-a} r_{s,a}^a (1+r_{s,a}^2)^{-a+(a-1)s/2}$$

for $a < 0$, where $r_{s,a}$ is defined uniquely by

$$r_{s,a}(1+r_{s,a}^2)^{s/2-1}\biggl(\int_0^{r_{s,a}} (1+\rho^2)^{-s/2} d\rho + 1\biggr) = \frac{|a|}{s}$$

and $C = 2^s$.

EXAMPLE 4. Let $w(r) = 1 + \log(1+r)$ and $a = -2$. Then,

$$w(r)\biggl(\int_0^r \frac{1}{w(\rho)} d\rho + 1\biggr)^{-2} \leq w(r)\biggl(\int_0^r \frac{1}{1+\rho} d\rho + 1\biggr)^{-2} = \frac{1}{w(r)} \to 0$$

as $r \to \infty$. This means $w \notin \mathcal{F}$.

EXAMPLE 5. The function w defined by $w(r) = (1+r)(1+\log(1+r))$ belongs to $\mathcal{F}$ with $C_a = 1$ for $a \geq 0$,

$$C_a = (-a)^{-a}(1+\tilde{r}_{s,a})^{-1}(1+\log(1+\tilde{r}_{s,a}))^{-1}(2+\log(\tilde{r}_{s,a}))^a$$

for $a < 0$, where $\tilde{r}_{s,a}$ is uniquely defined by

$$(2+\log(1+\tilde{r}_{s,a}))(1+\log(1+\log(1+\tilde{r}_{s,a}))) = |a|,$$

and $C = 2 + 2\log 2$.

REMARK 1.1. For $w \in \mathcal{F}$, we apply (A1) with $a = 0$ to obtain

$$\begin{aligned}
&\int_0^r \frac{1}{w(\rho)} d\rho \\
&\quad \leq \int_0^{2r} \frac{1}{w(\rho)} d\rho = \int_0^r \frac{1}{w(\rho)} d\rho + \int_0^r \frac{1}{w(\rho+r)} d\rho \\
&\quad \leq (1+C_0) \int_0^r \frac{1}{w(\rho)} d\rho. \qquad (1.12)
\end{aligned}$$

THEOREM 1.2. *Let $2 \leq p < \infty$ and let $w \in \mathcal{F}$. Let w_0, w_1, w_2 be defined by*

$$w_0(r) = (1+r)^{(n-1)/p} w(r)^{1/p}\biggl(\int_0^r \frac{1}{w(\rho)} d\rho + 1\biggr)^{-a},$$

$$w_1(r) = (1+r)^{-(n-1)/p} w(r)^{-1/p} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-b},$$

$$w_2(r) = (1+r)^{-(n-1)/p} w(r)^{-1/p} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-c}$$

with $a, b, c \in \mathbb{R}$ *satisfying either* (i) *or* (ii):

(i) $a + b + c \geq 1/p$ $\quad a + b > 0 \quad a + c > 0.$
(ii) $a + b + c > 1/p$ $\quad a + b \geq 0 \quad a + c \geq 0.$

Then, there exists $C > 0$ *such that the trilinear estimate*

$$\iint w_0(|x+y|) w_1(|x|) w_2(|y|) |f(x+y) g(x) h(y)| dx\, dy \leq C \|f\|_{L^p} \|g\|_{L^{p'}} \|h\|_{L^{p'}} \tag{1.13}$$

holds for all $f \in L^p(\mathbb{R}^n)$, $g, h \in L^{p'}(\mathbb{R}^n)$.

Remark 1.2. In the case $\int_0^\infty w^{-1}(\rho) d\rho < \infty$, we can choose any a, b, c for (1.13). In the case where $p = 2$ and $b = c = 0$, assumption (i) is equivalent to $a \geq 1/2$. In the case where $p = 2$ and $b = c > 0$, assumption (i) is equivalent to $a \geq 1/2 - 2b$ with $a > -b$. In the case where $p = 2$ and $-a = b = c$, assumption (i) breaks down and (ii) is equivalent to $-a = b = c > 1/2$.

Proof of Theorem 1.2. We prove that w_0, w_1, w_2 defined in the theorem satisfy the assumptions (1.9)–(1.11) in Corollary 1.1. Let r and R satisfy $0 \leq r \leq R$. By (A1),

$$w(r) \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-ap} \leq C_{-ap} w(R) \left(\int_0^R \frac{1}{w(\rho)} d\rho + 1 \right)^{-ap},$$

which yields

$$w_0(r) \leq C_{-ap}^{1/p} w_0(R). \tag{1.14}$$

By (A2) and (1.12),

$$\begin{aligned} w_0(2r) &= (1+2r)^{(n-1)/p} w(2r)^{1/p} \left(\int_0^{2r} \frac{1}{w(r)} d\rho + 1 \right)^{-a} \\ &\leq 2^{(n-1)/p} (1+r)^{(n-1)/p} C^{1/p} w(r)^{1/p} (1+C_0)^{(-a)_+} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-a} \\ &= 2^{(n-1)/p} C^{1/p} (1+C_0)^{(-a)_+} w_0(r), \end{aligned} \tag{1.15}$$

which yields

$$w_0(2r)w_1(r) \leq 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\left(\int_0^r \frac{1}{w(r)}d\rho + 1\right)^{-a-b}. \tag{1.16}$$

We estimate $w_2(|\cdot|)$ in $L^p(B(r))$ as

$$\|w_2(|\cdot|)\|_{L^p(B(r))} \leq \omega_{n-1}^{1/p}\left(\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho\right)^{1/p}, \tag{1.17}$$

where ω_{n-1} is the surface measure of the unit ball. To estimate the right hand side of (1.17) and M_1' of Corollary 1.1, we distinguish four cases:

(i) $c \leq 0$. (ii) $0 < c < 1/p$. (iii) $c = 1/p$. (iv) $c > 1/p$.

(i) In the case where $c \leq 0$, we estimate

$$\begin{aligned}\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho &\leq \int_0^r \frac{1}{w(\rho)}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho \\ &\leq \left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^{1-pc}.\end{aligned}$$

Then, M_1' is estimated as follows:

$$\begin{aligned}M_1' &\leq \sup_{r>0} 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\left(\int_0^r \frac{1}{w(r)}d\rho + 1\right)^{1/p-a-b-c} \\ &= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}.\end{aligned}$$

(ii) In the case where $0 < c < 1/p$, we estimate

$$\begin{aligned}\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho &= \frac{1}{1-pc}\left(\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{1-pc} - 1\right) \\ &\leq \frac{1}{1-pc}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{1-pc}.\end{aligned}$$

Then, M_1' is estimated as follows:

$$M_1' \leq \frac{1}{1-pc}2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}.$$

(iii) In the case where $c = 1/p$, we estimate

$$\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-1} d\rho = \log\left(1 + \int_0^r \frac{1}{w(\rho)}d\rho\right).$$

Since $a+b>0$, M_1' is estimated as follows:

$$\begin{aligned}
M_1' &\le C^{1/p}(1+C_0)^{(-a)_+}\sup_{r>0} 2^{(n-1)/p}\left(\int_0^r \frac{1}{w(r)}d\rho+1\right)^{-a-b}\log\left(1+\int_0^r\frac{1}{w(\rho)}d\rho\right)\\
&= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\sup_{r\ge 1} r^{-a-b}\log r\\
&= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\frac{1}{e(a+b)}.
\end{aligned}$$

(iv) In the case where $c>1/p$, we estimate

$$\begin{aligned}
&\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-pc}d\rho\\
&\quad=\frac{1}{1-pc}\left(\left(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\right)^{1-pc}d\rho-1\right)\\
&\quad\le\frac{1}{pc-1}.
\end{aligned}$$

Since $a+b\ge 0$, M_1' is estimated as follows:

$$M_1'\le\frac{1}{pc-1}2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}.$$

M_2' is estimated similarly. Then, the estimate (1.13) follows from Corollary 1.1. $\square$

In a way similar to the proof of Theorem 1.2, we have the following theorem for $p=\infty$.

Theorem 1.3. *Let $w\in\mathcal{F}$. Let w_0, w_1, w_2 be defined by*

$$\begin{aligned}
w_0(r)&=\left(\int_0^r\frac{1}{w(\rho)}d\rho+1\right)^{-a},\\
w_1(r)&=\left(\int_0^r\frac{1}{w(\rho)}d\rho+1\right)^{-b},\\
w_2(r)&=\left(\int_0^r\frac{1}{w(\rho)}d\rho+1\right)^{-c}
\end{aligned}$$

with $a,b,c\in\mathbb{R}$ satisfying

$$a+b+c_-\ge 0\quad\text{and}\quad a+b_-+c\ge 0,$$

where $b_-=-\max(0,-b)=\min(0,b)$, $c_-=-\max(0,-c)=\min(0,c)$.

Then, there exists $C>0$ such that the trilinear estimate

$$\iint w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\ dx\ dy \leq C\|f\|_\infty\|g\|_1\|h\|_1$$

holds for all $f \in L^\infty(\mathbb{R}^n)$, $g, h \in L^1(\mathbb{R}^n)$.

Theorem 1.2 shows the importance of the class $\mathcal{F}$ to the trilinear estimate such as (1.8). Accordingly, below we study the class $\mathcal{F}$ in details. In Section 2, we study a basic property of $\mathcal{F}$. In Section 3, we introduce arbitrarily and infinitely iterates of logarithm in connection with $\mathcal{F}$. A part of the arguments in Sections 2 and 3 are essentially given by Ando, Horiuchi, and Nakai [**1**]. We revisit them in the present framework for definiteness. In Section 4, we study optimality of Theorem 1.2.

2. A basic property of $\mathcal{F}$.

In this section we prove:

PROPOSITION 2.1. *For* $w \in \mathcal{F}$ *and* $a \in \mathbb{R}$, *we define* W_a *by*

$$W_a(r) = w(r)\biggl(\int_0^r \frac{1}{w(\rho)}d\rho + 1\biggr)^a, \qquad r \geq 0.$$

Then, $W_a \in \mathcal{F}$.

PROOF. By definition, we see that W_a is a positive, continuous function on $[0, \infty)$ satisfying $1/W_a \in L^1_{\mathrm{loc}}(0, \infty)$. By (A2) and Remark 1.1,

$$W_a(2r) \leq Cw(r)\biggl(\int_0^{2r} \frac{1}{w(\rho)}d\rho + 1\biggr)^a \leq C(C_0+1)^{a_+}W_a(r),$$

where $a_+ = \max(a, 0)$. It remains to prove that W_a satisfies (A1); For any $a, b \in \mathbb{R}$, there exists $C_{a,b}$ such that for any r and R with $0 \leq r \leq R$,

$$W_a(r)\biggl(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\biggr)^b \leq C_{a,b}W_a(R)\biggl(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\biggr)^b$$

holds. Let $0 \leq r \leq R$. We note that (A1) property of w is equivalent to $W_a(r) \leq C_aW_a(R)$. We distinguish three cases:

(i) $b \geq 0$. (ii) $b < 0$, $a \geq 0$. (iii) $b < 0$, $a < 0$.

(i) In the case where $b \geq 0$, we estimate

$$\begin{aligned} W_a(r)\biggl(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\biggr)^b &\leq C_aW_a(R)\biggl(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\biggr)^b \\ &\leq C_aW_a(R)\biggl(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\biggr)^b, \end{aligned}$$

as required.

(ii) In the case where $b<0$, $a\geq 0$, we first notice that

$$\begin{aligned}
&\frac{1}{W_a(R)}\left(\int_0^R \frac{1}{W_a(\rho)}d\rho+1\right)^{|b|}\\
&=\frac{1}{W_a(R)}\left(\int_0^r \frac{1}{W_a(\rho)}d\rho+\int_r^R \frac{1}{W_a(\rho)}d\rho+1\right)^{|b|}\\
&\leq \frac{2^{(|b|-1)_+}}{W_a(R)}\left(\left(\int_0^r \frac{1}{W_a(\rho)}d\rho+1\right)^{|b|}+\left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}\right)\\
&\leq \frac{C_a 2^{(|b|-1)_+}}{W_a(r)}\left(\int_0^r \frac{1}{W_a(\rho)}d\rho+1\right)^{|b|}+\frac{2^{(|b|-1)_+}}{W_a(R)}\left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}.
\end{aligned}\tag{2.1}$$

To estimate the second term on the right hand side of the last inequality of (2.1), we remark that

$$\begin{aligned}
\int_r^R \frac{1}{W_a(\rho)}d\rho&=\int_r^R \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-a}d\rho\\
&\leq \int_r^R \frac{1}{w(\rho)}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\right)^{-a}d\rho\\
&\leq \left(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\right)^{-a}\int_0^R \frac{1}{w(\rho)}d\rho
\end{aligned}$$

and

$$\frac{1}{W_a(R)}=\frac{1}{w(R)}\left(\int_0^R \frac{1}{w(\rho)}d\rho+1\right)^{-a}\leq \frac{1}{w(R)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{W_a(R)}\left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}\\
&\leq \frac{1}{w(R)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-a|b|}\left(\int_0^R \frac{1}{w(\rho)}d\rho\right)^{|b|}\\
&\leq \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-a|b|}\frac{1}{w(R)}\left(\int_0^R \frac{1}{w(\rho)}d\rho+1\right)^{|b|}\\
&\leq \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-a|b|}\cdot C_b\frac{1}{w(r)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{|b|}
\end{aligned}$$

$$\le C_b \frac{1}{w(r)\big(\int_0^r \frac{1}{w(\rho)} d\rho + 1\big)^a} \cdot \bigg(\int_0^r \frac{1}{w(\rho)\big(\int_0^r \frac{1}{w(\sigma)} d\sigma + 1\big)^a} d\rho + 1 \bigg)^{|b|}$$

$$\le C_b \frac{1}{W_a(r)} \bigg(\int_0^r \frac{1}{W_a(\rho)} d\rho + 1 \bigg)^{|b|}. \tag{2.2}$$

Combining (2.1) and (2.2) and taking the inverse of the resulting inequality, we find that W_a satisfies (A1).

(iii) In the case where $b < 0$, $a < 0$, we use the equality

$$\int_0^r \frac{1}{W_a(\rho)} d\rho + 1 = \frac{1}{|a|+1} \bigg(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \bigg)^{|a|+1} + \frac{|a|}{|a|+1}$$

to estimate

$$\begin{aligned} &W_a(r) \bigg(\int_0^r \frac{1}{W_a(\rho)} d\rho + 1 \bigg)^b \\ &\quad \le \frac{1}{(|a|+1)^b} w(r) \bigg(\int_0^r \frac{1}{w(\rho)} + 1 \bigg)^{a+(|a|+1)b} \\ &\quad \le (|a|+1)^{|b|} C_{a+b-ab} w(R) \bigg(\int_0^R \frac{1}{w(\rho)} d\rho + 1 \bigg)^{a+b-ab} \\ &\quad = (|a|+1)^{|b|} C_{a+b-ab} W_a(R) \bigg(\int_0^R \frac{1}{w(\rho)} d\rho + 1 \bigg)^{(|a|+1)b} \\ &\quad \le (|a|+1)^{|b|} C_{a+b-ab} W_a(R) \bigg(\int_0^R \frac{1}{W_a(\rho)} d\rho + 1 \bigg)^b, \end{aligned}$$

as required. □

3. Infinitely iterated logarithm.

In this section, we introduce arbitrarily and infinitely iterated logarithm functions in connection with class $\mathcal{F}$. The definition is different from that of [**1**] in the sense that convergence factors are introduced in terms of the parameter $\theta \in (0, 1]$.

Definition 3.1. Let $0 < \theta \le 1$. For nonnegative integers n, the following functions $l_{\theta,n} : [0, \infty) \to \mathbb{R}$ are defined successively by:

$$\begin{aligned} &l_{\theta,0}(r) = 1 + r, \\ &l_{\theta,k}(r) = 1 + \theta \log l_{\theta,k-1}(r), \quad k \ge 1. \end{aligned}$$

Moreover, we define $L_{\theta,k} : [0, \infty) \to \mathbb{R}$ by

$$L_{\theta,k}(r) = \prod_{j=0}^{k} l_{\theta,j}(r).$$

Remark 3.1. For any $k \geq 0$, $l_{\theta,k}(0) = L_{\theta,k}(0) = 1$. Moreover, $l_{\theta,k}(r) \geq 1$ and $L_{\theta,k}(r) \geq 1$ for all $r \geq 0$ since $l_{\theta,k}$ and $L_{\theta,k}$ are increasing functions. Explicitly, the derivative $l'_{\theta,k}$ is given by

$$l'_{\theta,k}(r) = \theta^k \cdot \frac{1}{L_{\theta,k-1}(r)}, \qquad r \geq 0.$$

By a successive use of the elementary inequality $\log(1+r) \leq r$ for $r \geq -1$,

$$0 \leq \log l_{\theta,k}(r) \leq \theta \log l_{\theta,k-1}(r) \leq \cdots \leq \theta^k \log l_{\theta,0}(r), \quad r \geq 0.$$

This implies that for any θ with $0 < \theta < 1$, the series $\sum_{k=0}^{\infty} \log l_{\theta,k}(r)$ converges with estimates

$$0 \leq \sum_{k=0}^{\infty} \log l_{\theta,k}(r) \leq \frac{1}{1-\theta} \log l_{\theta,0}(r), \quad r \geq 0.$$

Definition 3.2. For any θ with $0 < \theta < 1$, L_θ is defined by

$$L_\theta(r) = \prod_{k=0}^{\infty} l_{\theta,k}(r), \qquad r \geq 0.$$

Remark 3.2. By Remark 3.1, if $0 < \theta < 1$, L_θ converges with estimates

$$1 \leq L_\theta(r) \leq (1+r)^{1/(1-\theta)}, \quad r \geq 0.$$

If $\theta = 1$ and $r > 0$, we prove that $L_1(r) = \infty$ by contradiction. Assume that $L_1(r) < \infty$. Then, for any k we have

$$\begin{aligned}\log L_1(r) \geq \log L_{1,k}(r) &= \int_0^r \frac{d}{d\rho}\Bigg(\sum_{j=0}^{k} \log l_{1,j}(\rho)\Bigg) d\rho \\ &= \int_0^r \sum_{j=0}^{k} \frac{1}{L_{1,j}(\rho)} d\rho \\ &\geq \int_0^r \sum_{j=0}^{k} \frac{1}{L_{1,k}(r)} d\rho = r \sum_{j=0}^{k} \frac{1}{L_{1,k}(r)} \geq \frac{(k+1)r}{L_1(r)},\end{aligned}$$

which yields a contradiction for k sufficiently large.

The main theorem in this section now reads:

Theorem 3.1. *For any θ with $0 < \theta < 1$, $L_\theta \in \mathcal{F}$. Moreover,*

$$\int_0^\infty \frac{1}{L_\theta(r)} dr = \infty. \tag{3.0}$$

To prove Theorem 3.1, we introduce some preliminary propositions. From now on, θ denotes a real number with $0 < \theta < 1$ without particular comments.

Lemma 3.1. *For any $a \in \mathbb{R}$, there exists $C_{\theta,a} \geq 1$ such that for any r and R with $0 \leq r \leq R$*

$$(1+r)\biggl(\int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1\biggr)^a \leq C_{\theta,a}(1+R)\biggl(\int_0^R \frac{1}{L_\theta(\rho)} d\rho + 1\biggr)^a \tag{3.1}$$

holds.

Proof. For $a \geq 0$, (3.1) holds with $C_a = 1$ by monotonicity. Let $a < 0$ and let m_θ be defined by

$$m_\theta(r) = \int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1.$$

Then,

$$m_\theta'(R) = \frac{1}{L_\theta(R)} \leq \frac{m_\theta(r)}{l_{\theta,1}(R) l_{\theta,0}(R)} \leq \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} l_{\theta,1}'(R). \tag{3.2}$$

By (3.2), we have

$$\begin{aligned} m_\theta(R) &= m_\theta(r) + \int_r^R m_\theta'(\rho) d\rho \\ &\leq m_\theta(r) + \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} \int_r^R l_{\theta,1}'(\rho) d\rho \\ &= m_\theta(r) + \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} \bigl(l_{\theta,1}(R) - l_{\theta,1}(r)\bigr) \\ &\leq \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} l_{\theta,1}(R). \end{aligned} \tag{3.3}$$

By Remark 3.1 and (3.3), we obtain

$$\begin{aligned} (1+r) m_\theta(r)^a &= \biggl(\frac{m_\theta(r)}{l_{\theta,1}(r)}\biggr)^a (1+r)\bigl(l_{\theta,1}(r)\bigr)^a \\ &\leq C\biggl(\frac{m_\theta(r)}{l_{\theta,1}(r)}\biggr)^a (1+R)\bigl(l_{\theta,1}(R)\bigr)^a \\ &\leq C\theta^a (1+R) m_\theta(R)^a \end{aligned}$$

with some constant C, as required. □

Lemma 3.2. *For any $r, s \geq 0$,*

$$L_\theta(l_{\theta,0}(s)r) \leq L_\theta(s)L_\theta(r). \tag{3.4}$$

Proof. It is sufficient to prove that

$$l_{\theta,k}(l_{\theta,0}(s)r) \leq l_{\theta,k}(s)l_{\theta,k}(r) \qquad (3.5)_k$$

by induction on $k \geq 0$. For $k = 0$,

$$l_{\theta,0}(l_{\theta,0}(s)r) = 1 + l_{\theta,0}(s)r = 1 + (1+s)r \leq (1+s)(1+r) = l_{\theta,0}(s)l_{\theta,0}(r).$$

Let $k \geq 1$ and assume $(3.5)_{k-1}$. Then,

$$\begin{aligned} l_{\theta,k}(l_{\theta,0}(s)r) &= 1 + \theta \log\big(l_{\theta,k-1}(l_{\theta,0}(s)r)\big) \\ &\leq 1 + \theta \log\big(l_{\theta,k-1}(s)l_{\theta,k-1}(r)\big) \\ &\leq \big(1 + \theta \log l_{\theta,k-1}(s)\big)\big(1 + \theta \log l_{\theta,k-1}(r)\big) \\ &\leq l_{\theta,k}(s)l_{\theta,k}(r), \end{aligned}$$

which completes the induction argument. □

Lemma 3.3. *For any nonnegative integers k and j, $l_{\theta,k+j}$ is represented by $l_{\theta,k}$ and $l_{\theta,j}$ as*

$$l_{\theta,k+j}(r) = l_{\theta,j}\big(l_{\theta,k}(r) - 1\big) \tag{3.6}$$

for all $r \geq 0$.

Proof. We prove (3.6) by induction on j. For $j = 0$, we have

$$l_{\theta,k}(r) = l_{\theta,0}\big(l_{\theta,k}(r) - 1\big)$$

for all $k \geq 0$ by definition. Let $j \geq 1$ and assume that

$$l_{\theta,k+j-1}(r) = l_{\theta,j-1}\big(l_{\theta,k}(r) - 1\big)$$

holds for all $k \geq 0$ and $r \geq 0$. Then,

$$\begin{aligned} l_{\theta,k+j}(r) &= 1 + \theta \log\big(l_{\theta,k+j-1}(r)\big) \\ &= 1 + \theta \log\big(l_{\theta,j-1}(l_{\theta,k}(r) - 1)\big) \\ &= l_{\theta,j}\big(l_{\theta,k}(r) - 1\big) \end{aligned}$$

for all $k \geq 0$ and $r \geq 0$. This completes the induction argument. $\square$

Proof of Theorem 3.1. Let r, R satisfy $0 \leq r \leq R$. Then, by Lemma 3.1,

$$L_\theta(r)\biggl(\int_0^r \frac{1}{L_\theta(\rho)}d\rho + 1\biggr)^a \leq (1+r)\biggl(\prod_{k=1}^{\infty} l_{\theta,k}(R)\biggr)\biggl(\int_0^r \frac{1}{L_\theta(\rho)}d\rho + 1\biggr)^a$$
$$\leq C_{\theta,a} L_\theta(R)\biggl(\int_0^R \frac{1}{L_\theta(\rho)}d\rho + 1\biggr)^a.$$

Moreover, since $l_{\theta,0}(1) = 2$, we apply (3.4) with $s = 1$ to obtain

$$L_\theta(2r) \leq L_\theta(1) L_\theta(r).$$

Therefore, $L_\theta \in \mathcal{F}$. We prove (3.0). It suffices to prove that there exists a sequence $\{r_k; k \geq 0\}$ of positive numbers such that

$$\int_0^{r_k} \frac{1}{L_\theta(\rho)}d\rho \to \infty$$

as $k \to \infty$. Let $r_0 = 1$. Then, for any $k \geq 1$ there exists a unique $r_k > 0$ such that $l_{\theta,k}(r_k) = l_{\theta,0}(r_0) = 2$ since $l_{\theta,k}$ is an increasing function with $l_{\theta,k}(0) = 1$ and $\lim_{r\to\infty} l_{\theta,k}(r) = \infty$. Let $0 \leq \rho \leq r_k$. By Lemma 3.3,

$$\begin{aligned}
L_\theta(\rho) &= L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,k+j}(\rho) \\
&\leq L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,k+j}(r_k) \\
&= L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,j}\bigl(l_{\theta,k}(r_k) - 1\bigr) \\
&= L_{\theta,k-1}(\rho)\, L_\theta\bigl(l_{\theta,k}(r_k) - 1\bigr) \\
&= L_{\theta,k-1}(\rho)\, L_\theta\bigl(l_{\theta,0}(r_0) - 1\bigr) \\
&= L_{\theta,k-1}(\rho)\, L_\theta(1).
\end{aligned} \tag{3.7}$$

By (3.7),

$$\begin{aligned}
\int_0^{r_k} \frac{1}{L_\theta(\rho)}d\rho &\geq \frac{1}{L_\theta(1)} \int_0^{r_k} \frac{1}{L_{\theta,k-1}(\rho)}d\rho \\
&= \frac{1}{L_\theta(1)} \frac{1}{\theta^k}\bigl(l_{\theta,k}(r_k) - 1\bigr)
\end{aligned}$$

$$= \frac{1}{L_\theta(1)} \frac{1}{\theta^k} \to \infty$$

as $k \to \infty$, as required. □

4. Optimality of Theorems 1.2 and 1.3.

In this section, we consider optimality of Theorems 1.2 and 1.3. To this end, we divide weight functions $w \in \mathcal{F}$ into two cases:

$$\mathrm{I}: \int_0^\infty \frac{1}{w(r)} dr < \infty. \qquad \mathrm{II}: \int_0^\infty \frac{1}{w(r)} dr = \infty.$$

Theorem 4.1. *Let $2 \le p < \infty$ and let $w \in \mathcal{F}$. Let w_0, w_1, w_2 be as in Theorem* 1.2 *with a, b, $c \in \mathbb{R}$.*

(1) *In the case I, the trilinear estimate in Theorem* 1.2 *holds for any a, b, $c \in \mathbb{R}$.*
(2) *In the case II, let a, b, c satisfy one of the conditions* (iii), (iv), (v), (vi):

(iii) $a+b+c < 1/p$. (iv) $a+b<0$. (v) $a+c<0$.
(vi) $a+b+c = 1/p$ *and* "$a+b=0$ *or* $a+c=0$".

Then, the trilinear estimate in Theorem 1.2 *fails for some $f \in L^p(\mathbb{R}^n)$, g, $h \in L^{p'}(\mathbb{R}^n)$.*

Remark 4.1. The conditions (iii), (iv), (v), and (vi) in Theorem 4.1 consist of the negation of the condition "(i) or (ii)" in Theorem 1.2.

Proof. In the case I, we easily see the trilinear estimate holds with any a, b, and c. To give a counter example for the trilinear estimate in the case II, we divide the proof into three cases:

(i) $a+b+c<1/p$. (ii) $a+b<0$ or $a+c<0$.
(iii) $a+b+c=1/p$ and "$a+b=0$ or $a+c=0$".

(i) In the case where $a+b+c<1/p$, let $\delta > 0$ satisfy $\delta \neq 1/p - c$ and let

$$f(x) = (1+|x|)^{-(n-1)/p} w(|x|)^{-1/p} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-\delta},$$

$$g(x) = h(x) = (1+|x|)^{-(n-1)/p'} w(|x|)^{-1/p'} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-\delta}.$$

Then, $f \in L^p(\mathbb{R}^n)$ and $g, h \in L^{p'}(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$ with $|x| \ge 2$,

$$\int_{1 \le |y| \le |x|/2} w_0(|x+y|) f(x+y) w_2(|y|) h(y) dy$$
$$= \int_{1 \le |y| \le |x|/2} \left(\int_0^{|x+y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta}$$

$$\cdot (1+|y|)^{-(n-1)} \frac{1}{w(|y|)} \left(\int_0^{|y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-c-\delta} dy. \tag{4.1}$$

By (A1), if $1/p + a + \delta \geq 0$, then for any $y \in \mathbb{R}^n$ with $0 \leq |y| \leq |x|/2$,

$$\begin{aligned}
&\left(\int_0^{|x+y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad \geq \left(\int_0^{3|x|/2} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad = \left(\frac{3}{2} \int_0^{|x|} \frac{1}{w(3\rho/2)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad \geq \left(\frac{3C_0}{2} \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad \geq \left(\frac{3C_0}{2} + 1 \right)^{-1/p-a-\delta} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta}.
\end{aligned} \tag{4.2}$$

Similarly, if $1/p + a + \delta < 0$, then for any $y \in \mathbb{R}^n$ with $0 \leq |y| \leq |x|/2$,

$$\begin{aligned}
&\left(\int_0^{|x+y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad \geq \left(\int_0^{|x|/2} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad = \left(\frac{1}{2} \int_0^{|x|} \frac{1}{w(\rho/2)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad \geq \left(\frac{1}{2C_0} \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta} \\
&\quad \geq (2C_0)^{1/p+a+\delta} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a-\delta}.
\end{aligned} \tag{4.3}$$

In addition, if $1/p - c - \delta > 0$, then for any $x \in \mathbb{R}^n$ with $|x| \geq 4$,

$$\begin{aligned}
&\int_{1 \leq |y| \leq |x|/2} (1+|y|)^{-(n-1)} \frac{1}{w(|y|)} \left(\int_0^{|y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-c-\delta} dy \\
&= \omega_{n-1} \int_1^{|x|/2} \left(\frac{r}{1+r} \right)^{n-1} \frac{1}{w(r)} \left(\int_0^{r} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-c-\delta} dr \\
&\geq \frac{2^{1-n} \omega_{n-1}}{1/p - c - \delta} \left(\left(\int_0^{|x|/2} \frac{1}{w(r)} dr + 1 \right)^{1/p-c-\delta} - \left(\int_0^{1} \frac{1}{w(r)} dr + 1 \right)^{1/p-c-\delta} \right)
\end{aligned}$$

$$\geq \frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(1-\left(\frac{\int_0^1 \frac{1}{w(r)}dr+1}{\int_0^2 \frac{1}{w(r)}dr+1}\right)^{1/p-c-\delta}\right)\left(\int_0^{|x|/2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}$$

$$= \frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(1-\left(\frac{\int_0^1 \frac{1}{w(r)}dr+1}{\int_0^2 \frac{1}{w(r)}dr+1}\right)^{1/p-c-\delta}\right)\left(\frac{1}{2}\int_0^{|x|}\frac{1}{w(r/2)}dr+1\right)^{1/p-c-\delta}$$

$$\geq \frac{2^{1/p'-c-\delta-n}\omega_{n-1}}{(1/p-c-\delta)C_0^{1/p-c-\delta}}\left(1-\left(\frac{\int_0^1 \frac{1}{w(r)}dr+1}{\int_0^2 \frac{1}{w(r)}dr+1}\right)^{1/p-c-\delta}\right)$$
$$\cdot\left(\int_0^{|x|}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}. \tag{4.4}$$

If $1/p-c-\delta<0$, then

$$\int_{1\leq|y|\leq|x|/2}(1+|y|)^{-(n-1)}\frac{1}{w(|y|)}\left(\int_0^{|y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-c-\delta}dy$$
$$= \frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(\left(\int_0^1\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}-\left(\int_0^{|x|/2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\right)$$
$$\geq \frac{2^{1-n}\omega_{n}}{1/p-c-\delta}\left(\left(\int_0^1\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\right)$$
$$\cdot\left(\int_0^{|x|}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}. \tag{4.5}$$

By (4.1), (4.2), (4.3), (4.5), and (4.5), there exists a positive constant C such that for any $x\in\mathbb{R}^n$ with $|x|\geq 4$

$$\int_{1\leq|y|\leq|x|/2} w_0(|x+y|)f(x+y)w_2(|y|)h(y)dy$$
$$\geq C\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-a-c-2\delta}. \tag{4.6}$$

Finally by (4.6), we have

$$\iint w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy$$
$$\geq C\int_{|x|\geq 4}(|x|+1)^{-(n-1)}\frac{1}{w(|x|)}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-a-b-c-3\delta}dx$$
$$\geq C\omega_{n-1}\left(\frac{4}{5}\right)^{n-1}\int_4^\infty\frac{1}{w(r)}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-a-b-c-3\delta}dr$$

$$
\begin{aligned}
&\geq C\omega_{n-1}\left(\frac{4}{5}\right)^{n-1}\left(\log\left(\int_0^\infty \frac{1}{w(\rho)}d\rho+1\right)-\log\left(\int_0^4 \frac{1}{w(\rho)}d\rho+1\right)\right)\\
&=\infty
\end{aligned}
$$

with $\delta \leq (1/p-a-b-c)/3$.

(ii) In the case where $a+b<0$ or $a+c<0$, by symmetry, it is sufficient to give a counter example only in the case where $a+b<0$. Let f and g be as in the case (iii) with $\delta \leq -(a+b)/2$ and $a+1/p+\delta \neq 1$. Let

$$
h(x)=\chi_{B(1)}(x)\frac{1}{w_2(|x|)}.
$$

Then, by (1.15), (4.4), and (4.5),

$$
\begin{aligned}
&\iint w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\;dy\,dx\\
&\geq \int_{|x|\geq 2}\int_{|y|\leq 1}\left(\int_0^{|x+y|}\frac{1}{w(\rho)}d\rho+1\right)^{-a-1/p-\delta}dy\\
&\qquad\cdot(1+|x|)^{n-1}\frac{1}{w(|x|)}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-b-1/p'-\delta}dx\\
&\geq C\int_2^\infty \frac{1}{w(r)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-b-1-2\delta}dr\\
&\geq C\int_2^\infty \frac{1}{w(r)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-1}dr\\
&\geq C\left(\log\left(\int_0^\infty \frac{1}{w(\rho)}d\rho+1\right)-\log\left(\int_0^2 \frac{1}{w(\rho)}d\rho+1\right)\right)\\
&=\infty
\end{aligned}
$$

with some positive constant C, as required.

(iii) In the case where $a+b+c=1/p$ and $a+b=0$ or $a+b=c$, by symmetry, it is sufficient to give a counter example in the case where $a+b=0$. Let

$$
\begin{aligned}
J(r)&=\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-1}d\rho+1,\\
f(x)&=(1+|x|)^{-(n-1)/p}w(|x|)^{-1/p}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p}J(|x|)^{-1/p-\delta},
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= h(x) \\
&= (1+|x|)^{-(n-1)/p'} w(|x|)^{-1/p'} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'} J(|x|)^{-1/p'-\delta}
\end{aligned}
$$

for $\delta > 0$. By (A1),

$$
\begin{aligned}
J(2r) &= \int_0^{2r} \frac{1}{w(\rho)} \left(\int_0^{\rho} \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-1} d\rho + 1 \\
&\le \int_0^{r} \frac{1}{w(\rho)} \left(\int_0^{\rho} \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-1} d\rho + 1 \\
&\quad + \int_0^{r} \frac{1}{w(r+\rho)} \left(\int_0^{\rho} \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-1} d\rho + 1 \\
&\le (1+C_0) J(r). \qquad (4.7)
\end{aligned}
$$

In addition, with any $k \ge 0$ let $r_k > 0$ satisfy

$$
\int_0^{r_k} \frac{1}{w(\rho)} d\rho = 2^k - 1,
$$

where r_k is determined uniquely, since $\int_0^r 1/w(\rho) d\rho$ is a monotone increasing function of r. Then, we estimate

$$
\begin{aligned}
J(r_k) &= \sum_{j=1}^{k} \int_{r_{j-1}}^{r_j} \frac{1}{w(\rho)} \left(\int_0^{\rho} \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-1} d\rho \\
&\ge \sum_{j=1}^{k} \left(\int_0^{r_j} \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-1} \left(\int_0^{r_j} \frac{1}{w(\rho)} d\rho - \int_0^{r_{j-1}} \frac{1}{w(\rho)} d\rho \right) \\
&= \sum_{j=1}^{k} 2^{-j} (2^j - 2^{j-1}) = \frac{k}{2}. \qquad (4.8)
\end{aligned}
$$

This shows $\lim_{r\to\infty} J(r) = \infty$. By (4.4), (4.5), and (4.7), for any $x \in \mathbb{R}^n$ with $|x| \ge 4$ and $0 < \delta < 1/p$

$$
\begin{aligned}
&\int_{1\le|y|\le|x|/2} w_0(|x+y|) f(x+y) w_2(|y|) h(y) dy \\
&= \int_{1\le|y|\le|x|/2} \left(\int_0^{|x+y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} J(|x+y|)^{-1/p-\delta} \\
&\quad \cdot (1+|y|)^{-(n-1)} \frac{1}{w(|y|)} \left(\int_0^{|y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1} J(|y|)^{-1/p'-\delta} dy
\end{aligned}
$$

$$
\begin{aligned}
&\geq C\Big(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\Big)^{-1/p-a}J(2|x|)^{-1/p-\delta}\\
&\quad\cdot\int_1^{|x|/2}\frac{1}{w(r)}\Big(\int_0^r\frac{1}{w(\rho)}d\rho+1\Big)^{-1}J(r)^{-1/p'-\delta}dr\\
&\geq C(1+C_0)^{-1/p-\delta}\frac{1}{1/p-\delta}\Big(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\Big)^{-1/p-a}J(|x|)^{-1/p-\delta}\\
&\quad\cdot\big(J(|x|)^{1/p-\delta}-J(1)^{1/p-\delta}\big)\\
&\geq C(1+C_0)^{-1/p-\delta}\frac{1}{1/p-\delta}\big(1-(J(1)/J(2))^{1/p-\delta}\big)\Big(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\Big)^{-1/p-a}\\
&\quad\cdot J(|x|)^{-2\delta} \qquad (4.9)
\end{aligned}
$$

with some positive constant C. Then, by (4.9) and (4.8), for $0<\delta\leq 1/(3p)$, we estimate

$$
\begin{aligned}
&\iint w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\ dy\,dx\\
&\geq\int_{|x|\geq 4}\int_{1\leq|y|\leq|x|/2}w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\ dy\,dx\\
&\geq C(1+C_0)^{-1/p-\delta}\frac{1}{1/p-\delta}\big(1-(J(1)/J(2))^{1/p-\delta}\big)\\
&\quad\cdot\int_4^\infty\frac{1}{w(r)}\Big(\int_0^r\frac{1}{w(\rho)}d\rho+1\Big)^{-1}J(r)^{-1/p'-3\delta}dr\\
&\geq C(1+C_0)^{-1/p-\delta}\frac{1}{1/p-\delta}\big(1-(J(1)/J(2))^{1/p-\delta}\big)\Big(\lim_{r\to\infty}\log J(r)-\log J(4)\Big)\\
&=\infty,
\end{aligned}
$$

as required. □

Theorem 4.2. *Let $w\in\mathcal{F}$ and let w_0, w_1, w_2 be as in Theorem 1.2 with a, b, $c\in\mathbb{R}$.*

(1) *In the case I, the trilinear estimate in Theorem 1.2 holds for any a, b, $c\in\mathbb{R}$*
(2) *In the case II, let a, b, c satisfy either (iii) or (iv) or (v) in Theorem 4.1, then the trilinear estimate in Theorem 1.2 fails for some $f\in L^\infty(\mathbb{R}^n)$, g, $h\in L^1(\mathbb{R}^n)$.*
(3) *In the case II, let $a=b=c=0$. Then, the trilinear estimates holds.*

Proof. The proofs of (1) and (2) are the same as in the proof of Theorem 4.1, while (3) follows from the Hölder and Young inequalities as below:

$$
\begin{aligned}
&\iint w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy\\
&\quad\le \iint |f(x+y)g(x)h(y)|\,dx\,dy\\
&\quad= \iint |f(x)g(x-y)h(y)|\,dy\,dx\\
&\quad\le \|f\|_{L^\infty}\|g*h\|_{L^1}\\
&\quad\le \|f\|_{L^\infty}\ \|g\|_{L^1}\ \|h\|_{L^1}. \qquad \square
\end{aligned}
$$

Acknowledgments. The authors are grateful to the referee for important remarks.

References

[1] H. Ando, T. Horiuchi and E. Nakai, Construction of slowly increasing functions, Sci. Math. Japonicae Online, **e-2012** (2012), 207–221.
[2] H. Bahouri, J.-Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer, 2011.
[3] F. M. Christ and M. I. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation, J. Funct. Anal., **100** (1991), 87–109.
[4] P. D'Ancona, D. Foschi and S. Selberg, Atlas of products for wave-Sobolev space on $\mathbb{R}^{1+3}$, Trans. Amer. Math. Soc., **364** (2012), 31–63.
[5] J. Fan and T. Ozawa, Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the gradient of the pressure, J. Inequal. Appl., **2008**, Art. ID 412678, 6pp.
[6] A. E. Gatto, Product rule and chain rule estimates for fractional derivatives on spaces that satisfy the doubling condition, J. Funct. Anal., **188** (2002), 27–37.
[7] J. Ginibre, T. Ozawa and G. Velo, On the existence of the wave operators for a class of nonlinear Schrödinger equations, Ann. Inst. H. Pincaré Pys. Théor., **60** (1994), 211–239.
[8] T. Kato, On nonlinear Schrödinger equations, II. H^s solutions and unconditional well-posedness, J. Anal. Math., **67** (1995), 281–306.
[9] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., **41** (1988), 891–907.
[10] S. Klainerman and S. Selberg, Bilinear estimates and applications to nonlinear wave equations, Comm. Contemp. Math., **4** (2002), 223–295.
[11] H. Kozono and Y. Taniuchi, Bilinear estimates in BMO and the Navier-Stokes equations, Math. Z., **235** (2000), 173–194.
[12] F. Linares and G. Ponce, Introduction to Nonlinear Dispersive Equations, Springer, 2009.
[13] S. Machihara, K. Nakanishi and K. Tsugawa, Well-posedness for nonlinear Dirac equations in one dimension, Kyoto J. Math., **50** (2010), 403–451.
[14] T. Runst and W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, Walter de Gruyter, 1996.
[15] J.-C. Saut and R. Temam, Remarks on the Korteweg-de-Vries equation, Israel J. Math., **2014** (1976), 78–87.
[16] W. Sickel and H. Triebel, Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type, Z. Anal. Anwendungen, **14** (1995), 105–140.
[17] T. Tao, Multilinear weighted convolutions of L^2 functions, and applications to nonlinear dispersive equations, Amer. J. Math., **123** (2001), 839–908.
[18] T. Tao, Nonlinear Dispersive Equations, Local and Global Analysis, Amer. Math. Soc., 2006.

Kazumasa FUJIWARA
Department of Pure and Applied Physics
Waseda University
Tokyo 169-8555, Japan
E-mail: k-fujiwara@asagi.waseda.jp

Tohru OZAWA
Department of Applied Physics
Waseda University
Tokyo 169-8555, Japan
E-mail: txozawa@waseda.jp