J. Math. Soc. Japan
Vol. 66, No. 4 (2014) pp. 1303–1313
doi: 10.2969/jmsj/06641303

Tunnel number of tangles and knots

By Toshio SAITO

(Received Sep. 5, 2012)
(Revised Nov. 27, 2012)

Abstract. We study bridge number and tunnel number of tangles and knots, and also study their behavior under tangle decomposition of knots.

1. Introduction.

Let K be a *knot*, i.e., a simple closed curve embedded in the 3-sphere S^3 or in a more general 3-manifold. One of the classical and standard splittings of $K \subset S^3$ is a *bridge splitting* introduced by Schubert [**9**]. An *n-bridge splitting* of (S^3, K) is a splitting of a pair of S^3 and the knot K into two pairs of a 3-ball and n mutually trivial arcs. We denote such a bridge splitting by $(S^3, K) = (B_1, K_1) \cup_S (B_2, K_2)$, where each B_i is a 3-ball with $S = \partial B_1 = \partial B_2$ and each $K_i = B_i \cap K$ consists of n mutually trivial arcs in B_i. The *bridge number* $\mathrm{brg}_0(K)$ of $K \subset S^3$ is defined to be the minimal integer b for which (S^3, K) admits a b-bridge splitting. The bridge number is a knot invariant, and the following is well-known Schubert's equality on bridge number:

$$\mathrm{brg}_0(K \# K') = \mathrm{brg}_0(K) + \mathrm{brg}_0(K') - 1,$$

where $K \# K'$ means the connected sum of two knots K and K' in S^3.

The *tunnel number* is another knot invariant introduced by Clark [**1**]. Let K be a knot in a closed connected orientable 3-manifold M. The *tunnel number* $\mathrm{tnl}(K)$ of $K \subset M$ is the minimal number of mutually disjoint arcs τ properly embedded in the knot exterior $\mathrm{Ext}(K; M)$ such that the exterior of τ in $\mathrm{Ext}(K; M)$ is homeomorphic to a handlebody. The following is also well-known Clark's inequality on tunnel number:

$$\mathrm{tnl}(K \# K') \le \mathrm{tnl}(K) + \mathrm{tnl}(K') + 1.$$

It is shown by Morimoto, Sakuma and Yokota [**6**] and independently Moriah and Rubinstein [**4**] that there exist infinitely many pairs of knots $K, K' \subset S^3$ satisfying the equality. If K and K' are so-called $(1,1)$-knots, we see that $\mathrm{tnl}(K \# K') = \mathrm{tnl}(K) + \mathrm{tnl}(K')$. It is also proved by Kobayashi [**3**], by taking connected sum of examples due to Morimoto [**5**], that for any positive integer n, there are infinitely many pairs of knots K and K' with $\mathrm{tnl}(K \# K') < \mathrm{tnl}(K) + \mathrm{tnl}(K') - n$.

In 1970, Conway [**2**] introduced *tangle decomposition* of knots which is a generaliza-

---

2010 *Mathematics Subject Classification.* Primary 57N10; Secondary 57M25.
Key Words and Phrases. knot, tangle, bridge number, tunnel number.
The author was partially supported by Grant-in-Aid for Young Scientists (B) 24740041.

tion of connected sum decomposition of knots. Let K be a knot in a closed connected orientable 3-manifold M and P a separating 2-sphere in M which intersects K transversely in $2n$ points for a positive integer n. Then P cuts M into two 3-manifolds M_1 and M_2 and each $T_i := M_i \cap K$ $(i = 1, 2)$ consists of n mutually disjoint simple arcs properly embedded in M_i. Such a pair (M_i, T_i) is called an *n-tangle*, and $(M_1, T_1) \cup_P (M_2, T_2)$ is called an *n-tangle decomposition* of (M, K). We notice that a 1-tangle decomposition corresponds to connected sum decomposition. In this paper, we study bridge number and tunnel number of tangles (see Section 3 for definitions and details). The following is obtained as corollaries of Theorem 4.1.

Corollary 1.1. *Let K be a knot in S^3 and $(B_1, T_1) \cup_P (B_2, T_2)$ an n-tangle decomposition of (S^3, K). Then*

$$\mathrm{brg}_0(K) \leq \mathrm{brg}_0(T_1) + \mathrm{brg}_0(T_2) - n.$$

Corollary 1.2. *Let K be a knot in a closed connected orientable 3-manifold M and $(M_1, T_1) \cup_P (M_2, T_2)$ an n-tangle decomposition of (M, K). Then*

$$\mathrm{tnl}(K) \leq \mathrm{tnl}(T_1) + \mathrm{tnl}(T_2) + 2n - 1.$$

For example, Morimoto's knot $K_M(l, m, n) \subset S^3$ admits a 2-tangle decomposition $(B_1, T_1) \cup_P (B_2, T_2)$ illustrated in Figure 1. It follows from Ozawa's result [**7**] that this is a unique essential 2-tangle decomposition. We obtain in Section 3 that each 2-tangle (B_i, T_i) satisfies $\mathrm{brg}_0(T_i) = 3$. Hence we see $\mathrm{brg}_0(K_M(l, m, n)) \leq 4$ by Corollary 1.1 (or by deforming the diagram in Figure 1 directly). It follows from [**8**] that $\mathrm{brg}_0(K_M(2, 1, 1)) > 3$ and hence $\mathrm{brg}_0(K_M(2, 1, 1)) = 4$ which implies the equality holds for $K = K_M(2, 1, 1)$ and its essential tangle decomposition. We notice that each 2-tangle (B_i, T_i) in Figure 1 also satisfies $\mathrm{tnl}(T_i) = 0$. Hence Corollary 1.2 implies $\mathrm{tnl}(K_M(l, m, n)) \leq 3$. We, however, have already known that $K_M(l, m, n)$ is of tunnel number two. We give in Section 5 a sufficient condition not to satisfy the equality in Corollary 1.2.

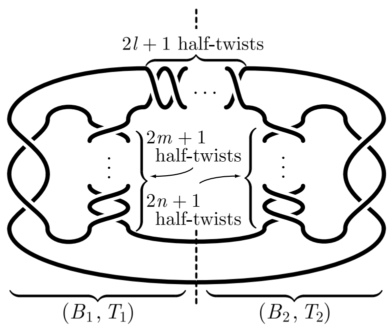

Figure 1. Morimoto's knot $K_M(l, m, n) \subset S^3$ with $l, m, n \in \mathbb{Z}_{>0}$.

2. Preliminaries.

Throughout this paper, we work in the piecewise linear category. Let B be a sub-manifold of a manifold A. The notation $\mathrm{Nbd}(B;A)$ denotes a (closed) regular neighborhood of B in A. By $\mathrm{Ext}(B;A)$, we mean the *exterior* of B in A, i.e., $\mathrm{Ext}(B;A) = \mathrm{cl}(A \setminus \mathrm{Nbd}(B;A))$, where $\mathrm{cl}(\cdot)$ means the closure. The notation $|\cdot|$ indicates the number of connected components. Let M be a compact connected orientable 3-manifold with non-empty boundary. Let J be a 1-manifold properly embedded in M and F a surface properly embedded in M. Here, a *surface* means a connected compact 2-manifold. We always assume that a surface intersects J transversely. Set $\mathcal{M} = (M, J)$ and $\mathcal{F} = (F, F \cap J)$. For convenience, we also call $\mathcal{F}$ a *surface*. A simple closed curve properly embedded in $F \setminus J$ is said to be *inessential* in $\mathcal{F}$ if it bounds a disk in F intersecting J in at most one point. A simple closed curve properly embedded in $F \setminus J$ is said to be *essential* in $\mathcal{F}$ if it is not inessential in $\mathcal{F}$. A surface $\mathcal{F}$ is *compressible* in $\mathcal{M}$ if there is a disk $D \subset M \setminus J$ such that $D \cap F = \partial D$ and ∂D is essential in $\mathcal{F}$. Such a disk D is called a *compressing disk* of $\mathcal{F}$. We say that $\mathcal{F}$ is *incompressible* in $\mathcal{M}$ if $\mathcal{F}$ is not compressible in $\mathcal{M}$.

A 3-manifold C is called a (genus g) *compression body* if there exists a closed surface F of genus g such that C is obtained from $F \times [0,1]$ by attaching 2-handles along mutually disjoint loops in $F \times \{0\}$ and filling in some resulting 2-sphere boundary components with 3-handles. We denote $F \times \{1\}$ by $\partial_+ C$ and $\partial C \setminus \partial_+ C$ by $\partial_- C$. A compression body C is called a *handlebody* if $\partial_- C = \emptyset$. The triplet $(C_1, C_2; S)$ is called a (genus g) *Heegaard splitting* of M if C_1 and C_2 are (genus g) compression bodies with $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = S$. The Heegaard genus $\mathrm{hg}(M)$ of M is the minimal integer g for which M admits a genus g Heegaard splitting.

A simple arc γ properly embedded in a compression body C is said to be *vertical* if γ is isotopic to an arc with $\{\text{a point}\} \times [0,1] \subset \partial_- C \times [0,1]$ relative to boundary. A simple arc γ properly embedded in C is said to be *trivial* if there is a disk δ in C with $\gamma \subset \partial\delta$ and $\partial\delta \setminus \gamma \subset \partial_+ C$. Such a disk δ is called a *bridge disk* of γ. A disjoint union of trivial arcs is said to be *mutually trivial* if they admit a disjoint union of bridge disks.

2.1. Bridge number and tunnel number of knots.

Let K be a *knot*, i.e., a closed connected 1-manifold in a compact connected orientable 3-manifold M. We say that K admits a $(g,0)$-*bridge splitting* if there is a genus g Heegaard splitting $(C_1, C_2; S)$ of M such that $K \subset C_i$ ($i = 1$ or 2), say $i = 2$, and that $\mathrm{cl}(C_2 \setminus K)$ is a compression body. We say that K admits a (g,b)-*bridge splitting* ($b > 0$) if there is a genus g Heegaard splitting $(C_1, C_2; S)$ of M such that $C_i \cap K$ consists of b arcs which are mutually trivial for each $i = 1, 2$. Set $\mathcal{C}_i = (C_i, C_i \cap K)$ and $\mathcal{S} = (S, S \cap K)$. We call the triplet $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a (g,b)-*bridge splitting* of (M, K) and $\mathcal{S}$ is called a (g,b)-*bridge surface*, or a *bridge surface* for short. The *genus g bridge number* $\mathrm{brg}_g(K)$ of $K \subset M$ is defined to be the minimal integer b for which (M, K) admits a (g,b)-bridge splitting. We notice that $\mathrm{brg}_0(K)$ is well-defined only if $K \subset S^3$ and is the classical bridge number.

Definition 2.1. Let K be a knot in a closed connected orientable 3-manifold M. A disjoint union of simple arcs $\tau = \tau_1 \cup \cdots \cup \tau_n$ properly embedded in $\mathrm{Ext}(K;M)$ is called an *unknotting tunnel system* if $\mathrm{cl}(\mathrm{Ext}(K;M) \setminus \mathrm{Nbd}(\tau;M))$ is a handlebody. The *tunnel* *number* $\mathrm{tnl}(K)$ of $K \subset M$ is the minimal number of components of such unknotting tunnel systems.

The tunnel number $\mathrm{tnl}(K)$ of $K \subset M$ is equivalent to the minimal integer t for which (M, K) admits a $(t+1, 0)$-bridge splitting.

2.2. C-compression bodies and c-Heegaard splittings.

We now recall definitions of a *c-compression body* and a *c-Heegaard splitting* given by Tomova [**10**]. Let J be a 1-manifold properly embedded in a compact connected orientable 3-manifold M with non-empty boundary. A surface $\mathcal{F} = (F, F \cap J)$ is *c-compressible in* $\mathcal{M} = (M, J)$ if there is a disk $D \subset M \setminus J$ such that $D \cap F = \partial D$, ∂D is essential in $\mathcal{F}$ and D intersects J in at most one point. If $|D \cap J| = 1$, then D is called a *cut disk of* $\mathcal{F}$. We say that $\mathcal{F}$ is *c-incompressible in* $\mathcal{M}$ if $\mathcal{F}$ is not c-compressible in $\mathcal{M}$. A *c-disk* is a compressing disk or a cut disk.

Let $\mathcal{C}$ be a pair of a genus g compression body C and a 1-manifold J properly embedded in C. Then $\mathcal{C} = (C, J)$ is called a (genus g) *c-compression body* if there is a disjoint union $\mathbb{D}$ of c-disks and bridge disks which cuts $\mathcal{C}$ into some 3-balls and a 3-manifold homeomorphic to $\partial_{-}C \times [0, 1]$ with vertical arcs. Then $\mathbb{D}$ is called a *complete c-disk system* of $\mathcal{C}$. If $\mathbb{D}$ contains a compressing disk, then $\mathcal{C}$ is said to be *compressible*. We set $\partial_{\pm}\mathcal{C} = (\partial_{\pm}C, \partial_{\pm}C \cap J)$.

Definition 2.2. Let J be a 1-manifold properly embedded in a compact connected orientable 3-manifold M. The triplet $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ is a (genus g) *c-Heegaard splitting* of $\mathcal{M} = (M, J)$ if $\mathcal{C}_1$ and $\mathcal{C}_2$ are (genus g) c-compression bodies with $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{M}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \partial_+\mathcal{C}_1 = \partial_+\mathcal{C}_2 = \mathcal{S}$. The surface $\mathcal{S}$ is called a *c-Heegaard surface* of $\mathcal{M}$.

3. Bridge number and tunnel number of tangles.

Let M be a compact connected orientable 3-manifold with $\partial M \cong S^2$ and T a 1-manifold properly embedded in M. We say that (M, T) is an *n-tangle* if T consists of n arcs. An n-tangle (M, T) is said to be *essential* if the surface $(\partial M, \partial M \cap T)$ is incompressible in (M, T). An n-tangle (M, T) is said to be *free* if $\mathrm{Ext}(T; M)$ is a handlebody. A free n-tangle (M, T) admits a c-Heegaard splitting $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ such that $\mathcal{C}_i$ is ambient isotopic to $\mathrm{Nbd}(\partial M \cup T; M)$ and that $\mathcal{C}_j$ is a genus n handlebody disjoint from T for $(i, j) = (1, 2)$ or $(2, 1)$.

Definition 3.1. Let (M, T) be an n-tangle. A disjoint union of simple arcs $\tau = \tau_1 \cup \cdots \cup \tau_n$ properly embedded in $\mathrm{Ext}(T; M)$ is called an *unknotting tunnel system* if $\mathrm{cl}(\mathrm{Ext}(T; M) \setminus \mathrm{Nbd}(\tau; M))$ is a handlebody. The *tunnel number* $\mathrm{tnl}(T)$ of (M, T) is the minimal number of components of such unknotting tunnel systems. In particular, we define $\mathrm{tnl}(T) = 0$ if (M, T) is a free tangle.

Proposition 3.2. *Let M be a closed connected orientable 3-manifold and K a knot in M with $\mathrm{tnl}(K) = t + 1$. Then there is an open 3-ball $B \subset M$ such that $(M \setminus B, K \setminus B)$ is a 2-tangle with $\mathrm{tnl}(T) = t$, where $T = K \setminus B$.*

Proof. Let τ be an unknotting tunnel system of $K \subset M$ realizing the tunnel number and τ_0 a component of τ. We can naturally extend each component τ_i of τ into $\mathrm{Nbd}(K; M)$ so that τ_i is a simple arc in M joining K to itself. A small regular neighborhood B_0, which is a 3-ball, of τ_0 cuts off two sub-arcs γ_1 and γ_2 from K. Removing the interior of $(B_0, \gamma_1 \cup \gamma_2)$ from (M, K), we obtain a 2-tangle (M', T_0). Since $K \subset M$ is of tunnel number $t + 1$, we see that the 2-tangle (M', T_0) must be of tunnel number t and hence we have a desired 2-tangle. □

EXAMPLE 3.3. (1) Let $K_{l,m} \subset S^3$ be the $(-2, 2l + 1, 2m + 1)$-pretzel knot with $l > 0$. It is known that $K_{l,m}$ is of tunnel number one and that τ illustrated in Figure 2(a) is an unknotting tunnel of $K_{l,m}$. For any integer m, by removing a regular neighborhood of τ, we get a 2-tangle (B^3, T_l) as in Figure 2(a). By Proposition 3.2, we have $\mathrm{tnl}(T_l) = 0$ and hence (B^3, T_l) is a free tangle.

(2) The 2-tangle (B^3, T'_n) in Figure 2(b) comes from the knot K_n $(n > 0)$ illustrated in Figure 2(b). We notice that K_1 is the knot 8_{16} in the Rolfsen's knot table and that $\tau_1 \cup \tau_2$ in Figure 2(b) is an unknotting tunnel system of K_n. Since K_n admits an essential 2-tangle decomposition, we see that K_n is of tunnel number two. This implies that $\mathrm{tnl}(T'_n) = 1$.

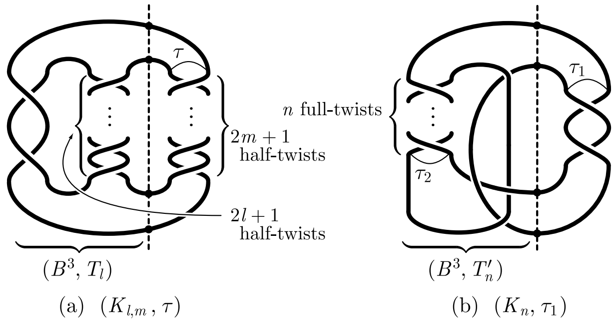

Figure 2. (a) The 2-tangles (B^3, T_l) with $l > 0$ are of tunnel number zero.
(b) The 2-tangles (B^3, T'_n) with $n > 0$ are of tunnel number one.

Let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a c-Heegaard splitting of an n-tangle (M, T) with $\partial M = \partial_- C_i$ for $i = 1$ or 2, say $i = 2$, where $\mathcal{C}_i = (C_i, C_i \cap T)$ and $\mathcal{S} = (S, S \cap T)$. Then we notice that C_1 is a handlebody and $C_1 \cap T$ consists of mutually trivial arcs. Such a c-Heegaard splitting $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ is called a *(g, b, c)-splitting* of (M, T), where g is the genus of the closed surface S, b is the number of trivial arcs $C_1 \cap T$ and c is the number of the components of T each of which is contained in C_2. For example, a free n-tangle admits an $(n, 0, n)$-splitting, and an n-tangle of tunnel number t admits a $(t + n, 0, n)$-splitting. Using these words, we can say that the tunnel number $\mathrm{tnl}(T)$ of an n-tangle (M, T) is the minimal integer t for which (M, T) admits a $(t + n, 0, n)$-splitting. The *genus g bridge number* $\mathrm{brg}_g(T)$ of an n-tangle (M, T) is defined to be the minimal integer b for which (M, T) admits a $(g, b, 0)$-splitting. We notice that $\mathrm{brg}_0(T) \geq n$ for any n-tangle (B^3, T). Moreover an n-tangle T with $\mathrm{brg}_0(T) = n$ is *trivial*, i.e., T is n mutually trivial arcs in B^3.

EXAMPLE 3.4. Each of the 2-tangles (B^3, T_l) and (B^3, T'_n) in Figure 2 admits a $(0, 3, 0)$-splitting. The 2-spheres S and S' in Figure 3 give $(0, 3, 0)$-splittings. Since both

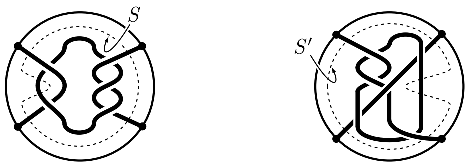

Figure 3. Non-trivial 2-tangles each with a $(0, 3, 0)$-splitting.

tangles are non-trivial, we see that $\mathrm{brg}_0(T_l) = 3$ and $\mathrm{brg}_0(T'_n) = 3$.

Suppose $c > 0$ for a (g, b, c)-splitting $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$. Then we obtain a $(g, b+1, c-1)$-splitting of by *push-out operation* as follows. Since $c > 0$, there is an arc component γ of $C_2 \cap T$ which is entirely contained in C_2. Let p be a single point in the interior of γ. Then we can isotope γ (relative to boundary) so that $\mathrm{Nbd}(p; \gamma)$ is out of C_2. This implies that we obtain a $(g, b+1, c-1)$-splitting of (M, T) from its (g, b, c)-splitting.

Lemma 3.5. *Let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a (g, b, c)-splitting of an n-tangle (M, T) with $\partial M = \partial_- C_2$, where $\mathcal{C}_i = (C_i, C_i \cap T)$ and $\mathcal{S} = (S, S \cap T)$. Then*

1. *the number of vertical arc components in $C_2 \cap T$ is $2n - 2c$, and*
2. *the number of trivial arc components in $C_2 \cap T$ is $b + c - n$.*

Proof. We first notice that $\partial M \cap T (\subset \partial_- C_2)$ consists of $2n$ points. Hence $2n - 2c$ points of them are endpoints of vertical arc components in $C_2 \cap T$. Since T intersects S in $2b$ points, we see that $2b - (2n - 2c)$ points of them are endpoints of trivial arc components in $C_2 \cap T$. $\square$

Definition 3.6. Let K be a knot in a closed connected orientable 3-manifold M and $P \subset M$ a separating 2-sphere which intersects K transversely in $2n (> 0)$ points. Then P cuts M into two 3-manifolds M_1 and M_2 so that (M_i, T_i) $(i = 1, 2)$ are n-tangles, where $T_i = M_i \cap K$. The decomposition $(M_1, T_1) \cup_P (M_2, T_2)$ is called an *n-tangle decomposition*, or a *tangle decomposition* for short. A tangle decomposition $(M_1, T_1) \cup_P (M_2, T_2)$ is said to be *essential* if each tangle (M_i, T_i) is essential.

4. Amalgamating c-Heegaard splittings of tangle decompositions.

Theorem 4.1. *Let K be a knot in a closed connected orientable 3-manifold M and $(M_1, T_1) \cup_P (M_2, T_2)$ an n-tangle decomposition of (M, K). If each (M_i, T_i) $(i = 1, 2)$ admits a (g_i, b_i, c_i)-splitting, then (M, K) admits a $(g_1 + g_2, b_1 + b_2 + \min\{c_1, c_2\} - n)$-bridge splitting.*

Proof. Without loss of generality, we may assume $c_1 \leq c_2$. We notice that $T_i = M_i \cap K$ $(i = 1, 2)$. Since (M_1, T_1) admits a (g_1, b_1, c_1)-splitting, we obtain a $(g_1, b_1 + c_1, 0)$-splitting of (M_1, T_1) by push-out operation. Let $(\mathcal{C}_{11}, \mathcal{C}_{12}; \mathcal{S}_1)$ be a $(g_1, b_1 + c_1, 0)$-splitting of (M_1, T_1) such that $\mathcal{C}_{11}$ is a pair of a genus g_1 handlebody C_{11} and $C_{11} \cap K$, and that $\mathcal{C}_{12}$ is a pair of a compression body C_{12} with $\partial_- C_{12} = \partial M_1$ and $C_{12} \cap K$. We notice that $C_{11} \cap K$ consists of $b_1 + c_1$ mutually trivial arcs and that $C_{12} \cap K$ consists of $2n$ vertical arcs and $b_1 + c_1 - n$ (possibly zero) mutually trivial arcs (cf. Lemma 3.5). Similarly, let $(\mathcal{C}_{21}, \mathcal{C}_{22}; \mathcal{S}_2)$ be a (g_2, b_2, c_2)-splitting of (M_2, T_2) such that $\mathcal{C}_{21}$ is a pair of a compression body C_{21} with $\partial_- C_{21} = \partial M_2$ and $C_{21} \cap K$, and that $\mathcal{C}_{22}$ is a pair of a genus g_2 handlebody C_{22} and $C_{22} \cap K$. Then $C_{21} \cap K$ consists of $2n - 2c_2$ vertical arcs and $b_2 + c_2 - n$ (possibly zero) mutually trivial arcs, and $C_{22} \cap K$ consists of b_2 (possibly zero) mutually trivial arcs (cf. Lemma 3.5). Using these c-Heegaard splittings, we have a decomposition of (M, K):

$$(M, K) = (\mathcal{C}_{11} \cup_{\mathcal{S}_1} \mathcal{C}_{12}) \cup_P (\mathcal{C}_{21} \cup_{\mathcal{S}_2} \mathcal{C}_{22}),$$

where $\partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ is a 2-sphere P giving the tangle decomposition $(M_1, T_1) \cup_P (M_2, T_2)$ of (M, K).

We now amalgamate these c-Heegaard splittings to obtain the desired splitting of (M, K). Suppose that $\mathcal{C}_{12}$ is compressible. Then there is a compressing disk D_{12} of $\mathcal{C}_{12}$ which cuts $\mathcal{C}_{12}$ into $\mathcal{V}_{12}$ and $\mathcal{W}_{12}$, where $\mathcal{V}_{12}$ is a pair of a genus g_1 handlebody V_{12} and $b_1 + c_1 - n$ (possibly zero) mutually trivial arcs (cf. Lemma 3.5), and $\mathcal{W}_{12}$ is a pair of a compression body W_{12} homeomorphic to $S^2 \times [0, 1]$ and $2n$ vertical arcs (cf. Figure 4). Let α_{12} be a vertical arc in $\mathcal{W}_{12}$ which is disjoint from K and joins $\partial_- W_{12}$ to the interior of $D_{12} \subset \partial_+ W_{12}$. Set $\overline{\mathcal{V}}_{12} = \mathcal{V}_{12} \cup \mathrm{Nbd}(\alpha_{12}; W_{12})$ and $\overline{\mathcal{W}}_{12} = \mathrm{Ext}(\overline{\mathcal{V}}_{12}; \mathcal{C}_{12})$. If $\mathcal{C}_{12}$ is not compressible, then C_{12} is homeomorphic to $S^2 \times [0, 1]$ and $C_{12} \cap K$ consists only of vertical arcs. We set $\overline{\mathcal{V}}_{12} = \emptyset$ and $\overline{\mathcal{W}}_{12} = \mathcal{C}_{12}$ in this case.

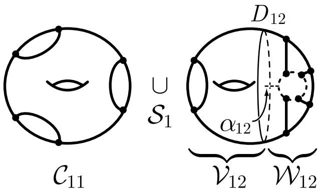

Figure 4. An example of $(\mathcal{C}_{11}, \mathcal{C}_{12}; \mathcal{S}_1)$ if $\mathcal{C}_{12}$ is compressible.

In summery,

$$\overline{\mathcal{V}}_{12} = \begin{cases} \mathcal{V}_{12} \cup \mathrm{Nbd}(\alpha_{12}; W_{12}) & (\text{if } \mathcal{C}_{12} \text{ is compressible}) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$\overline{\mathcal{W}}_{12} = \begin{cases} \mathrm{Ext}(\overline{\mathcal{V}}_{12}; \mathcal{C}_{12}) & (\text{if } \mathcal{C}_{12} \text{ is compressible}) \\ \mathcal{C}_{12} & (\text{otherwise}). \end{cases}$$

Let T_2' be a (possibly empty) disjoint union of the components of $T_2 = M_2 \cap K$ which are contained in C_{21}. Set $\mathcal{V}_{21} = (\mathrm{Nbd}(T_2'; C_{21}), T_2')$ and $\mathcal{W}_{21} = \mathrm{Ext}(\mathcal{V}_{21}; \mathcal{C}_{21})$. We notice that $\mathcal{V}_{21}$ is a disjoint union of c_2 (possibly zero) 3-balls each with a single trivial arc. Suppose that $\mathcal{W}_{21}$ is compressible. Then there is a compressing disk D_{21} of $\mathcal{W}_{21}$ which cuts $\mathcal{W}_{21}$ into $\mathcal{W}_{21}'$ and $\mathcal{W}_{21}''$, where $\mathcal{W}_{21}'$ is a pair of a genus $g_2 - c_2$ handlebody W_{21}' and $b_2 + c_2 - n$ (possibly zero) mutually trivial arcs (cf. Lemma 3.5), and $\mathcal{W}_{21}''$ is a pair of a compression body W_{21}'' homeomorphic to {a closed connected orientable surface of genus $c_2\} \times [0,1]$ and $2n-2c_2$ vertical arcs (cf. Figure 5). Let α_{21} be a vertical arc in $\mathcal{W}''_{21}$ which is disjoint from K and joins $\partial_- W''_{21}$ to the interior of $D_{21} \subset \partial_+ W''_{21}$. We, if necessary, move an endpoint of α_{21} slightly so that α_{21} does not share an endpoint with α_{12}. Set $\overline{\mathcal{V}}_{21} = \mathcal{V}_{21} \cup \mathcal{W}'_{21} \cup \mathrm{Nbd}(\alpha_{21}; W''_{21})$ and $\overline{\mathcal{W}}_{21} = \mathrm{Ext}(\overline{\mathcal{V}}_{21}; \mathcal{C}_{21})$. If $\mathcal{W}_{21}$ is not compressible, then W_{21} is homeomorphic to {a closed connected orientable surface of genus $c_2\} \times [0,1]$ and $W_{21} \cap K$ consists only of vertical arcs. We set $\overline{\mathcal{V}}_{21} = \mathcal{V}_{21}$ and $\overline{\mathcal{W}}_{21} = \mathcal{W}_{21}$ in this case.

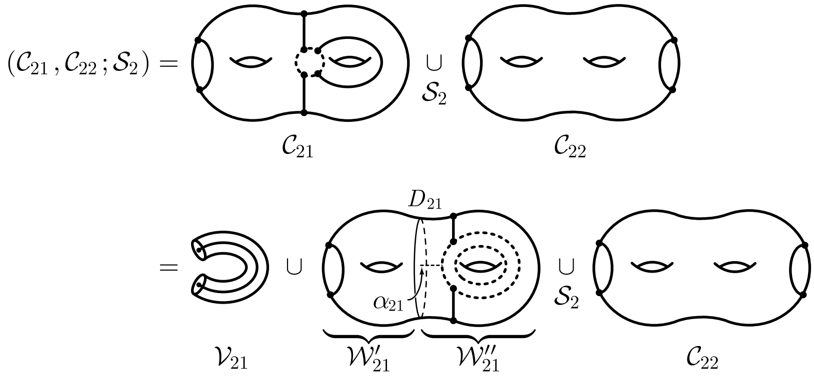

Figure 5. An example of $(\mathcal{C}_{21}, \mathcal{C}_{22}; \mathcal{S}_2)$ if $\mathcal{W}_{21}$ is compressible.

In summery,

$$\overline{\mathcal{V}}_{21} = \begin{cases} \mathcal{V}_{21} \cup \mathcal{W}'_{21} \cup \mathrm{Nbd}(\alpha_{21}; W''_{21}) & (\text{if } \mathcal{W}_{21} \text{ is compressible}) \\ \mathcal{V}_{21} & (\text{otherwise}), \end{cases}$$

$$\overline{\mathcal{W}}_{21} = \begin{cases} \mathrm{Ext}(\overline{\mathcal{V}}_{21}; \mathcal{C}_{21}) & (\text{if } \mathcal{W}_{21} \text{ is compressible}) \\ \mathcal{W}_{21} & (\text{otherwise}). \end{cases}$$

Set $\mathcal{C}_1 = \mathcal{C}_{11} \cup \overline{\mathcal{W}}_{12} \cup \overline{\mathcal{V}}_{21}$ and $\mathcal{C}_2 = \overline{\mathcal{V}}_{12} \cup \overline{\mathcal{W}}_{21} \cup \mathcal{C}_{22}$. Since K is a knot in M, i.e., K is a connected simple closed curve, we see that $\mathcal{C}_1$ is a pair of a genus g_1+g_2 handlebody and $(b_1+c_1)-c_2+(b_2+c_2-n) = b_1+b_2+c_1-n$ mutually trivial arcs. We also see that $\mathcal{C}_2$ is a pair of a genus g_1+g_2 handlebody and $(b_1+c_1-n)+b_2 = b_1+b_2+c_1-n$ mutually trivial arcs. Hence $\{\mathcal{C}_1, \mathcal{C}_2\}$ gives a $(g_1+g_2, b_1+b_2+c_1-n)$-bridge splitting of (M,K). □

Let K be a knot in a closed connected orientable 3-manifold M and $(M_1,T_1) \cup_P (M_2,T_2)$ an n-tangle decomposition of (M,K). We recall that each n-tangle (M_i,T_i) $(i=1,2)$ admits a $(g_i, \mathrm{brg}_{g_i}(T_i), 0)$-splitting. It follows from Theorem 4.1 that (M,K) admits a $(g_1+g_2, \mathrm{brg}_{g_1}(T_1)+\mathrm{brg}_{g_2}(T_2)-n)$-bridge splitting. Hence we have:

Corollary 4.2. *Let K be a knot in a closed connected orientable* 3*-manifold M and $(M_1,T_1) \cup_P (M_2,T_2)$ an n-tangle decomposition of (M,K). Then*

$$\mathrm{brg}_{g_1+g_2}(K) \leq \mathrm{brg}_{g_1}(T_1) + \mathrm{brg}_{g_2}(T_2) - n.$$

We notice that Corollary 1.1 is a special case of the above. Similarly, each n-tangle (M_i, T_i) $(i = 1, 2)$ admits a $(\mathrm{tnl}(T_i) + n, 0, n)$-splitting. Hence (M, K) admits a $(\mathrm{tnl}(T_1) + \mathrm{tnl}(T_2) + 2n, 0)$-bridge splitting. Hence we have the inequality in Corollary 1.2.

5. Meridional destabilizing number of tangles.

Let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a c-Heegaard splitting of an n-tangle (M, T) with $\partial M = \partial_- C_i$ for $i = 1$ or 2, say $i = 2$, where $\mathcal{C}_i = (C_i, C_i \cap T)$ and $\mathcal{S} = (S, S \cap T)$. Let T' be a (possibly empty) disjoint union of the components of T which are contained in C_2. We say that $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ is *meridionally stabilized* if there are a compressing disk D_1 of $\mathcal{C}_1$ and a cut disk D_2 of $\mathcal{C}_2$ such that $|D_2 \cap T'| = 1$ and $|\partial D_1 \cap \partial D_2| = 1$. Such a pair of disks (D_1, D_2) is called a *meridional cancelling pair.* Suppose that $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ is a meridionally stabilized (g, b, c)-splitting of an n-tangle (M, T) with $\partial M = \partial_- C_2$. Then we can obtain $(g-1, b+1, c-1)$-splitting of $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ as follows. Let (D_1, D_2) be a meridional cancelling pair. We recall that D_2 is a cut disk of $\mathcal{C}_2$ which intersects a single component, say γ, of T' entirely contained in C_2. Set $\mathcal{N} = (\mathrm{Nbd}(D_2; C_2), \mathrm{Nbd}(D_2; C_2) \cap T)$. Then $\mathrm{Nbd}(D_2; C_2)$ can be regarded as a 2-handle with $\mathrm{Nbd}(D_2; C_2) \cap T$ its co-core. Set $\mathcal{C}_1' = \mathcal{C}_1 \cup \mathcal{N}$. Since (D_1, D_2) is a meridional cancelling pair, we see that $\mathcal{C}_1'$ is a c-compression body which is a pair of a genus $g-1$ handlebody and $b+1$ mutually trivial arcs. Set $\mathcal{C}_2' = \mathrm{Ext}(\mathcal{N}; \mathcal{C}_2)$. Then $\mathcal{C}_2'$ is a c-compression body which is a pair of a genus $g-1$ compression body C_2' with $\partial_- C_2' = \partial M$ and $C_2' \cap T$. Let T'' be a (possibly empty) disjoint union of the components of T which are contained in C_2'. Since $T'' = T' \setminus \gamma$, we see that $|T''| = c-1$. Hence $\{\mathcal{C}_1', \mathcal{C}_2'\}$ gives a $(g-1, b+1, c-1)$-splitting of (M, T). Such an operation is called *meridional destabilization.* The *meridional destabilizing number* is the maximal number of times of meridional destabilization we can do from minimal genus Heegaard splittings, i.e., $(t(T) + n, 0, n)$-splittings of (M, T).

Definition 5.1. Let (M, T) be an n-tangle. The *meridional destabilizing number* $\mathrm{md}(T)$ of (M, T) is defined to be the maximal integer m for which (M, T) admits a $(\mathrm{tnl}(T) + n - m, m, n - m)$-splitting.

Example 5.2. Recall that the 2-tangle (B^3, T_l) in Figure 2 is of tunnel number zero. The torus S illustrated in Figure 6 gives a $(1, 1, 1)$-splitting of (B^3, T_l). This implies that $\mathrm{md}(T_l) \geq 1$. Since (B^3, T_l) is a non-trivial 2-tangle, we see that $\mathrm{md}(T_l) < 2$ and hence $\mathrm{md}(T_l) = 1$. It also follows from the lemma below that the 2-tangle (B^3, T_n') in Figure 2 satisfies $\mathrm{md}(T_n') = 2$.

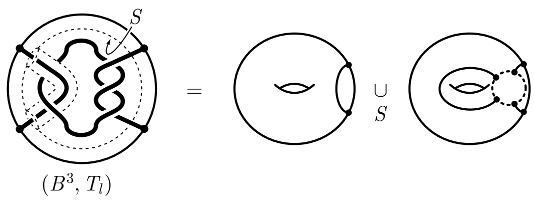

Figure 6. The 2-tangle (B^3, T_l) satisfies $\mathrm{md}(T_l) = 1$.

Lemma 5.3. *Let* (B^3, T) *be a* 2*-tangle which admits a* $(0,3,0)$*-splitting. Then* (B^3, T) *also admits a* $(1,2,0)$*-splitting.*

Proof. Let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a $(0,3,0)$-splitting of (B^3, T), where $\mathcal{C}_1 = (C_1, T_1)$ is a pair of a 3-ball and three trivial arcs, and $\mathcal{C}_2 = (C_2, T_2)$ is a pair of a 3-manifold homeomorphic to $S^2 \times [0,1]$ and five arcs such that four of them are vertical and the other is trivial. Let γ_2 be the trivial arc component of T_2. Set $C_1' = C_1 \cup \mathrm{Nbd}(\gamma_2, C_2)$ and $C_2' = \mathrm{Ext}(C_1'; B^3)$. Then $C_1' \cap T$ consists of two trivial arcs and $C_2' \cap T$ consists of four vertical arcs. This implies that $\mathcal{C}_i' = (C_i', C_i' \cap T)$ $(i = 1, 2)$ give a $(1,2,0)$-splitting. □

Let K be a knot in a closed connected orientable 3-manifold M and $(M_1, T_1) \cup_P (M_2, T_2)$ an n-tangle decomposition of (M, K). Then each (M_i, T_i) $(i = 1, 2)$ admits a $(\mathrm{tnl}(T_i) + n - \mathrm{md}(T_i), \mathrm{md}(T_i), n - \mathrm{md}(T_i))$-splitting. Hence (M, K) admits a $(\mathrm{tnl}(T_1) + \mathrm{tnl}(T_2) + 2n - \mathrm{md}(T_1) - \mathrm{md}(T_2), \min\{\mathrm{md}(T_1), \mathrm{md}(T_2)\})$-splitting by Theorem 4.1. Therefore we have the following which implies that an upper bound of tunnel number could be improved by meridional destabilizing number of tangles.

Corollary 5.4. *Let* K *be a knot in a closed connected orientable* 3*-manifold* M *and* $(M_1, T_1) \cup_P (M_2, T_2)$ *an* n*-tangle decomposition of* (M, K)*. Then*

$$\mathrm{tnl}(K) \leq \mathrm{tnl}(T_1) + \mathrm{tnl}(T_2) + 2n - 1 - \max\{\mathrm{md}(T_1), \mathrm{md}(T_2)\}.$$

We notice that Morimoto's knot $K_M(l, m, n)$ and its 2-tangle decomposition $(B_1, T_1) \cup_P (B_2, T_2)$ in Figure 1 satisfy the equality in Corollary 5.4 because of $\mathrm{md}(T_i) = 1$ for each $i = 1, 2$ (cf. Example 5.2).

Acknowledgements. The author would like to thank the referee for careful reading and useful comments.

References

[1] B. Clark, The Heegaard genus of manifolds obtained by surgery on links and knots, Internat. J. Math. Math. Sci., **3** (1980), 583–589.

[2] J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, In: Computational Problems in Abstract Algebra, Oxford, 1967, (ed. J. Leech), Pergamon Press, New York, 1970, pp. 329–358.

[3] T. Kobayashi, A construction of arbitrarily high degeneration of tunnel numbers of knots under connected sum, J. Knot Theory Ramifications, **3** (1994), 179–186.

[4] Y. Moriah and H. Rubinstein, Heegaard structures of negatively curved 3-manifolds, Comm. Anal. Geom., **5** (1997), 375–412.

[5] K. Morimoto, There are knots whose tunnel numbers go down under connected sum, Proc. Amer. Math. Soc., **123** (1995), 3527–3532.

[6] K. Morimoto, M. Sakuma and Y. Yokota, Examples of tunnel number one knots which have the property '$1 + 1 = 3$', Math. Proc. Cambridge Philos. Soc., **119** (1996), 113–118.

[7] M. Ozawa, On uniqueness of essential tangle decompositions of knots with free tangle decompositions, In: Proc. Appl. Math. Workshop 8, (eds. G. T. Jin and K. H. Ko), KAIST, Taejon, 1998, pp. 227–232.

[8] T. Saito, Free 2-tangles from the point of view of c-Heegaard splittings, Topology Appl., **160** (2013), 74–81.

[9] H. Schubert, Über eine numerische Knoteninvariante, Math. Z., **61** (1954), 245–288.
[10] M. Tomova, Thin position for knots in a 3-manifold, J. Lond. Math. Soc. (2), **80** (2009), 85–98.

Toshio SAITO
Department of Mathematics
Joetsu University of Education
1 Yamayashiki, Joetsu
Niigata 943-8512, Japan
E-mail: toshio@juen.ac.jp