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Characterizing non-separable sigma-locally compact infinite-dimensional manifolds and its applications

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Abstract. For an infinite cardinal τ, let $\ell_2^f(\tau)$ be the linear span of the canonical orthonormal basis of the Hilbert space $\ell_2(\tau)$ of weight $= \tau$. In this paper, we give characterizations of topological manifolds modeled on $\ell_2^f(\tau)$ and $\ell_2^f(\tau) \times \boldsymbol{Q}$, where $\boldsymbol{Q} = [-1,1]^{\mathbb{N}}$ is the Hilbert cube. We denote the full simplicial complex of cardinality $= \tau$ and the hedgehog of weight $= \tau$ by $\Delta(\tau)$ and $J(\tau)$, respectively. Using our characterization of $\ell_2^f(\tau)$, we prove that both the metric polyhedron of $\Delta(\tau)$ and the space

$$J(\tau)_f^{\mathbb{N}} = \{x \in J(\tau)^{\mathbb{N}} \mid x(n) = 0 \text{ except for finitely many } n \in \mathbb{N}\}$$

are homeomorphic to $\ell_2^f(\tau)$.

1. Introduction.

Throughout the paper, all spaces are paracompact Hausdorff spaces and maps are continuous maps. Given a space E, an *E-manifold* is a topological manifold modeled on E, that is, a space such that each point has an open neighborhood homeomorphic to an open subset of E, where E is called a model space. The Hilbert space of weight $= \tau$ is denoted by $\ell_2(\tau)$, that is,

$$\ell_2(\tau) = \left\{ x \in \mathbb{R}^\tau \;\middle|\; \sum_{\gamma\in\tau} x(\gamma)^2 < \infty \right\},$$

where τ is an infinite cardinal.[1] Let $\ell_2^f(\tau)$ stand for the linear span of the canonical orthonormal basis of the Hilbert space $\ell_2(\tau)$, that is,

$$\ell_2^f(\tau) = \{x \in \ell_2(\tau) \mid x(\gamma) = 0 \text{ except for finitely many } \gamma \in \tau\}.$$

In case $\tau = \aleph_0$, the linear spaces $\ell_2(\aleph_0)$ and $\ell_2^f(\aleph_0)$ are simply denoted by ℓ_2 and ℓ_2^f, respectively. Moreover, we denote the Hilbert cube by $\boldsymbol{Q} = [-1,1]^{\mathbb{N}}$.

For an open cover $\mathcal{U}$ of a space Y, a map $f : X \to Y$ is *$\mathcal{U}$-close* to a map $g : X \to Y$, which is denoted by $f \sim_{\mathcal{U}} g$, if for each $x \in X$, both $f(x)$ and $g(x)$ are contained in some

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[1] We can use each cardinal itself as a set. Actually, it is the smallest ordinal of given cardinality.

member $U \in \mathcal{U}$. In addition, when $Y = (Y, d)$ is a metric space, for each $\epsilon > 0$, we say that f is *ϵ-close* to g, provided $d(f(x), g(x)) < \epsilon$ for every $x \in X$. A closed subset A of a space X is said to be a *Z-set* (or a *strong Z-set*) in X if for each open cover $\mathcal{U}$ of X, there exists a map $f : X \to X$ such that f is $\mathcal{U}$-close to the identity id_X and $f(X) \cap A = \emptyset$ (or $\mathrm{cl}\, f(X) \cap A = \emptyset$). A countable union of Z-sets (or strong Z-sets) in X is called a *Z_σ-set* (or a *strong Z_σ-set*). In addition, a *Z-embedding* is an embedding whose image is a Z-set in the range. It is said that a space X is *strongly universal* for a class $\mathcal{C}$ when the following condition is satisfied:

> For each space $A \in \mathcal{C}$, each closed subset B of A, each map $f : A \to X$ such that the restriction $f|_B$ is a Z-embedding and each open cover $\mathcal{U}$ of X, there exists a Z-embedding $g : A \to X$ such that $g \sim_{\mathcal{U}} f$ and $g|_B = f|_B$.

In 1984, J. Mogilski [**14**] characterized ℓ_2^f-manifolds as follows:

THEOREM 1.1. *A connected space X is an ℓ_2^f-manifold if and only if the following conditions are satisfied*:

(1) *X is an ANR and a countable union of finite-dimensional* (*briefly, f.d.*) *compact metrizable spaces.*
(2) *X is strongly universal for f.d. compact metrizable spaces.*
(3) *Every f.d. compact subset of X is a strong Z-set in X.*

By removing "finite-dimensionality" from the above conditions, the characterization of $(\ell_2^f \times \boldsymbol{Q})$-manifolds can be obtained, see [**14**].

In 2003, Theorem 1.1 was generalized to the non-separable case by K. Sakai and M. Yaguchi [**19**].

THEOREM 1.2. *Let τ be an infinite cardinal. A connected space X is an $\ell_2^f(\tau)$-manifold if and only if the following conditions hold*:

(1) *X is an ANR of weight $= \tau$ and a strongly countable-dimensional* (*briefly, s.c.d.*) *σ-locally compact*[2] *strong Z_σ-space.*
(2) *X is strongly universal for s.c.d. locally compact metrizable spaces of weight $\leq \tau$.*

Similar to the characterizations of J. Mogilski, removing "strongly countable-dimensionality" allows us to characterize $(\ell_2^f(\tau) \times \boldsymbol{Q})$-manifolds, see [**19**].

Clearly, the strong universality for s.c.d. locally compact metrizable spaces (the condition (2) of Theorem 1.2) is more difficult to verify than the strong universality for f.d. compact metrizable spaces (the condition (2) of Theorem 1.1). In this paper, we shall improve Theorem 1.2. For an infinite cardinal τ and a non-negative integer n, a space X has the *τ-discrete n-cells property* if the following condition is satisfied:

> Let $D = \bigoplus_{\gamma \in \tau} D_\gamma$ be a discrete union of n-cube D_γ's and $f : D \to X$ be a map. Then, for each open cover $\mathcal{U}$ of X, there exists a map $g : D \to X$ such that $g \sim_{\mathcal{U}} f$ and $\{g(D_\gamma) \mid \gamma \in \tau\}$ is discrete in X.

[2] A space X is said to be *σ-locally compact* if X is a countable union of locally compact **closed** subsets.

We shall use this property to give the following useful characterization to $\ell_2^f(\tau)$-manifolds.

MAIN THEOREM. *For every infinite cardinal τ, a connected space X is an $\ell_2^f(\tau)$-manifold if and only if the following conditions hold*:

(1) *X is an ANR of weight $= \tau$ and a countable union of closed sets which are discrete unions of f.d. compact metrizable spaces.*
(2) *X has the τ-discrete n-cells property for every non-negative integer n.*
(3) *X is strongly universal for f.d. compact metrizable spaces.*
(4) *Every f.d. compact subset of X is a strong Z-set in X.*

A characterization of $(\ell_2^f(\tau) \times \boldsymbol{Q})$-manifolds can be obtained by the same argument as the above, see Theorem 5.3.

For an infinite cardinal τ, let

$$\ell_1(\tau) = \left\{ x \in \mathbb{R}^\tau \;\middle|\; \sum_{\gamma \in \tau} |x(\gamma)| < \infty \right\},$$

which has the norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{\gamma\in\tau} |x(\gamma)|$. For a simplicial complex K of cardinality $\leq \tau$, the *metric polyhedron* $|K|_m$ of K is realized in $\ell_1(\tau)$ with the all vertices of K in one-to-one correspondence to the unit vectors of $\ell_1(\tau)$, where $|K|_m$ admits the metric induced by the norm $\|\cdot\|_1$. A *full simplicial complex* K is a simplicial complex such that any finite vertices of K spans a simplex of K. We denote the full simplicial complex of cardinality of the vertices $= \tau$ by $\Delta(\tau)$. The following assertion was proved by K. Sakai in 1987 (cf. Proposition 4.1 of [**15**]).

PROPOSITION 1.3. *The metric polyhedron $|\Delta(\aleph_0)|_m$ is homeomorphic to ℓ_2^f.*

By using our new characterization of $\ell_2^f(\tau)$-manifolds, we can extend the above assertion to the non-separable case.

THEOREM A. *For every infinite cardinal τ, the metric polyhedron $|\Delta(\tau)|_m$ is homeomorphic to $\ell_2^f(\tau)$.*

For an infinite cardinal τ, the *hedgehog* $J(\tau)$ is the closed subspace in $\ell_1(\tau)$ defined as follows:

$$J(\tau) = \{x \in \ell_1(\tau) \cap \boldsymbol{I}^\tau \mid x(\gamma) \neq 0 \text{ at most one } \gamma \in \tau\}.$$

We define the subspace $J(\tau)^{\mathbb{N}}_f$ in the countable product of $J(\tau)$ as follows:

$$J(\tau)^{\mathbb{N}}_f = \{x \in J(\tau)^{\mathbb{N}} \mid x(n) = \mathbf{0} \text{ except for finitely many } n \in \mathbb{N}\}.$$

Applying our characterization to $J(\tau)^{\mathbb{N}}_f$, we can also prove the following theorem.

THEOREM B. *For each infinite cardinal τ, the space $J(\tau)^{\mathbb{N}}_f$ is homeomorphic to $\ell_2^f(\tau)$.*

For spaces $Y \subset X$ and $F \subset E$, the pair (X, Y) is an (E, F)-*manifold pair* if each point of X has an open neighborhood U such that the pair $(U, U \cap Y)$ is homeomorphic to a pair $(V, V \cap F)$ for some open subset V of E. R. D. Anderson [**1**] characterized the pair (ℓ_2, ℓ_2^f) by using the notion of f.d. cap sets, which was generalized to (ℓ_2, ℓ_2^f)-manifold pairs by T. A. Chapman in [**7**], [**8**]. In 1970, J. E. West extended this to the non-separable case in his paper [**24**]. A subset A of a space X is said to be *homotopy dense* in X if there exists a homotopy $h : X \times \boldsymbol{I} \to X$ such that $h(x, 0) = x$ for all $x \in X$ and $h(X \times (0, 1]) \subset A$. In the last section, combining West's characterization with Main Theorem, we shall obtain the following:

Theorem C. *For spaces $Y \subset X$, the pair (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, Y is an $\ell_2^f(\tau)$-manifold and Y is homotopy dense in X.*

Then, Theorem A and B can be strengthened as follows:

Corollary A. *For every infinite cardinal τ, the pair $(\mathrm{cl}_{\ell_1(\tau)} |\Delta(\tau)|, |\Delta(\tau)|_m)$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$.*

Corollary B. *Let τ be an infinite cardinal. The pair $(J(\tau)^{\mathbb{N}}, J(\tau)^{\mathbb{N}}_f)$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$.*

2. Preliminaries.

In this section, we shall prepare some notation and results which are used later. We denote the set of all non-negative integers by ω. Let X and Y be spaces, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X. When $\mathcal{A}$ is a refinement (or a star-refinement) of $\mathcal{B}$, we write $\mathcal{A} \prec \mathcal{B}$ (or $\mathcal{A} \prec^\star \mathcal{B}$). Moreover, let $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. For a subset $C \subset X$, the collection $\mathcal{A} \wedge \{C\}$ is denoted by $\mathcal{A}|_C$. We denote the collection of all open covers of X by $\mathrm{cov}(X)$. For maps $f, g : X \to Y$, we write $f \simeq g$ if there is a homotopy $h : X \times \boldsymbol{I} \to Y$ linking f and g. For each $t \in \boldsymbol{I}$, the map $h_t : X \to Y$ is defined by $h_t(x) = h(x, t)$ for all $x \in X$. Moreover, for an open cover $\mathcal{U} \in \mathrm{cov}(Y)$, when h is a $\mathcal{U}$-homotopy, that is, $\{h(\{x\} \times \boldsymbol{I}) \mid x \in X\} \prec \mathcal{U}$, we write $f \simeq_{\mathcal{U}} g$.

The following proposition can be proved by the same way as Corollary 1.8 of [**6**], which is useful to us for detecting Z-sets in ANR's.

Proposition 2.1. *Let X be an ANR. If X has the $\aleph_0$-discrete n-cells property for every $n \in \omega$, then every compact subset of X is a Z-set.*

The following properties of (strong) Z-sets in ANR's are well-known.

Proposition 2.2. *Let X be an ANR.*

(1) *For every* (*strong*) *Z-set A in X and every open subset U of X, $A \cap U$ is a* (*strong*) *Z-set in U.*
(2) *A locally finite union of* (*strong*) *Z-sets in X is a* (*strong*) *Z-set.*

The following proposition is very useful to estimate the distance between two maps to a metric space (cf. (A) of Section 2 in [**14**]).

PROPOSITION 2.3. *Let $Y = (Y, d)$ be a metric space. For each open cover $\mathcal{U}$, there is a map $\alpha : Y \to (0, 1)$ such that for every space X and arbitrary maps $f, g : X \to Y$, if $d(f(x), g(x)) < \alpha(g(x))$ for all $x \in X$, then $f \sim_{\mathcal{U}} g$.*

We shall use the following lemma to construct a homeomorphism which approximates a map in the next section. Refer to (D) of Section 2 in [**14**].

LEMMA 2.4. *Let X and $Y = (Y, d)$ be metric spaces and $\{Y_n\}_{n\in\mathbb{N}}$ be a closed cover of Y such that $Y_1 \subset Y_2 \subset \cdots$. Suppose that $\{g_n : X \to Y\}_{n\in\mathbb{N}}$ is a sequence of surjective maps satisfying the following conditions*:

(I) $g_n|_{g_n^{-1}(Y_n)} : g_n^{-1}(Y_n) \to Y_n$ *is bijective and for every point $y \in Y_n$ and every neighborhood V of $g_n^{-1}(y)$ in X, there exists an open neighborhood U of y in Y such that $g_n^{-1}(U) \subset V$.*
(II) $g_{n+1}|_{g_n^{-1}(Y_n)} = g_n|_{g_n^{-1}(Y_n)}$.
(III) *$d(g_{n+1}(x), g_n(x)) < \alpha_n(g_n(x))$ for all $x \in X \setminus g_n^{-1}(Y_n)$, where $\alpha_n(y) = 2^{-n}\min\{1, d(y, Y_n)\}$, $n \in \mathbb{N}$, and $\alpha_0(y) = 1$.*

Then, a homeomorphism $g : \bigcup_{n\in\mathbb{N}} g_n^{-1}(Y_n) \to Y$ can be defined as follows:

$$g(x) = \lim_{n\to\infty} g_n(x) \textit{ for all } x \in \bigcup_{n\in\mathbb{N}} g_n^{-1}(Y_n),$$

where $d(g(x), g_1(x)) < 1$ for each $x \in \bigcup_{n\in\mathbb{N}} g_n^{-1}(Y_n)$.

Let X and Y be spaces and A be a closed subset of X. The product of X and Y *reduced over* A, which is denoted by $(X \times Y)_A$, is the space $((X \setminus A) \times Y) \cup A$ endowed with the topology generated by open subsets of the product space $(X \setminus A) \times Y$ and sets $((U \setminus A) \times Y) \cup (U \cap A)$, where U is an open subset of X. Then, the product space $(X \setminus A) \times Y$ is an open subspace in $(X \times Y)_A$. Moreover, the projection $\mathrm{pr}_X : X \times Y \to X$ is factored into the two natural maps $q : X \times Y \to (X \times Y)_A$ and $p : (X \times Y)_A \to X$ defined as follows:

$$\begin{cases} q(x, y) = (x, y) & \text{if } (x, y) \in (X \setminus A) \times Y, \\ q(x, y) = x & \text{if } (x, y) \in A \times Y, \end{cases}$$

$$\begin{cases} p(x, y) = x & \text{if } (x, y) \in (X \setminus A) \times Y, \\ \quad p(x) = x & \text{if } x \in A. \end{cases}$$

Note that if both X and Y are metrizable spaces, then $(X \times Y)_A$ is also a metrizable space by the Bing Metrization Theorem (Theorem 4.4.8 of [**12**]). We shall prove the following lemma used the next section.

LEMMA 2.5. *Let X and Y be metrizable spaces and let $A_1 \subset A_2$ be closed subsets* *in X. Then, there exists $\mathcal{U} \in \mathrm{cov}(X \setminus A_1)$ with the following property*:

(∗) *For a subspace B of $(X \setminus A_1) \times Y$ and an embedding $g : B \to (X \times Y)_{A_2} \setminus A_1$, if $g \sim_{p^{-1}(\mathcal{U})} q|_B$, then g extends to the embedding $\tilde{g} : B \cup A_1 \to (X \times Y)_{A_2}$ by $\tilde{g}|_{A_1} = \mathrm{id}_{A_1}$,*

where p, q are the natural maps, that is,

$$p : (X \times Y)_{A_2} \setminus A_1 = ((X \setminus A_1) \times Y)_{A_2 \setminus A_1} \to X \setminus A_1,$$
$$q : (X \setminus A_1) \times Y \to ((X \setminus A_1) \times Y)_{A_2 \setminus A_1} = (X \times Y)_{A_2} \setminus A_1.$$

Moreover, if g is a closed embedding, that is, $g(B)$ is closed in $(X \times Y)_{A_2} \setminus A_1$, then $\tilde{g}$ is also a closed embedding.

Proof. Taking an admissible metric d for X, we can define the desired open cover $\mathcal{U}$ as follows:

$$\mathcal{U} = \{B_d(x, d(x, A_1)/2) \mid x \in X \setminus A_1\} \in \mathrm{cov}(X \setminus A_1).$$

To show that $\mathcal{U}$ has the property (∗), let $g : B \to (X \times Y)_{A_2} \setminus A_1$ be an embedding of $B \subset (X \setminus A_1) \times Y$, which is $p^{-1}(\mathcal{U})$-close to $q|_B$. We extend g to $\tilde{g}$ by $\tilde{g}|_{A_1} = \mathrm{id}_{A_1}$. Then, it is enough to show the continuity of both $\tilde{g}$ and $\tilde{g}^{-1} : g(B) \cup A_1 \to B \cup A_1$. Since $(X \setminus A_1) \times Y$ and $(X \times Y)_{A_2} \setminus A_1$ are respectively open subspaces of $(X \times Y)_{A_1}$ and $(X \times Y)_{A_2}$, we need to check that both $\tilde{g}$ and $\tilde{g}^{-1}$ are continuous at each $a \in A_1$.

First, to verify that $\tilde{g}$ is continuous at $a \in A_1$, let $\epsilon > 0$. Fix a point $x \in B_d(a, \epsilon/3) \subset X$. In case $x \in A_1$, we have

$$\tilde{g}(x) = x \in B_d(a, \epsilon/3) \cap A_1 \subset B_d(a, \epsilon) \cap A_2.$$

In case $x \notin A_1$, we have $\tilde{g}(x, y) = g(x, y)$ for all $y \in Y$ with $(x, y) \in B$. Since $g \sim_{p^{-1}(\mathcal{U})} q|_B$, there exists a point $x_0 \in X \setminus A_1$ such that

$$p\tilde{g}(x, y) = pg(x, y), \quad pq(x, y) = x \in B_d(x_0, d(x_0, A_1)/2).$$

Then, we get

$$d(x_0, A_1) \leq d(x_0, a) \leq d(x_0, x) + d(x, a) < \frac{1}{2} d(x_0, A_1) + \frac{\epsilon}{3},$$

hence $d(x_0, A_1) < 2\epsilon/3$. It follows that

$$d(p\tilde{g}(x, y), a) \leq d(pg(x, y), x) + d(x, a) \leq d(x_0, A_1) + \frac{\epsilon}{3} \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

so $\tilde{g}(x, y) \in (B_d(a, \epsilon) \setminus A_2) \times Y \cup (B_d(a, \epsilon) \cap A_2)$. Therefore

$$\begin{aligned}&\tilde{g}\big((((B_d(a,\epsilon/3)\setminus A_1)\times Y)\cap B)\cup(B_d(a,\epsilon/3)\cap A_1)\big)\\&\quad\subset(B_d(a,\epsilon)\setminus A_2)\times Y\cup(B_d(a,\epsilon)\cap A_2),\end{aligned}$$

which implies that $\tilde{g}$ is continuous at a.

Next, we show that $\tilde{g}^{-1}$ is continuous at $a \in A_1$. Given $\epsilon > 0$, take any point

$$x \in ((B_d(a,\epsilon/3)\setminus A_2)\times Y\cup(B_d(a,\epsilon/3)\cap A_2))\cap(g(B)\cup A_1).$$

When $x \in A_1$, we get

$$\tilde{g}^{-1}(x) = x \in B_d(a,\epsilon/3)\cap A_1 \subset B_d(a,\epsilon)\cap A_1.$$

When $x \in g(B) \subset (X\times Y)_{A_2}\setminus A_1$, we have $\tilde{g}(x',y') = g(x',y') = x$ for the unique point $(x',y') \in B$. We can choose a point $x_0 \in X\setminus A_1$ so that

$$p(x) = p\tilde{g}(x',y') = pg(x',y'),\ pq(x',y') = x' \in B(x_0, d(x_0,A_1)/2)$$

because $g \sim_{p^{-1}(\mathcal{U})} q|_B$. It follows that

$$d(x_0,A_1) \le d(x_0,a) \le d(x_0,p(x)) + d(p(x),a) < \frac{1}{2}d(x_0,A_1) + \frac{\epsilon}{3},$$

so $d(x_0,A_1) \le 2\epsilon/3$. Therefore, we have

$$d(x',a) \le d(x',p(x)) + d(p(x),a) < d(x_0,A_1) + \frac{\epsilon}{3} \le \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

that is, $\tilde{g}^{-1}(x) = (x',y') \in (B_d(a,\epsilon)\setminus A_1)\times Y$. Hence

$$\begin{aligned}&\tilde{g}^{-1}\big(((B_d(a,\epsilon/3)\setminus A_2)\times Y\cup(B_d(a,\epsilon/3)\cap A_2))\cap(g(B)\cup A_1)\big)\\&\quad\subset(B_d(a,\epsilon)\setminus A_1)\times Y\cup(B_d(a,\epsilon)\cap A_1),\end{aligned}$$

so $\tilde{g}^{-1}$ is continuous at a.

To prove the additional assertion, assume that $g(B)$ is closed in $(X\times Y)_{A_2}\setminus A_1$. Then $\mathrm{cl}_{(X\times Y)_{A_2}} g(B)\cap((X\times Y)_{A_2}\setminus A_1) = g(B)$. Therefore, we have

$$\begin{aligned}\tilde{g}(B\cup A_1) &= g(B)\cup A_1\\&= \big(\mathrm{cl}_{(X\times Y)_{A_2}} g(B)\cap((X\times Y)_{A_2}\setminus A_1)\big)\cup A_1\\&= \mathrm{cl}_{(X\times Y)_{A_2}} g(B)\cup A_1,\end{aligned}$$

that is, $\tilde{g}(B\cup A_1)$ is closed in $(X\times Y)_{A_2}$. Hence $\tilde{g}$ is a closed embedding. □

REMARK 1. In the above lemma, if g is a continuous map, then so the extension $\tilde{g}$

is. When $B = (X \setminus A_1) \times Y$ and $g : (X \setminus A_1) \times Y \to (X \times Y)_{A_2} \setminus A_1$ is a homeomorphism, $\tilde{g} : (X \times Y)_{A_1} \to (X \times Y)_{A_2}$ is a homeomorphism.

3. $\boldsymbol{E}$-Manifold Factors being $\boldsymbol{E}$-Manifolds.

Throughout the section, let $\mathfrak{C}$ be a class of spaces which has the following properties:

($*$) $\mathfrak{C}$ is *topological*, that is, every space homeomorphic to some member of $\mathfrak{C}$ is also a member of $\mathfrak{C}$.

($**$) $\mathfrak{C}$ is *closed hereditary*, that is, every closed subspace of a member of $\mathfrak{C}$ is also a member of $\mathfrak{C}$.

Moreover, let E be a locally convex topological linear metric space such that E is homeomorphic to $E^{\mathbb{N}}$ or

$$E_f^{\mathbb{N}} = \{x \in E^{\mathbb{N}} \mid x(n) = \mathbf{0} \text{ except for finitely many } n \in \mathbb{N}\},$$

and E satisfies the following conditions:

($\star$) E is a countable union of closed subspaces which belong to $\mathfrak{C}$.

($\star\star$) For any closed subset C of E, if $C \in \mathfrak{C}$, then C is a strong Z-set.

We shall use the following notation for subclasses of the class $\mathfrak{M}$ of all metrizable spaces:

$$\mathfrak{M}_0 = \text{the class of compact metrizable spaces,}$$

$$\mathfrak{M}_0^f = \text{the class of f.d. compact metrizable spaces and}$$

$$\mathfrak{M}_0(n) = \text{the class of compact metrizable spaces of dimension } \leq n.$$

For a cardinal τ and a class $\mathcal{C}$, we denote by $\bigoplus_\tau \mathcal{C}$, the class of spaces $X = \bigoplus_{\gamma \in \tau} X_\gamma$ which are discrete unions of spaces $X_\gamma \in \mathcal{C}$. Note that the classes $\bigoplus_\tau \mathfrak{M}_0$, $\bigoplus_\tau \mathfrak{M}_0^f$ and $\bigoplus_\tau \mathfrak{M}_0(n)$ are topological and closed hereditary. It is known that the locally convex topological linear metric space $\ell_2^f(\tau)$ is homeomorphic to $(\ell_2^f(\tau))_f^{\mathbb{N}}$. Let $\ell_2^{\boldsymbol{Q}}$ be the linear subspace in ℓ_2 spanned by $\prod_{n \in \mathbb{N}}[-2^{-n}, 2^{-n}]$. Then, it is also known that $\ell_2^f(\tau) \times \boldsymbol{Q}$ is homeomorphic to the locally convex topological linear metric space $\ell_2^f(\tau) \times \ell_2^{\boldsymbol{Q}}$, which is homeomorphic to $(\ell_2^f(\tau) \times \ell_2^{\boldsymbol{Q}})_f^{\mathbb{N}}$. Furthermore, $\ell_2^f(\tau)$ (respectively, $\ell_2^f(\tau) \times \boldsymbol{Q}$) satisfies the conditions ($\star$) and ($\star\star$) with respect to $\bigoplus_\tau \mathfrak{M}_0^f$ (respectively, $\bigoplus_\tau \mathfrak{M}_0$), which will be seen in the proof of Theorem 5.2 (cf. Remark 4).

Remark 2. Let M be a connected E-manifold. Then M is a countable union of strong Z-sets which belong to the class $\mathfrak{C}$. Indeed, Theorem 4 of [**13**] allows us to regard an E-manifold M as an open subspace in E, that is, an F_σ-set, so we have $M = \bigcup_{m \in \mathbb{N}} D_m$, where each D_m is regarded as a closed subspace in E. On the other hand, by the conditions ($\star$) and ($\star\star$) of E, we can write $E = \bigcup_{n \in \mathbb{N}} E_n$ such that every E_n is a strong Z-set belonging to $\mathfrak{C}$. Since $\mathfrak{C}$ is closed hereditary, $D_m \cap E_n \in \mathfrak{C}$ for all $m, n \in \mathbb{N}$. Furthermore, $D_m \cap E_n$ is a strong Z-set in M due to ($\star\star$) and Proposition

2.2(1). Therefore $M = \bigcup_{m,n\in\mathbb{N}} D_m \cap E_n$ is a countable union of strong Z-sets which are members of $\mathfrak{C}$.

The following proposition, which was proved by H. Toruńczyk in Theorem B1 of [**23**] (cf. Proposition 5.1 of [**21**]), shall play an important role in the proof of Theorem 3.3.

PROPOSITION 3.1. *Suppose that A is a strong Z-set in a space X. If $X \times E$ is an E-manifold, then for each open cover $\mathcal{U} \in \operatorname{cov}((X\times E)_A)$, there exists a homeomorphism $h : X \times E \to (X \times E)_A$ such that $h \sim_{\mathcal{U}} q$ and $h(x,0) = x$ for all $x \in A$, where $q : X \times E \to (X \times E)_A$ is the natural map.*

LEMMA 3.2. *Let X be a strongly universal ANR for a class $\mathfrak{C}$. Suppose that $f : A \to X$ is a map from a space $A \in \mathfrak{C}$ to X and U is an open subset of X. Given any open cover $\mathcal{U}$ of U, there exists a Z-embedding $g : f^{-1}(U) \to U$ such that $g \sim_{\mathcal{U}} f|_{f^{-1}(U)}$.*

PROOF. We write $U = \bigcup_{n\in\omega} C_n$, where C_n is a closed subset of X and

$$\emptyset = C_0 \subset \operatorname{int}_X C_1 \subset C_1 \subset \operatorname{int}_X C_2 \subset C_2 \subset \cdots .$$

Let $A_n = f^{-1}(C_n)$ and $B_n = f^{-1}(X \setminus \operatorname{int}_X C_{n+1})$ for each $n \in \mathbb{N}$. Then $A_1 \subset A_2 \subset \cdots$ and $B_1 \supset B_2 \supset \cdots$ are closed in A, $A_n \cap B_n = \emptyset$ for each $n \in \mathbb{N}$, $f^{-1}(U) = \bigcup_{n\in\mathbb{N}} A_n$ and $A \setminus f^{-1}(U) = \bigcap_{n\in\mathbb{N}} B_n$.

Let $\mathcal{V} \in \operatorname{cov}(U)$ be a star-refinement of $\mathcal{U}$. Give an admissible metric for X and take a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X so that $\operatorname{mesh}\mathcal{U}_n \leq 2^{-n}$ and

$$\mathcal{U}_n \prec (\mathcal{V} \wedge \{\operatorname{int}_X C_{i+1} \setminus C_{i-1} \mid i \in \mathbb{N}\}) \bigcup \{X \setminus C_{n+2}\}.$$

By induction, we shall construct a sequence $\{f_n : A \to X\}_{n\in\mathbb{N}}$ so as to satisfy the following conditions:

$(1)_n$ $f_n|_{B_n} = f|_{B_n}$,
$(2)_n$ $f_n|_{A_n} : A_n \to U$ is a Z-embedding,
$(3)_n$ $f_n|_{A_{n-1}\cup B_n} = f_{n-1}|_{A_{n-1}\cup B_n}$,
$(4)_n$ $f_n \sim_{\mathcal{U}_n} f_{n-1}$ and
$(5)_n$ $f_n(A_n \setminus \operatorname{int}_A A_{n-1}) \subset \operatorname{int}_X C_{n+2} \setminus C_{n-3}$,

where $A_0 = C_{-1} = C_{-2} = \emptyset$, $B_0 = A$ and $f_0 = f$. Assume that f_m has been constructed for all $m \leq n-1$. Since X is an ANR and X is strongly universal for $\mathfrak{C}$, we can obtain a $\mathcal{U}_n$-homotopy $h : A \times \boldsymbol{I} \to X$ such that $h_0 = f_{n-1}$, h_1 is a Z-embedding and $h_1|_{A_{n-1}} = f_{n-1}|_{A_{n-1}}$. Taking an Urysohn map $k : A \to \boldsymbol{I}$ so that $k(B_n) = 0$ and $k(A_n) = 1$, we define the map $f_n : A \to X$ by $f_n(x) = h(x, k(x))$. Immediately, the conditions $(1)_n$, $(3)_n$ and $(4)_n$ hold from the definition. Observe that

$$A_n \setminus \operatorname{int}_A A_{n-1} = A_n \setminus \operatorname{int}_A f^{-1}(C_{n-1}) \subset A_n \setminus f^{-1}(\operatorname{int}_X C_{n-1}) \subset A_n \cap B_{n-2}.$$

By the inductive assumption $(1)_{n-2}$,

$$f_{n-2}(A_n \cap B_{n-2}) = f(A_n \cap B_{n-2}) \subset f(A_n) \cap f(B_{n-2}) \subset C_n \setminus \operatorname{int}_X C_{n-1},$$

where $B_{-1} = A$ and $f_{-1} = f$. Furthermore, $f_n(A_n \cap B_{n-2}) \subset \operatorname{int}_X C_{n+2} \setminus C_{n-3}$ due to the condition $(4)_{n-1}$ and $(4)_n$. It follows that

$$f_n(A_n \setminus \operatorname{int}_A A_{n-1}) \subset f_n(A_n \cap B_{n-2}) \subset \operatorname{int}_X C_{n+2} \setminus C_{n-3},$$

hence $(5)_n$ holds. Since $f_n|_{A_n} = h_1|_{A_n}$ is a Z-embedding into X and $f_n(A_n) \subset \operatorname{int}_X C_{n+2} \subset U$, it follows from Proposition 2.2(1) that $f_n(A_n)$ is a Z-set in U, that is, $(2)_n$ also holds.

Now, we can define the desired map $g : f^{-1}(U) \to U$ by $g|_{A_n} = f_n|_{A_n}$ because of $(3)_n$, where the continuity of g is guaranteed by $(4)_n$ and the condition $\operatorname{mesh} \mathcal{U}_n < 2^{-n}$ for all $n \in \mathbb{N}$. To verify that $g \sim_{\mathcal{U}} f|_{f^{-1}(U)}$, let $x \in f^{-1}(U)$. Then, we have $x \in A_n \setminus \operatorname{int}_A A_{n-1} \subset A_n \cap B_{n-2}$ for some $n \in \mathbb{N}$, so

$$f_{n-2}(x) = f(x) \in C_n \text{ and } g(x) = f_n(x) \in \operatorname{int}_X C_{n+2}.$$

Since $f_{n-1} \sim_{\mathcal{U}_{n-1}} f_{n-2}$ and $f_n \sim_{\mathcal{U}_n} f_{n-1}$ by $(4)_{n-1}$ and $(4)_n$, respectively, we can choose $V, V' \in \mathcal{V}$ so that $f_{n-2}(x), f_{n-1}(x) \in V$ and $f_{n-1}(x), f_n(x) \in V'$. Therefore,

$$f(x), g(x) \in V \cup V' \subset W \in \mathcal{U} \text{ for some } W \in \mathcal{U}$$

because $\mathcal{V}$ is a star-refinement of $\mathcal{U}$, that is, $g \sim_{\mathcal{U}} f|_{f^{-1}(U)}$. It remains to show that g is a Z-embedding into U. It is clear that g is injective because $f^{-1}(U) = \bigcup_{n \in \mathbb{N}} A_n$ and $g|_{A_n} = f_n|_{A_n}$ is injective. For any closed subset $D \subset f^{-1}(U)$ and $n \in \mathbb{N}$, due to $(5)_n$,

$$g(D \cap A_n \setminus \operatorname{int}_A A_{n-1}) = f_n(D \cap A_n \setminus \operatorname{int}_A A_{n-1}) \subset \operatorname{int}_X C_n + 2 \setminus C_{n-3}.$$

It follows from $(2)_n$ that

$$g(D) = \bigcup_{n \in \mathbb{N}} g(D \cap A_n \setminus \operatorname{int}_A A_{n-1}) = \bigcup_{n \in \mathbb{N}} f_n(D \cap A_n \setminus \operatorname{int}_A A_{n-1})$$

is a locally finite union of closed sets in U, that is, a closed subset of $g(f^{-1}(U))$. Thus, $g : f^{-1}(U) \to g(f^{-1}(U))$ is a closed map. Moreover,

$$g(f^{-1}(U)) = \bigcup_{n \in \mathbb{N}} g(A_n \setminus \operatorname{int}_X A_{n-1}) = \bigcup_{n \in \mathbb{N}} f_n(A_n \setminus \operatorname{int}_X A_{n-1})$$

is a locally finite union of Z-sets in U, that is, a Z-set by Proposition 2.2(2). As a result, g is a Z-embedding. $\square$

A map $f : X \to Y$ is a *near-homeomorphism* provided that for each open cover $\mathcal{U} \in \operatorname{cov}(Y)$, there exists a homeomorphism $h : X \to Y$ with $h \sim_{\mathcal{U}} f$. The following theorem is proved by analogy with Theorem 4 of [**14**].

THEOREM 3.3. *Suppose that X is a connected ANR satisfying the following conditions*:

(i) *X is a countable union of closed subspaces which belong to $\mathfrak{C}$.*
(ii) *X is strongly universal for $\mathfrak{C}$.*
(iii) *For every closed subset $C \subset X$, if $C \in \mathfrak{C}$, then C is a strong Z-set in X.*

If $X \times E$ is an E-manifold, then the projection $\mathrm{pr}_X : X \times E \to X$ is a near-homeomorphism.

PROOF. According to Remark 2 and the conditions (i) and (iii), we can write $X \times E = \bigcup_{n\in\mathbb{N}} A_n$ and $X = \bigcup_{n\in\mathbb{N}} B_n$, where A_n and B_n are strong Z-sets which belong to $\mathfrak{C}$. For any open cover $\mathcal{U} \in \mathrm{cov}(X)$, X admits a metric d such that $\{B_d(x,1) \mid x \in X\} \prec \mathcal{U}$ due to Theorem 4.1 in Chapter II of [**5**]. Then, it is sufficient to construct a homeomorphism $k : X \times E \to X$ which is 1-close to the projection pr_X.

To begin with, we shall inductively construct a sequence of strong Z-sets $C_1 \subset C_2 \subset \cdots \subset X$ with $X = \bigcup_{n\in\omega} C_n$ and homeomorphisms $h_n : X \times E \to (X \times E)_{C_n}$, $n \in \mathbb{N}$, such that:

$(1)_n$ $B_n \cup C_{n-1} \subset C_n$,
$(2)_n$ $h_n(A_n) \subset C_n$,
$(3)_n$ $h_n|_{h_{n-1}^{-1}(C_{n-1})} = h_{n-1}|_{h_{n-1}^{-1}(C_{n-1})}$ and
$(4)_n$ $d(p_n h_n(x), p_{n-1}h_{n-1}(x)) < \alpha_{n-1}(p_{n-1}h_{n-1}(x))$ for all $x \in (X \times E) \setminus h_{n-1}^{-1}(C_{n-1})$,

where $C_0 = \emptyset$, $h_0 : X \times E \to X \times E$ is the identity, $p_0 : X \times E \to X$ is the projection onto X, $p_n : (X \times E)_{C_n} \to X$ is the natural map and $\alpha_n : X \setminus C_n \to (0,1)$ is the map defined by $\alpha_n(y) = 2^{-n}\min\{1, d(y, C_n)\}$, $n \in \mathbb{N}$, and $\alpha_0(y) = 1$.

Suppose that C_i and h_i satisfying $(1)_i$, $(2)_i$, $(3)_i$ and $(4)_i$ have been obtained for all $i \le n$. We define the map $\alpha_n : X \setminus C_n \to (0,1)$ by $\alpha_n(y) = 2^{-n}\min\{1, d(y, C_n)\}$. Due to Lemma 2.5, we can choose $\mathcal{U}_n \in \mathrm{cov}(X \setminus C_n)$ so that:

(a) For a map $f : (X \times E) \setminus h_n^{-1}(C_n) \to X$, if $f \sim_{\mathrm{st}^2 \mathcal{U}_n} p_n h_n|_{(X\times E)\setminus h_n^{-1}(C_n)}$, then $d(f(x), p_n h_n(x)) < \alpha_n(p_n h_n(x))$ for all $x \in (X \times E) \setminus h_n^{-1}(C_n)$.
(b) For a homeomorphism $f' : (X \setminus C_n) \times E \to (X \setminus C_n) \times E$, if $f' \sim_{p_n^{-1}(\mathrm{st}\,\mathcal{U}_n)} \mathrm{id}_{(X\setminus C_n)\times E}$, then f' extends to the homeomorphism $f : (X \times E)_{C_n} \to (X \times E)_{C_n}$ by $f|_{C_n} = \mathrm{id}_{C_n}$.
(c) For a closed embedding $v : h_n(A_{n+1}) \setminus C_n \to X \setminus C_n$, if $v \sim_{\mathrm{st}\,\mathcal{U}_n} p_n|_{h_n(A_{n+1})\setminus C_n}$, then v extends to the closed embedding $\tilde{v} : h_n(A_{n+1}) \cup C_n \to X$ by $v|_{C_n} = \mathrm{id}_{C_n}$.

Since h_n is a homeomorphism and $\mathfrak{C}$ is topological, $h_n(A_{n+1}) \in \mathfrak{C}$ is a strong Z-set in $(X \times E)_{C_n}$. Applying Lemma 3.2 to the map $p_n|_{h_n(A_{n+1})} : h_n(A_{n+1}) \to X$ and the open subset $X \setminus C_n \subset X$, we can find a Z-embedding $v : h_n(A_{n+1}) \setminus C_n \to X \setminus C_n$ such that $v \simeq_{\mathcal{U}_n} p_n|_{h_n(A_{n+1})\setminus C_n}$. Let $i : X \setminus C_n \to (X \setminus C_n) \times \{0\} \subset (X \setminus C_n) \times E$ be the natural inclusion. Then $iv(h_n(A_{n+1}) \setminus C_n)$ is a Z-set in $(X \setminus C_n) \times E$. Hence $iv : h_n(A_{n+1}) \setminus C_n \to (X \setminus C_n) \times E$ is a Z-embedding such that $iv \simeq_{p_n^{-1}(\mathcal{U}_n)} \mathrm{id}_{h_n(A_{n+1})\setminus C_n}$ in $(X \setminus C_n) \times E$ because $v \simeq_{\mathcal{U}_n} p_n|_{h_n(A_{n+1})}$ and E is contractible. On the other hand, $(X \setminus C_n) \times E$ is an E-manifold as an open subspace of the E-manifold $X \times E$. By Proposition 2.2(1), $h_n(A_{n+1}) \setminus C_n = h_n(A_{n+1}) \cap (X \setminus C_n) \times E$ is a strong Z-set in

$(X \setminus C_n) \times E$. Applying the Z-set Unknotting Theorem (cf. Theorem 2 of [**9**][3]) to the E-manifold $(X \setminus C_n) \times E$ and using the condition (b), we can obtain a homeomorphism $f : (X \times E)_{C_n} \to (X \times E)_{C_n}$ so that

$$f|_{h_n(A_{n+1}) \setminus C_n} = iv, \quad f|_{(X \setminus C_n) \times E} \simeq_{p_n^{-1}(\mathrm{st}\,\mathcal{U}_n)} \mathrm{id}_{(X \setminus C_n) \times E}$$

and $f|_{C_n} = \mathrm{id}_{C_n}$. Then $f \sim_{p_n^{-1}(\mathrm{st}\,\mathcal{U}_n)} \mathrm{id}_{(X \times E)_{C_n}}$.

By the way, due to (c), the Z-embedding v extends to a closed embedding $\tilde{v} : h_n(A_{n+1}) \cup C_n \to X$ by $v|_{C_n} = \mathrm{id}_{C_n}$, so $\tilde{v}(h_n(A_{n+1})) \in \mathfrak{C}$ is a closed subspace in X, which implies that $\tilde{v}(h_n(A_{n+1}))$ is a strong Z-set in X by (iii). Since C_n and B_{n+1} are strong Z-sets, it follows from Proposition 2.2 that

$$C_{n+1} = \tilde{v}(h_n(A_{n+1}) \cup C_n \cup B_{n+1}$$

is a strong Z-set in X, so $C_{n+1} \setminus C_n$ is a strong Z-set in $X \setminus C_n$. Let $q : (X \times E)_{C_n} \to (X \times E)_{C_n+1}$ be the natural map defined by $p_n = p_{n+1}q$. Lemma 2.5 allows us to choose $\mathcal{V}_n \in \mathrm{cov}(X \setminus C_n)$ so that:

(d) $\mathcal{V}_n \prec \mathcal{U}_n$ and
(e) For a homeomorphism $g' : (X \setminus C_n) \times E \to (X \times E)_{C_{n+1}} \setminus C_n$, if $g' \sim_{p_{n+1}^{-1}(\mathcal{V}_n)} q|_{(X \setminus C_n) \times E}$, then g' extends to the homeomorphism $g : (X \times E)_{C_n} \to (X \times E)_{C_n+1}$ by $g|_{C_n} = \mathrm{id}_{C_n}$.

Then, applying Proposition 3.1 and (e), we can find a homeomorphism $g : (X \times E)_{C_n} \to (X \times E)_{C_{n+1}}$ such that

$$g|_{(X \setminus C_n) \times E} \sim_{p_{n+1}^{-1}(\mathcal{V}_n)} q|_{(X \setminus C_n) \times E}, \ g(x, 0) = x \text{ for all } x \in C_{n+1} \setminus C_n$$

and $g|_{C_n} = \mathrm{id}_{C_n}$. Then $g \sim_{p_{n+1}^{-1}(\mathcal{U}_n)} q$ by (d).

Now, we have the homeomorphism $h_{n+1} = gfh_n : X \times E \to (X \times E)_{C_{n+1}}$. By the definition of C_{n+1}, we have $(1)_{n+1}$. It follows that

$$\begin{aligned} h_{n+1}(A_{n+1}) = gfh_n(A_{n+1}) &= g(v(h_n(A_{n+1}) \setminus C_n) \times \{0\}) \cup (h_n(A_{n+1}) \cap C_n) \\ &\subset g((C_{n+1} \setminus C_n) \times \{0\}) \cup C_n = (C_{n+1} \setminus C_n) \cup C_n = C_{n+1}, \end{aligned}$$

that is, $(2)_{n+1}$ holds. Moreover, we get

$$h_{n+1}(x) = gfh_n(x) = h_n(x) \text{ for every } x \in h_n^{-1}(C_n),$$

which means $(3)_{n+1}$. Observe that

$$\begin{aligned} p_{n+1}h_{n+1}|_{(X \times E) \setminus h_n^{-1}(C_n)} &= p_{n+1}gfh_n|_{(X \times E) \setminus h_n^{-1}(C_n)} \sim_{\mathcal{U}_n} p_{n+1}qfh_n|_{(X \times E) \setminus h_n^{-1}(C_n)} \\ &= p_nfh_n|_{(X \times E) \setminus h_n^{-1}(C_n)} \sim_{\mathrm{st}\,\mathcal{U}_n} p_nh_n|_{(X \times E) \setminus h_n^{-1}(C_n)}, \end{aligned}$$

[3]Theorem 2 of [**9**] holds for a locally convex topological linear metric space E not only such that E is homeomorphic to $E^{\mathbb{N}}$ but also such that E is homeomorphic to $E^{\mathbb{N}}_f$.

hence $p_{n+1}h_{n+1}|_{(X\times E)\setminus h_n^{-1}(C_n)} \sim_{\mathrm{st}^2\mathcal{U}_n} p_nh_n|_{(X\times E)\setminus h_n^{-1}(C_n)}$. By (a), we have

$$d(p_{n+1}h_{n+1}(x), p_nh_n(x)) < \alpha_n(p_nh_n(x)) \text{ for every } x \in (X\times E)\setminus h_n^{-1}(C_n),$$

so $(4)_{n+1}$ holds. Thus, we complete the inductive step.

Finally, we shall construct the desired homeomorphism $k : X \times E \to X$ by using Lemma 2.4. Define the surjective maps $k_n = p_nh_n : X \times E \to X$, $n \in \omega$. Since $B_n \subset C_n$ by $(1)_n$ for all $n \in \mathbb{N}$, the increasing sequence $\{C_n\}_{n\in\omega}$ is a closed cover of X. It follows from $(2)_n$ that

$$A_n \subset h_n^{-1}(C_n) = h_n^{-1}p_n^{-1}(C_n) = k_n^{-1}(C_n),$$

which means that $X\times E = \bigcup_{n\in\omega} k_n^{-1}(C_n)$. It remains to show that the sequence $\{k_n\}_{n\in\omega}$ satisfies the conditions (I), (II) and (III) of Lemma 2.4.

(I): Note that

$$k_n|_{k_n^{-1}(C_n)} = p_nh_n|_{k_n^{-1}(C_n)} = h_n|_{k_n^{-1}(C_n)},$$

so $k_n|_{k_n^{-1}(C_n)}$ is bijective. Given a point $x \in C_n$ and a neighborhood V of $k_n^{-1}(x)$ in $X\times E$, $h_n(V)$ is a neighborhood of $h_n(k_n^{-1}(x)) = p_n^{-1}(x) = x$ in $(X\times E)_{C_n}$. Then, there exists an open neighborhood U of x in X such that

$$p_n^{-1}(U) = (U \cap C_n) \cup (U \setminus C_n) \times E \subset h_n(V),$$

hence it follows that $k_n^{-1}(U) = h_n^{-1}p_n^{-1}(U) \subset V$.

(II): By $(3)_n$, we have

$$\begin{aligned} k_{n+1}|_{k_n^{-1}(C_n)} &= p_{n+1}h_{n+1}|_{h_n^{-1}p_n^{-1}(C_n)} = p_{n+1}h_{n+1}|_{h_n^{-1}(C_n)} \\ &= p_nh_n|_{h_n^{-1}(C_n)} = k_n|_{k_n^{-1}(C_n)}. \end{aligned}$$

(III): It follows from $(4)_{n+1}$ that for all $x \in (X\times E)\setminus k_n^{-1}(C_n)$,

$$\begin{aligned} d(k_{n+1}(x), k_n(x)) &= d(p_{n+1}h_{n+1}(x), p_nh_n(x)) \\ &< \alpha_n(p_nh_n(x)) = \alpha_n(k_n(x)). \end{aligned}$$

In conclusion, we can obtain the desired homeomorphism $k : X\times E \to X$ as follows:

$$k(x) = \lim_{n\to\infty} k_n(x) \text{ for every } x \in X\times E,$$

where k is 1-close to $k_0 = p_0h_0 = \mathrm{pr}_X$. The proof is complete. □

4. The Discrete Approximation Property.

For a cardinal $\tau > 1$, a space X has the *τ-discrete approximation property* (or the *τ-locally finite approximation property*) for a class $\mathcal{C}$ if the following condition is satisfied:

> Let $A = \bigoplus_{\gamma\in\tau} A_\gamma$ be a discrete union of a collection $\{A_\gamma \in \mathcal{C} \mid \gamma \in \tau\}$ and $f : A \to X$ be a map. Then, for each open cover $\mathcal{U}$, there exists a map $g : A \to X$ such that $g \sim_{\mathcal{U}} f$ and $\{g(A_\gamma) \mid \gamma \in \tau\}$ is discrete (or locally finite) in X.

For the sake of convenience, we abbreviate the τ-discrete approximation property for $\mathcal{C}$ and the τ-locally finite approximation property for $\mathcal{C}$ to *τ-DAP($\mathcal{C}$)* and *τ-LFAP($\mathcal{C}$)*, respectively. When $\mathcal{C} = \{C\}$, we simply write τ-DAP(C) and τ-LFAP(C). The τ-discrete n-cells property is no other than τ-DAP($\boldsymbol{I}^n$). Moreover, τ-DAP($\{\boldsymbol{I}^n \mid n \in \omega\}$) is called the *$\tau$-discrete cells property*. The τ-discrete cells property is stronger than the τ-discrete n-cells property for all $n \in \omega$, but the same as τ-DAP($\boldsymbol{Q}$).

Lemma 4.1. *For a cardinal $\tau > 1$, a space X has the τ-discrete cells property if and only if X has τ-DAP($\boldsymbol{Q}$).*

Proof. Let $\boldsymbol{Q}_\gamma$ be a copy of $\boldsymbol{I}^{\mathbb{N}}$ for all $\gamma \in \tau$ and $\mathcal{U} \in \operatorname{cov}(X)$, where each $\boldsymbol{Q}_\gamma$ admits the following metric d:

$$d(x, y) = \sup_{i\in\mathbb{N}} i^{-1}|x(i) - y(i)| \text{ for } x, y \in \boldsymbol{Q}_\gamma.$$

For each $n \in \mathbb{N}$, the inclusion $i_n : \boldsymbol{I}^n \to \boldsymbol{I}^{\mathbb{N}}$ and the projection $p_n : \boldsymbol{I}^{\mathbb{N}} \to \boldsymbol{I}^n$ are respectively defined as follows:

$$i_n(x) = (x(1), \dots, x(n), 0, 0, \dots) \text{ for } x = (x(i))_{1\le i\le n} \text{ and}$$
$$p_n(x) = (x(1), \dots, x(n)) \text{ for } x = (x(i))_{i\in\mathbb{N}}.$$

Moreover, let $i_0 : \boldsymbol{I}^0 = \{0\} \ni 0 \mapsto (0, 0, \dots) \in \boldsymbol{I}^{\mathbb{N}}$ and $p_0 : \boldsymbol{I}^{\mathbb{N}} \ni x \mapsto 0 \in \boldsymbol{I}^0 = \{0\}$.

First, to show the "if" part, take any map $f : D = \bigoplus_{\gamma\in\tau} \boldsymbol{I}^{n(\gamma)} \to X$, where $n(\gamma) \in \omega$ for all $\gamma \in \tau$. Define a map $g : \bigoplus_{\gamma\in\tau} \boldsymbol{Q}_\gamma \to X$ by $g|_{\boldsymbol{Q}_\gamma} = f|_{\boldsymbol{I}^{n(\gamma)}} p_{n(\gamma)}$ for each $\gamma \in \tau$. Since X has τ-DAP($\boldsymbol{Q}$), there is a map $g' : \bigoplus_{\gamma\in\tau} \boldsymbol{Q}_\gamma \to X$ such that $g' \sim_{\mathcal{U}} g$ and $\{g'(\boldsymbol{Q}_\gamma) \mid \gamma \in \tau\}$ is discrete in X. Then, we define a map $f' : D \to X$ by $f'|_{\boldsymbol{I}^{n(\gamma)}} = g'|_{\boldsymbol{Q}_\gamma} i_{n(\gamma)}$ for each $\gamma \in \tau$. It follows that

$$f'|_{\boldsymbol{I}^{n(\gamma)}} = g'|_{\boldsymbol{Q}_\gamma} i_{n(\gamma)} \sim_{\mathcal{U}} g|_{\boldsymbol{Q}_\gamma} i_{n(\gamma)} = f|_{\boldsymbol{I}^{n(\gamma)}} p_{n(\gamma)} i_{n(\gamma)} = f|_{\boldsymbol{I}^{n(\gamma)}} \text{ for every } \gamma \in \tau,$$

hence $f' \sim_{\mathcal{U}} f$. Moreover, $f'(\boldsymbol{I}^{n(\gamma)}) = g'|_{\boldsymbol{Q}_\gamma} i_{n(\gamma)}(\boldsymbol{I}^{n(\gamma)}) \subset g'(\boldsymbol{Q}_\gamma)$ for each $\gamma \in \tau$, so the collection $\{f'(\boldsymbol{I}^{n(\gamma)}) \mid \gamma \in \tau\}$ is discrete in X. As a result, X has the τ-discrete cells property.

Next, to prove the "only if" part, take any map $f : \bigoplus_{\gamma\in\tau} \boldsymbol{Q}_\gamma \to X$. Let $\mathcal{V} \in \operatorname{cov}(X)$ be a star-refinement of $\mathcal{U}$ and ϵ_γ be a Lebesgue number for $(f|_{\boldsymbol{Q}_\gamma})^{-1}(\mathcal{V}) \in \operatorname{cov}(\boldsymbol{Q}_\gamma)$. Then, we can choose $n(\gamma) \in \mathbb{N}$ so that $n(\gamma)^{-1} < \epsilon_\gamma$. It is easy to see that $\mathrm{id}_{\boldsymbol{Q}_\gamma}$ is $n(\gamma)^{-1}$-close to $i_{n(\gamma)} p_{n(\gamma)}$, hence $f|_{\boldsymbol{Q}_\gamma} \sim_{\mathcal{V}} f|_{\boldsymbol{Q}_\gamma} i_{n(\gamma)} p_{n(\gamma)}$. Define a map $g : D = \bigoplus_{\gamma\in\tau} \boldsymbol{I}^{n(\gamma)} \to X$ by

$g|_{\boldsymbol{I}^{n(\gamma)}} = f|_{\boldsymbol{Q}_\gamma} i_{n(\gamma)}$ for each $\gamma \in \tau$. Due to the τ-discrete cells property of X, we can find a map $g' : D \to X$ such that $g' \sim_{\mathcal{V}} g$ and $\{g'(\boldsymbol{I}^{n(\gamma)}) \mid \gamma \in \tau\}$ is discrete in X. Then, we define a map $f' : \bigoplus_{\gamma\in\tau} \boldsymbol{Q}_\gamma \to X$ by $f'|_{\boldsymbol{Q}_\gamma} = g'|_{\boldsymbol{I}^{n(\gamma)}} p_{n(\gamma)}$ for each $\gamma \in \tau$. Observe that for every $\gamma \in \tau$,

$$f'|_{\boldsymbol{Q}_\gamma} = g'|_{\boldsymbol{I}^{n(\gamma)}} p_{n(\gamma)} \sim_{\mathcal{V}} g|_{\boldsymbol{I}^{n(\gamma)}} p_{n(\gamma)} = f|_{\boldsymbol{Q}_\gamma} i_{n(\gamma)} p_{n(\gamma)} \sim_{\mathcal{V}} f|_{\boldsymbol{Q}_\gamma},$$

which means that $f' \sim_{\mathcal{U}} f$. Furthermore, $f'(\boldsymbol{Q}_\gamma) = g'|_{\boldsymbol{I}^{n(\gamma)}} p_{n(\gamma)}(\boldsymbol{Q}_\gamma) = g'(\boldsymbol{I}^{n(\gamma)})$ for all $\gamma \in \tau$, so the collection $\{f'(\boldsymbol{Q}_\gamma) \mid \gamma \in \tau\}$ is discrete in X. Consequently, X has τ-DAP($\boldsymbol{Q}$). $\square$

For a topological subclass $\mathcal{C} \subset \mathfrak{M}_0$, by the same argument as Lemma 4 of [**2**] (cf. [**10**]) we can show that τ-LFAP($\mathcal{C}$) coincides with τ-DAP($\mathcal{C}$), that is:

LEMMA 4.2. *Let τ be an infinite cardinal and let $\mathcal{C}$ be a topological subclass of $\mathfrak{M}_0$. A space X has τ-LFAP($\mathcal{C}$) if and only if X has τ-DAP($\mathcal{C}$).*

PROPOSITION 4.3. *Let τ be a cardinal > 1 and $n \in \omega$. Suppose that W is an open set in an ANR X which is contractible in X. If X has the τ-discrete cells property* (*respectively, the τ-discrete* $(2n+1)$*-cells property*), *then W has τ-DAP($\mathfrak{M}_0$)* (*respectively, τ-DAP($\mathfrak{M}_0(n)$)*).

PROOF. We may only prove the case when X has the τ-discrete $(2n+1)$-cells property because the other case is similarly proved by virtue of Lemma 4.1. Suppose that $f : A = \bigoplus_{\gamma\in\tau} A_\gamma \to W$ is a map, where $A_\gamma \in \mathfrak{M}_0(n)$ for all $\gamma \in \tau$, and $\mathcal{U} \in \operatorname{cov}(X)$. Due to Lemma 4.2, we may construct a map $h : A \to W$ such that $h \sim_{\mathcal{U}} f$ and $\{h(A_\gamma) \mid \gamma \in \tau\}$ is locally finite in W. Denote $D = \bigoplus_{\gamma\in\tau} D_\gamma$, where $D_\gamma = \boldsymbol{I}^{2n+1}$ for each $\gamma \in \tau$. We may assume that $A_\gamma \subset D_\gamma$ for all $\gamma \in \tau$.

Since W is an ANR, f extends to a map $\tilde{f} : V \to W$ from an open neighborhood V of A in D to W. Take an open neighborhood V' of A in D so that $\operatorname{cl} V' \subset V$ and let $k : D \to \boldsymbol{I}$ be an Urysohn map such that $k^{-1}(0) = A$ and $k^{-1}(1) = D \setminus V'$. By the hypothesis, we have a contraction $\phi : W \times \boldsymbol{I} \to X$ so that $\phi_0 = \mathrm{id}_W$ and $\phi_1(W) = \{x_0\}$ for some $x_0 \in X$. Then, we can define the map $\bar{f} : D \to X$ as follows:

$$\bar{f}(x) = \phi(\tilde{f}(x), k(x)) \text{ for each } x \in V \text{ and } \bar{f}(D \setminus V) = \{x_0\}.$$

Now, we can write $W = \bigcup_{i\in\mathbb{N}} W_i$, where W_i is an open set in X and $\operatorname{cl} W_i \subset W_{i+1}$ for every $i \in \mathbb{N}$. Let $\mathcal{U}_0 \in \operatorname{cov}(X)$ such that $\mathcal{U}_0 \prec^\star \mathcal{U}$. We define closed subsets $R_i \subset A$, $i \in \mathbb{N}$, an open cover $\mathcal{U}' \in \operatorname{cov}(W)$ and open covers $\mathcal{U}_i \in \operatorname{cov}(X)$, $i \in \mathbb{N}$, as follows:

$$R_i = f^{-1}(\operatorname{cl} W_i \setminus W_{i-1}), \quad \mathcal{U}' = \bigcup_{i\in\mathbb{N}} \mathcal{U}_0|_{W_i \setminus \operatorname{cl} W_{i-2}}$$
$$\text{and } \mathcal{U}_i = \mathcal{U}'|_{W_{2i}} \cup \{X \setminus \operatorname{cl} W_{2i-1}\},$$

where $W_{-1} = W_0 = \emptyset$. Using the τ-discrete $(2n+1)$-cells property of X, we can obtain a map $g_i : D \to X$ such that $g_i \simeq_{\mathcal{U}_i} \bar{f}$ and $\{g_i(D_\gamma) \mid \gamma \in \tau\}$ is discrete in X. Then $g_i|_{R_{2i-1}} \simeq_{\mathcal{U}'} f|_{R_{2i-1}}$ for all $i \in \mathbb{N}$. By the Homotopy Extension Theorem, we can take a map $g : A \to W$ such that $g \simeq_{\mathcal{U}'} f$ and $g|_{R_{2i-1}} = g_i|_{R_{2i-1}}$ for each $i \in \mathbb{N}$. It is easy to see that $\{g(A_\gamma \cap R_{2i-1}) \mid \gamma \in \tau\}$ is discrete in $W_{2i} \setminus \operatorname{cl} W_{2i-3}$. Therefore $\{g(A_\gamma \cap R_{2i-1}) \mid \gamma \in \tau, i \in \mathbb{N}\}$ is locally finite in W.

Next, we can find an open refinement $\mathcal{V} \in \operatorname{cov}(W)$ of $\mathcal{U}_0$ so as to satisfy the following:

> For every map $h : A \to W$, $h \sim_{\mathcal{V}} g$ implies that $\{h(A_\gamma \cap R_{2i-1}) \mid \gamma \in \tau, i \in \mathbb{N}\}$ is locally finite in W.

By the same construction as g, we can obtain a map $h : A \to W$ so that $h \simeq_{\mathcal{V}} g$ and $\{h(A_\gamma \cap R_{2i}) \mid \gamma \in \tau, i \in \mathbb{N}\}$ is locally finite in W. It is follows from the definition of $\mathcal{V}$ that $\{h(A_\gamma \cap R_{2i-1}) \mid \gamma \in \tau, i \in \mathbb{N}\}$ is locally finite in W. Therefore $\{h(A_\gamma \cap R_i) \mid \gamma \in \tau, i \in \mathbb{N}\}$ is locally finite in W, which means that $\{h(A_\gamma) \mid \gamma \in \tau\}$ is locally finite in W. Moreover, $h \sim_{\mathcal{V}} g \sim_{\mathcal{U}'} f$, so $h \sim_{\mathcal{U}} f$. Thus, the proof is complete. □

A little stronger condition than τ-DAP will be introduced in the following proposition.

PROPOSITION 4.4. *Let τ be a cardinal > 1 and $\mathcal{C}$ be a topological and closed hereditary subclass of $\mathfrak{M}_0$. Suppose that X is an ANR with τ-DAP($\mathcal{C}$) and that any closed set $C \in \mathcal{C}$ in X is a strong Z-set. Then, for every map $f : A = \bigoplus_{\gamma\in\tau} A_\gamma \to X$ from a discrete union of A_γ's to X, where $A_\gamma \in \mathcal{C}$, for every closed subset $B \subset A$ such that the restriction $f|_B$ is a closed embedding and for every $\mathcal{U} \in \operatorname{cov}(X)$, there exists a map $g : A \to X$ such that $g \sim_{\mathcal{U}} f$, $g|_B = f|_B$ and the collection $\{g(A_\gamma) \mid \gamma \in \tau\}$ is discrete in X.*

PROOF. We take $\mathcal{U}_1, \mathcal{U}_2 \in \operatorname{cov}(X)$ so that $\mathcal{U} \succ^\star \mathcal{U}_1 \succ^\star \mathcal{U}_2$. Let $B_\gamma = A_\gamma \cap B$ for each $\gamma \in \tau$. Since $f|_B$ is a closed embedding, $\{f(B_\gamma) \mid \gamma \in \tau\}$ is a discrete collection in X. Then, we can find a pairwise disjoint collection $\{U_\gamma \mid \gamma \in \tau\}$ of open subsets of X so that $f(B_\gamma) \subset U_\gamma$ for each $\gamma \in \tau$.

Take $\mathcal{U}_2' \in \operatorname{cov}(X)$ such that

$$\mathcal{U}_2' \prec \mathcal{U}_2 \wedge \{U_\gamma, X \setminus f(B) \mid \gamma \in \tau\}.$$

Since $f(B_\gamma) \in \mathcal{C}$ for every $\gamma \in \tau$, it follows from Proposition 2.2(2) that $f(B) = \bigcup_{\gamma\in\tau} f(B_\gamma)$ is a strong Z-set in X. Then, we can obtain a $\mathcal{U}_2'$-homotopy $h' : X \times \boldsymbol{I} \to X$ and an open neighborhood W of $f(B)$ in X such that $h_0' = f$ and $h_1'(X) \subset X \setminus W$. We write $W_\gamma = W \cap U_\gamma$ for each $\gamma \in \tau$. Let $h = h'(f \times \mathrm{id}_{\boldsymbol{I}}) : A \times \boldsymbol{I} \to X$, so h is a $\mathcal{U}_2'$-homotopy and $h_0 = h_0' f = f$. Observe that $h(B_\gamma \times \boldsymbol{I}) \subset U_\gamma$ for each $\gamma \in \tau$. Since each B_γ is compact, we can find an open neighborhood V_γ of B_γ in A_γ so that $h(V_\gamma \times \boldsymbol{I}) \subset U_\gamma$. Take an Urysohn map $k : A \to \boldsymbol{I}$ such that $k^{-1}(0) = B$ and $k^{-1}(1) = A \setminus \bigcup_{\gamma\in\tau} V_\gamma$ and define the map $f' : A \to X$ by $f'(x) = h(x, k(x))$ for $x \in A$. It is easy to see that $f' \sim_{\mathcal{U}_2'} f$ and $f'|_B = h_0|_B = f|_B$. Moreover, f' satisfies the following condition:

(1) $f'(A \setminus V_\gamma) \cap W_\gamma = \emptyset$ for any $\gamma \in \tau$.

Indeed, take any point $x \in A \setminus V_\gamma$. When $x \in A \setminus \bigcup_{\gamma\in\tau} V_\gamma$,

$$f'(x) = h_1(x) = h'_1 f(x) \in X \setminus W \subset X \setminus W_\gamma.$$

When $x \in V_{\gamma'}$ for some $\gamma' \neq \gamma$,

$$f'(x) = h_{k(x)}(x) \in U_{\gamma'} \subset X \setminus U_\gamma \subset X \setminus W_\gamma.$$

We take an open neighborhood W'_γ of $f(B_\gamma)$ for each $\gamma \in \tau$ so that $\operatorname{cl} W'_\gamma \subset W_\gamma$. Let $\mathcal{U}'_1 \in \operatorname{cov}(X)$ such that

$$\mathcal{U}'_1 \prec \mathcal{U}_1 \wedge \Big\{ W'_\gamma, W_\gamma \setminus f(B_\gamma), X \setminus \bigcup_{\gamma' \in \tau} \operatorname{cl} W'_{\gamma'} \mid \gamma \in \tau \Big\}.$$

Applying τ-DAP($\mathcal{C}$) of X to f', we can obtain a $\mathcal{U}'_1$-homotopy $h'' : A \times \boldsymbol{I} \to X$ so that $h''_0 = f'$ and

(2) $\{h''_1(A_\gamma) \mid \gamma \in \tau\}$ is discrete in X.

Since h'' is a $\mathcal{U}'_1$-homotopy and $h''_0|_B = f'|_B = f|_B$, it follows that $h''(B_\gamma \times \boldsymbol{I}) \subset W'_\gamma$ for each $\gamma \in \tau$. Because of the compactness, each B_γ has an open neighborhood G_γ in A_γ such that $h''(G_\gamma \times \boldsymbol{I}) \subset W'_\gamma$. Let $k' : A \to \boldsymbol{I}$ be an Urysohn map such that $(k')^{-1}(0) = B$ and $(k')^{-1}(1) = A \setminus \bigcup_{\gamma \in \tau} G_\gamma$. Now, we can define the desired map $g : A \to X$ by $g(x) = h''(x, k'(x))$ for all $x \in A$. Observe that $g \sim_{\mathcal{U}'_1} f'$ and the restriction $g|_B = h''_0|_B = f'|_B$, hence $g \sim_{\mathcal{U}} f$ and $g|_B = f|_B$. Thus, it remains to show that $\{g(A_\gamma) \mid \gamma \in \tau\}$ is discrete in X.

Fix a point $x \in X$. Due to (2), the collection $\{g(A_\gamma \setminus G_\gamma) \mid \gamma \in \tau\}$ is discrete in X, hence there exists an open neighborhood U_x of x in X such that

$$\operatorname{card}(\{\gamma \in \tau \mid g(A_\gamma \setminus G_\gamma) \cap U_x \neq \emptyset\}) \leq 1.$$

(Case 1) $\operatorname{card}(\{\gamma \in \tau \mid g(A_\gamma \setminus G_\gamma) \cap U_x \neq \emptyset\}) = 0$.

When $x \in X \setminus \bigcup_{\gamma \in \tau} \operatorname{cl} W'_\gamma$, the subset $U'_x = U_x \setminus \bigcup_{\gamma \in \tau} \operatorname{cl} W'_\gamma$ is an open neighborhood of x in X. Since $g(G_\gamma) \subset W'_\gamma$, we have $U'_x \cap g(G_\gamma) = \emptyset$, so $U'_x \cap g(A_\gamma) = \emptyset$ for any $\gamma \in \tau$. When $x \in \bigcup_{\gamma \in \tau} \operatorname{cl} W'_\gamma$, $x \in \operatorname{cl} W'_{\gamma_0}$ for the unique $\gamma_0 \in \tau$. Then $U'_x = U_x \setminus \bigcup_{\gamma \neq \gamma_0} \operatorname{cl} W'_\gamma$ is an open neighborhood of x in X such that $U'_x \cap g(A_\gamma) = \emptyset$ for all $\gamma \neq \gamma_0$.

(Case 2) $\operatorname{card}(\{\gamma \in \tau \mid g(A_\gamma \setminus G_\gamma) \cap U_x \neq \emptyset\}) = 1$.

We may assume that $g(A_{\gamma_0} \setminus G_{\gamma_0}) \cap U_x \neq \emptyset$ for the unique $\gamma_0 \in \tau$. Note that $g(A_{\gamma_0} \setminus G_{\gamma_0})$ is a closed set in X because of the compactness of A_{γ_0}, so we can turn the case when $x \notin g(A_{\gamma_0} \setminus G_{\gamma_0})$ into Case 1 by replacing U_x by $U_x \setminus g(A_{\gamma_0} \setminus G_{\gamma_0})$. When $x \in g(A_{\gamma_0} \setminus G_{\gamma_0})$, we have $x \in X \setminus \bigcup_{\gamma \neq \gamma_0} \operatorname{cl} W'_\gamma$. Otherwise $x \in \operatorname{cl} W'_{\gamma_1}$ for some $\gamma_1 \neq \gamma_0$. As $x \in g(A_{\gamma_0} \setminus G_{\gamma_0})$, the point $x = g(a)$ for a point $a \in A_{\gamma_0} \setminus G_{\gamma_0}$. Then $f'(a) \in W_{\gamma_1}$ because $g \sim_{\mathcal{U}'_1} f'$. On the other hand, since $A_{\gamma_0} \subset A \setminus V_{\gamma_1}$, it follows from (1) that $f'(A_{\gamma_0}) \cap W_{\gamma_1} = \emptyset$, which is a contradiction. Now x has the open neighborhood $U'_x = U_x \setminus \bigcup_{\gamma \neq \gamma_0} \operatorname{cl} W'_\gamma$ in X such that $U'_x \cap g(A_\gamma) = \emptyset$ for every $\gamma \neq \gamma_0$. □

5. A Proof of Main Theorem.

This section is devoted to proving Main Theorem. The following proposition follows from A. H. Stone's Theorem (Theorem 4.4.1 of [**12**]).

Proposition 5.1. *Let X be a metrizable space. Then X is a countable union of closed subsets which are discrete unions of f.d. compact metrizable spaces if and only if X is a countable union of locally compact locally f.d. closed subsets.*

Proof. The "only if" part is obvious. To prove the "if" part, we assume that $X = \bigcup_{n\in\omega} X_n$, where X_n is a locally compact locally f.d. closed subsets for all $n \in \omega$. By the local compactness and the local finite-dimensionality, each X_n has an open cover $\mathcal{U}_n$ such that for every $U \in \mathcal{U}_n$, the closure of U is compact and finite-dimensional. Due to A. H. Stone's Theorem, each $\mathcal{U}_n$ has a σ-discrete open refinement $\mathcal{V}_n = \bigcup_{m\in\omega} \mathcal{V}_n^m \in \operatorname{cov}(X_n)$, where $\mathcal{V}_n^m$ is discrete in X_n. Then, $A_n^m = \bigcup_{V\in\mathcal{V}_n^m} \operatorname{cl} V$ is a closed subset of X_n which is a discrete union of f.d. compact metrizable spaces. Evidently $X = \bigcup_{n,m\in\omega} A_n^m$. The proof is complete. □

Remark 3. When X is a countable union of closed subsets which are discrete unions of f.d. compact metrizable spaces, we can write $X = \bigcup_{n\in\omega} X_n$, where each X_n is a closed subspace which is discrete unions of compact metrizable spaces of dimension $\leq n$. Moreover, it is should be noted that a metrizable space X satisfies this condition if and only if X is s.c.d. σ-locally compact.

Now, we shall show the following characterization.

Theorem 5.2. *Let τ be an infinite cardinal. For a connected space X, the following conditions* (1), (2) *and* (3) *are equivalent*:

(1) *X is an $\ell_2^f(\tau)$-manifold.*
(2) (a) *X is an ANR of weight $= \tau$ and a countable union of closed sets which are discrete unions of f.d. compact metrizable spaces.*
    (b) *X is strongly universal for $\bigoplus_\tau \mathfrak{M}_0(n)$ for all $n \in \omega$.*
    (c) *For every subset $C \subset X$, if $C \in \mathfrak{M}_0^f$, then C is a strong Z-set in X.*
(3) (a) *X is an ANR of weight $= \tau$ and a countable union of closed sets which are discrete unions of f.d. compact metrizable spaces.*
    (b) (i) *X has τ-DAP($\mathfrak{M}_0(n)$) for all $n \in \omega$.*
    (ii) *X is strongly universal for $\mathfrak{M}_0^f$.*
    (c) *For every subset $C \subset X$, if $C \in \mathfrak{M}_0^f$, then C is a strong Z-set in X.*

Proof. The implication (2) ⇒ (3) is clear. According to Proposition 4.4, the condition (b) of (3) implies the condition (b) of (2), so the implication (3) ⇒ (2) also holds. Now, we shall show the equivalence (1) ⇔ (2).

(1) ⇒ (2): Due to Proposition 4.5 of [**20**], X is an ANR which is a countable union of locally compact locally f.d. closed subsets. By Proposition 5.1, X is a countable union of closed subsets which are discrete unions of f.d. compact metrizable spaces. Moreover, since X is connected, we have $\mathrm{w}(X) = \mathrm{w}(\ell_2^f(\tau)) = \tau$. Therefore X satisfies the condition

(a).

By 1.1 of [**20**], every space in $\bigoplus_\tau \mathfrak{M}_0(n)$, $n \in \omega$, can be embedded into $\ell_2^f(\tau)$ as a closed subspace. Hence, the condition (b) follows from the Strong Universality Theorem (cf. Lemma 5.1 of [**9**][4]). Furthermore, since the condition (b) implies that X has the τ-discrete n-cells property for all $n \in \omega$, any f.d. compact subset $C \subset X$ is a Z-set in X by Proposition 2.1. Then C is a strong Z-set in X due to A1 of [**23**], which means that the condition (c) holds.

(2) $\Rightarrow$ (1): Obviously, the class $\mathfrak{C} = \bigcup_{n\in\omega} \bigoplus_\tau \mathfrak{M}_0(n)$ is topological and closed hereditary. As is seen in the proof of (1) $\Rightarrow$ (2), the locally convex topological linear metric space $\ell_2^f(\tau)$ satisfies the condition (2). Due to the condition (a) and Remark 3, with respect to $\mathfrak{C}$ the space $\ell_2^f(\tau)$ and the connected ANR X satisfy ($\star$) in Section 3 and (i) in Theorem 3.3, respectively. Combining the condition (c) with Proposition 2.2(2) implies that $\ell_2^f(\tau)$ and X satisfy ($\star\star$) in Section 3 and (iii) in Theorem 3.3 with respect to $\mathfrak{C}$, respectively. The condition (b) is no other than the condition (ii) in Theorem 3.3. On the other hand, since X is an ANR of weight $= \tau$ and a countable union of locally compact locally f.d. closed subsets, applying Theorem 4.3 of [**20**] to $X \times \ell_2^f(\tau)$, we have $X \times \ell_2^f(\tau)$ is an $\ell_2^f(\tau)$-manifold. According to Theorem 3.3, X is homeomorphic to $X \times \ell_2^f(\tau)$, that is, an $\ell_2^f(\tau)$-manifold. □

Remark 4. As is seen in the above, the space $\ell_2^f(\tau)$ has the properties ($\star$) and ($\star\star$) in Section 3 with respect to the class $\mathfrak{C} = \bigcup_{n\in\omega} \bigoplus_\tau \mathfrak{M}_0(n)$. Then, it follows from $\mathfrak{C} \subset \bigoplus_\tau \mathfrak{M}_0^f$ that $\ell_2^f(\tau)$ satisfies ($\star$) with respect to $\bigoplus_\tau \mathfrak{M}_0^f$, immediately. Moreover, combining (c) of Theorem 5.2 with Proposition 2.2(2) implies the stronger assertion that $\ell_2^f(\tau)$ satisfies ($\star\star$) with respect to $\bigoplus_\tau \mathfrak{M}_0^f$, actually. In addition, removing "finite-dimensionality", we have $\ell_2^f(\tau) \times \boldsymbol{Q}$ satisfies ($\star$) and ($\star\star$) with respect to the class $\bigoplus_\tau \mathfrak{M}_0$.

Using the above characterization, we shall prove Main Theorem.

Proof of Main Theorem. Using the condition (3) of Theorem 5.2, we can obtain the "only if" part immediately. Now, we shall prove the "if" part. Since X is locally contractible, each point $x \in X$ has an open neighborhood W which is contractible in X. It is enough to show that W is an $\ell_2^f(\tau)$-manifold, that is, W satisfies (3) of Theorem 5.2.

It follows from Proposition 2.2(1) that W satisfies the condition (c). To verify the condition (b-ii), suppose that $f : A \to W$ is a map from $A \in \mathfrak{M}_0^f$ such that the restriction $f|_B$ on a closed subset B of A is a Z-embedding. For each open cover $\mathcal{W} \in \mathrm{cov}(W)$, the collection $\mathcal{U} = \mathcal{W} \cup \{X \setminus f(A)\} \in \mathrm{cov}(X)$ because A is compact. Then, applying the strong universality of X to f allows us to find a Z-embedding $g : A \to X$ such that $g \sim_\mathcal{U} f$ and $g|_B = f|_B$. Due to the definition of $\mathcal{U}$, we have $g(A) \subset W$ and $g \sim_\mathcal{W} f$. Thus, W satisfies (b-ii). The contractibility of W in X and the τ-discrete n-cells property of X, $n \in \omega$, imply that W has τ-DAP($\mathfrak{M}_0(n)$) for all $n \in \omega$ by Proposition 4.3, namely, the condition (b-i) is satisfied. It remains to check the condition (a). It follows from

[4]Lemma 5.1 of [**9**] holds for a locally convex topological linear metric space E not only such that E is homeomorphic to $E^\mathbb{N}$ but also such that E is homeomorphic to $E_f^\mathbb{N}$.

τ-DAP($\mathfrak{M}_0(n)$) of W that $\tau \le \mathrm{w}(W) \le \mathrm{w}(X) = \tau$, hence $\mathrm{w}(W) = \tau$. Since W is an open subset in X, it is an ANR and an F_σ-set in X. Then, because X is a countable union of closed subsets which are discrete unions of f.d. compact metrizable space, so an F_σ-set W is. Therefore, the condition (a) holds. $\square$

By removing "finite-dimensionality" from the characterization of $\ell_2^f(\tau)$-manifolds, we can similarly prove the following characterization of $(\ell_2^f(\tau) \times \boldsymbol{Q})$-manifolds.

THEOREM 5.3. *Let τ be an infinite cardinal. A connected space X is an $(\ell_2^f(\tau) \times \boldsymbol{Q})$-manifold if and only if the following conditions are satisfied*:

(1) *X is an ANR of weight $= \tau$ and a countable union of closed sets which are discrete unions of compact metrizable spaces.*
(2) *X has the τ-discrete cells property.*
(3) *X is strongly universal for $\mathfrak{M}_0$.*
(4) *For every subset $C \subset X$, if $C \in \mathfrak{M}_0$, then C is a strong Z-set in X.*

6. A Proof of Theorem A.

Throughout Sections 6 and 7, we consider τ an infinite cardinal. Then, we can regard $\ell_1(\tau)$ as a linear subspace in $\mathbb{R}^\tau$. Now, we shall fix some notation for the sake of convenience in Sections 6 and 7. Define the vector $\boldsymbol{e}_\gamma \in \ell_1(\tau)$ for each $\gamma \in \tau$ as follows:

$$\boldsymbol{e}_\gamma(\gamma') = \begin{cases} \boldsymbol{e}_\gamma(\gamma') = 1 & \text{if } \gamma' = \gamma, \\ \boldsymbol{e}_\gamma(\gamma') = 0 & \text{if } \gamma' \neq \gamma, \end{cases}$$

that is, $\boldsymbol{e}_\gamma$ is an unit vector of $\ell_1(\tau)$. For a subset $\Gamma \subset \tau$, we identify

$$\mathbb{R}^\Gamma = \{x \in \mathbb{R}^\tau \mid x(\gamma) = 0 \text{ for all } \gamma \notin \Gamma\}.$$

Then, the projection $p_\Gamma : \ell_1(\tau) \to \ell_1(\tau) \cap \mathbb{R}^\Gamma$ is defined by

$$p_\Gamma(x)(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in \Gamma, \\ 0 & \text{if } \gamma \notin \Gamma. \end{cases}$$

Moreover, let $B_1(x, \epsilon) = \{y \in \ell_1(\tau) \mid \|x - y\|_1 < \epsilon\}$ for each $x \in \ell_1(\tau)$ and $\epsilon > 0$. For a simplicial complex K and $n \in \omega$, let $K^{(n)}$ be the n-skeleton of K. The set of all vertices of a simplex σ is denoted by $\sigma^{(0)}$.

PROPOSITION 6.1. *For any simplicial complex K, the metric polyhedron $|K|_m$ is a countable union of closed sets which are discrete unions of f.d. compact metrizable spaces.*

PROOF. For each simplex $\sigma \in K$, let $\hat{\sigma}$ and $\partial\sigma$ be the barycenter and the boundary of σ, respectively. Given $\sigma \in K \setminus K^{(0)}$ and $t \in \boldsymbol{I}$,

$$\sigma[t] = \{(1 - s)\hat{\sigma} + sx \mid x \in \partial\sigma, 0 \le s \le t\}$$

is a closed subset of σ. Let $\mathcal{A}_0 = K^{(0)}$ and $\mathcal{A}_n = \{\sigma[1-2^{-n}] \mid \sigma \in K^{(n)} \setminus K^{(0)}\}$ for all $n \in \mathbb{N}$, so $\mathcal{A}_n$ is a discrete collection of f.d. compact metrizable spaces in $|K|_m$. Then $|K|_m = \bigcup_{n\in\omega}(\bigcup \mathcal{A}_n)$. Consequently, the assertion holds. $\square$

The following lemma is a reformulation of Lemma 6.1 of [**4**], which is a non-separable version of Lemma 1 of [**11**].

Lemma 6.2. *Let $G = (G, d)$ be a topological group with a left-invariant metric d and $\theta \in G$ be the unit element. Suppose that S is a submonoid*[5] *in G which is locally path-connected at θ. If S satisfies the following condition*:

($*$) *For any neighborhood V of θ in S, there is a subset $T \subset V$ with* $\operatorname{card}(T) = \tau$ *so that there exists $\delta > 0$ such that $d(x, y) \geq \delta$ for every distinct points $x, y \in T$,*

then S has the τ-discrete n-cells property for all $n \in \omega$.

Using the above lemma and the technique of Proposition 7.1 of [**4**], we shall prove the following.

Proposition 6.3. *For every $n \in \omega$, the metric polyhedron $|\Delta(\tau)|_m$ has the τ-discrete n-cells property.*

Proof. For the sake of convenience, we denote $X = |\Delta(\tau)|_m$ and $L = \ell_1(\tau)$, and define the admissible metric d on $L \times \mathbb{R}$ as follows:

$$d((x,s),(y,t)) = \|x-y\|_1 + |s-t| \text{ for every } x, y \in L \text{ and } s, t \in \mathbb{R}.$$

Furthermore, we use the following notation for subsets and a point of $L \times \mathbb{R}$:

$$S = \{(tx, t) \mid x \in X, t \in [0,\infty)\} \subset L \times \mathbb{R},$$
$$X[t_1, t_2] = \{(tx, t) \mid x \in X, t_1 \leq t \leq t_2\} \subset S \text{ for } 0 \leq t_1 \leq t_2 < \infty$$
$$\text{and } \theta = (\mathbf{0}, 0) \in S.$$

Note that $X[1,1]$ is homeomorphic to X. It is easy to see that the map $r : S \setminus \{\theta\} \ni (x,t) \mapsto x/t \in X$ is a retraction of $S \setminus \{\theta\}$ onto X and the restriction $r|_{X[1/2,3/2]}$ is a perfect map.

We shall show that if S has the τ-discrete n-cells property, then X has the τ-discrete n-cells property. Let $D = \bigoplus_{\gamma\in\tau} D_\gamma$, where $D_\gamma = \boldsymbol{I}^n$. Given any map $f : D \to X$ and any open cover $\mathcal{U} \in \operatorname{cov}(X)$, we define a map $f' : D \to X[1,1] \subset S$ and an open cover $\tilde{\mathcal{U}} \in \operatorname{cov}(S)$ as follows:

$$f'(x) = (f(x), 1) \text{ for all } x \in D \text{ and}$$
$$\tilde{\mathcal{U}} = \left\{ S \setminus X\left[\frac{3}{4}, \frac{5}{4}\right],\ U\left(\frac{1}{2}, \frac{3}{2}\right) \;\middle|\; U \in \mathcal{U} \right\},$$

[5] We call S a submonoid of G if $\theta \in S$ and $xy \in S$ for any $x, y \in S$.

$$\text{where } U\left(\frac{1}{2}, \frac{3}{2}\right) = \left\{(tx, t) \,\middle|\, x \in U,\ \frac{1}{2} < t < \frac{3}{2}\right\}.$$

Then, we can find a map $\tilde{f} : D \to S$ such that $\tilde{f} \sim_{\tilde{\mathcal{U}}} f'$ and $\{\tilde{f}(D_\gamma) \mid \gamma \in \tau\}$ is discrete in S. Note that $\tilde{f}(D) \subset X[1/2, 3/2]$, so we can define the map $g : D \to X$ as the composition $g = r\tilde{f}$. Then $g \sim_{\mathcal{U}} f$. Moreover, it follows from the discreteness of $\{\tilde{f}(D_\gamma) \mid \gamma \in \tau\}$ and the perfectness of $r|_{X[1/2,3/2]}$ that $\{g(D_\gamma) \mid \gamma \in \tau\}$ is locally finite in X. Consequently, X has τ-LFAP($\boldsymbol{I}^n$). Due to Lemma 4.2, X has the τ-discrete n-cells property.

Now, using Lemma 6.2, we shall show that S has the τ-discrete n-cells property. First, S is a submonoid in $L \times \mathbb{R}$. Let $x, y \in X$ and $s, t \in [0, \infty)$. Since X is a convex subset of L, we have $z = tx/(t+s) + sy/(t+s) \in X$. Then,

$$\begin{aligned}(tx, t) + (sy, s) &= (tx + sy, t + s) \\ &= \left((t+s)\left(\frac{tx}{t+s} + \frac{sy}{t+s}\right), t + s\right) \\ &= ((t+s)z, t+s) \in S,\end{aligned}$$

so S is a submonoid in $L \times \mathbb{R}$.

Second, to check the local path-connectedness of S at θ, let V be a neighborhood of θ in S. Then, we can choose $t > 0$ so that $W = S \cap (B_1(\mathbf{0}, t) \times (-t, t)) \subset V$. Given $x_1, x_2 \in X$ and $s_1, s_2 \in [0, t)$, the two points $(s_1x_1, s_1), (s_2x_2, s_2) \in W$ can be connected by the path $\alpha : \boldsymbol{I} \to W$ defined as follows:

$$\alpha(x) = \begin{cases} ((1-2s)s_1x_1, (1-2s)s_1) & \text{if } 0 \le s \le \dfrac{1}{2}, \\[2ex] ((2s-1)s_2x_2, (2s-1)s_2) & \text{if } \dfrac{1}{2} \le s \le 1. \end{cases}$$

Hence S is locally path-connected at θ.

Finally, we show that S satisfies the condition $(*)$ in Lemma 6.2. Note that X is contained in the unit sphere of L. For each neighborhood V of θ, we can take $t > 0$ so that $T = \{(tv, t) \mid v \in \Delta(\tau)^{(0)}\} \subset V$. Then $\operatorname{card}(T) = \operatorname{card}(\Delta(\tau)^{(0)}) = \tau$. Moreover, for every distinct vertices $v, v' \in \Delta(\tau)^{(0)}$,

$$d((tv, t), (tv', t)) = \|tv - tv'\|_1 + |t - t| = t\|v - v'\|_1 = 2t,$$

that is, T is the desired subset. Thus, the proof is complete. □

Proposition 6.4. *Every simplex σ of the full simplicial complex $\Delta(\tau)$ is a strong Z-set in $|\Delta(\tau)|_m$.*

Proof. Let $\alpha : |\Delta(\tau)|_m \to (0, 1)$ be a map. Due to Proposition 2.3, it is sufficient to show that there exists a map $f : |\Delta(\tau)|_m \to |\Delta(\tau)|_m$ such that $\|x - f(x)\|_1 \le \alpha(x)$ for $x \in |\Delta(\tau)|_m$ and $\operatorname{cl} f(|\Delta(\tau)|_m) \in |\Delta(\tau)|_m \setminus \sigma$. Fix a vertex $v_0 \in \Delta(\tau)^{(0)} \setminus \sigma^{(0)}$. Since $\Delta(\tau)$ is a full simplicial complex, we can define the desired map $f : |\Delta(\tau)|_m \to |\Delta(\tau)|_m$

as follows:

$$f(x) = \left(1 - \frac{\alpha(x)}{2}\right)x + \frac{\alpha(x)}{2}v_0 \text{ for every } x \in |\Delta(\tau)|_m.$$

Indeed, for each $x \in |\Delta(\tau)|_m$,

$$\begin{aligned}\|x - f(x)\|_1 &= \left\| x - \left(1 - \frac{\alpha(x)}{2}\right)x + \frac{\alpha(x)}{2}v_0 \right\|_1 \\ &\leq \frac{\alpha(x)}{2}(\|x\|_1 + \|v_0\|_1) = \frac{\alpha(x)}{2}2 = \alpha(x).\end{aligned}$$

Then, it remains to prove that $\mathrm{cl}\, f(|\Delta(\tau)|_m) \in |\Delta(\tau)|_m \setminus \sigma$. On the contrary, we assume that there is a point $x \in \mathrm{cl}\, f(|\Delta(\tau)|_m) \cap \sigma$. It follows from the compactness of σ that we can choose

$$0 < \alpha_0 = \min\{\alpha(y) \mid y \in \sigma\} \leq \alpha(x).$$

By the continuity of α, there is $\delta > 0$ such that $\alpha(y) > 2\alpha_0/3$ for all $y \in B_1(x, \delta) \cap |\Delta(\tau)|_m$. On the other hand, we have a sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subset |\Delta(\tau)|_m$ converges to x because $x \in \mathrm{cl}\, f(|\Delta(\tau)|_m)$. Remark that for every $n \in \mathbb{N}$,

$$\|x - f(x_n)\|_1 = \left\| x - \left(1 - \frac{\alpha(x_n)}{2}\right)x_n - \frac{\alpha(x_n)}{2}v_0 \right\|_1 \geq \frac{\alpha(x_n)}{2}$$

since $v_0 \notin \sigma^{(0)}$. Then, we can take $n_0 \in \mathbb{N}$ so that $\|x - f(x_{n_0})\|_1 < \min\{\delta, \alpha_0\}/3$, hence we get

$$\alpha(x_{n_0}) \leq 2\|x - f(x_{n_0})\|_1 < \frac{2}{3}\min\{\delta, \alpha_0\}.$$

It follows that

$$\begin{aligned}\|x - x_{n_0}\|_1 &\leq \|x - f(x_{n_0})\|_1 + \|x_{n_0} - f(x_{n_0})\|_1 \\ &\leq \|x - f(x_{n_0})\|_1 + \alpha(x_{n_0}) \\ &< \frac{\delta}{3} + \frac{2\delta}{3} = \delta,\end{aligned}$$

which implies $\alpha(x_{n_0}) > 2\alpha_0/3$. This is a contradiction. Consequently, $\mathrm{cl}\, f(|\Delta(\tau)|_m) \cap \sigma = \emptyset$. Thus, the proof is complete. □

While every compact subspace of the polyhedron endowed with the weak topology is covered by finitely many simplexes, the following holds with respect to the metric polyhedron.

Lemma 6.5. *Let K be an infinite simplicial complex and A be a compact subset of $|K|_m$. Then, there exists a countable subcomplex $L \subset K$ such that $A \subset |L|$.*

Proof. By Theorem 1 of [**16**], the identity $\mathrm{id} : |K|_w \to |K|_m$ is a fine homotopy equivalence, where $|K|_w$ is the polyhedron of K endowed with the weak topology. Then, for each $n \in \mathbb{N}$, we can find a map $f_n : |K|_m \to |K|_w$ such that $\mathrm{id}\, f_n$ is $1/n$-close to the identity map $\mathrm{id}_{|K|_m}$. Since $f_n(A)$ is compact in $|K|_w$, there exists a finite subcomplex L_n of K such that $f_n(A) \subset |L_n|$.

Let $L = \bigcup_{n\in\mathbb{N}} L_n$. Clearly, L is a countable subcomplex of K. Moreover, $A \subset |L|$. On the contrary, we suppose that there is a point $x \in A \setminus |L|$. Since A is compact and $|L|$ is closed in $|K|_m$, we can choose $n \in \mathbb{N}$ so that $\inf_{y\in|L|} \|x - y\|_1 > 1/n$. On the other hand, we have $f_n(x) \in |L_n| \subset |L|$ and $\|x - f_n(x)\|_1 < 1/n$ because $\mathrm{id}\, f_n$ is $1/n$-close to $\mathrm{id}_{|K|_m}$. This is a contradiction. Hence $A \subset |L|$, which means that the proof is completed. □

Combining the results which we have obtained, we can establish the following proposition.

Proposition 6.6. *If C is a compact subset of the metric polyhedron $|\Delta(\tau)|_m$, then C is a strong Z-set in $|\Delta(\tau)|_m$.*

Proof. Due to Lemma 6.5, we have a countable subcomplex L of $\Delta(\tau)$ so that $C \subset |L|$. Since all simplexes of $\Delta(\tau)$ are strong Z-sets in $|\Delta(\tau)|_m$ by Proposition 6.4, it follows that $C = \bigcup_{\sigma\in L}(C \cap \sigma)$ is a strong Z_σ-set in $|\Delta(\tau)|_m$. On the other hand, $|\Delta(\tau)|_m$ has the τ-discrete n-cells property for every $n \in \omega$ by Proposition 6.3. It follows from Proposition 2.1 that C is a Z-set in $|\Delta(\tau)|_m$. Furthermore, according to Proposition 3.1 of [**19**], C is a strong Z-set in $|\Delta(\tau)|_m$. □

We can prove the next proposition by using the technique of Theorem 4 of [**24**].

Proposition 6.7. *The metric polyhedron $|\Delta(\tau)|_m$ is strongly universal for $\mathfrak{M}_0^f$.*

Proof. For simplicity, let X stand for $|\Delta(\tau)|_m$. To verify the strong universality of X, take any space $A \in \mathfrak{M}_0^f$, any closed subset B in A, any map $f : A \to X$ such that the restriction $f|_B$ is a Z-embedding and an arbitrary open cover $\mathcal{U} \in \mathrm{cov}(X)$. We shall construct an embedding $\tilde{g} : A \to X$ such that $\tilde{g} \sim_{\mathcal{U}} f$ and $\tilde{g}|_B = f|_B$. Then, remark that $\tilde{g}$ is a Z-embedding because of Proposition 6.6.

We can write $A \setminus B = \bigcup_{n\in\mathbb{N}} A_n$, where $A_1 \subset A_2 \subset \cdots$ are closed subsets in A. Applying Theorem 2.4 of [**21**][6] to a Z-set $f(B)$, we can obtain a homotopy $\phi : X \times \boldsymbol{I} \to X$ so that $\phi_0 = \mathrm{id}_X$ and $\phi(X \times (0,1]) \subset X \setminus f(B)$. Let $k : A \to \boldsymbol{I}$ be a map such that $k^{-1}(0) = B$ and for each $x \in A$, there exists $U \in \mathcal{U}'$ such that $\{f(x)\} \times [0, k(x)] \subset \phi^{-1}(U)$, where $\mathcal{U} \succ^\star \mathcal{U}' \in \mathrm{cov}(X)$. We define the map $f' : A \to X$ by $f'(x) = \phi(f(x), k(x))$. Observe that $f' \sim_{\mathcal{U}'} f$, $f'|_B = f|_B$ and $f'(A \setminus B) \subset X \setminus f(B)$. We have a Lebesgue number $\lambda < 1$ for $\mathcal{U}'$ with respect to $f'(A)$. By the same argument of Lemma 2.5, we can find an open cover $\mathcal{V} \in \mathrm{cov}\,(X \setminus f(B))$ with $\mathrm{mesh}\,\mathcal{V} < \lambda$ so as to satisfy the following

[6]Remark that a Z-set in an ANR is a closed locally homotopy negligible set.

condition:

> For a map $h : (f')^{-1}(X \setminus f(B)) = A \setminus B \to X \setminus f(B)$, if $h \sim_{\mathcal{V}} f'|_{A \setminus B}$, then h extends to the map $\tilde{h} : A \to X$ by $\tilde{h}|_B = f|_B$.

We take a sequence of open covers $\mathcal{V} \succ^\star \mathcal{V}_0 \succ^\star \mathcal{V}_1 \succ^\star \cdots \in \operatorname{cov}(X \setminus f(B))$ with $\operatorname{mesh} \mathcal{V}_n < 2^{-n}$ for every $n \in \omega$. Moreover, since $X \setminus f(B)$ is an ANR, we can choose $\mathcal{V}'_n \in \operatorname{cov}(X \setminus f(B))$ for each $n \in \omega$ so that $\mathcal{V}'_n \prec \mathcal{V}_n$ and has the following property:

> Given a space Y and maps $h_1, h_2 : Y \to X \setminus f(B)$, if $h_1 \sim_{\mathcal{V}'_n} h_2$, then $h_1 \simeq_{\mathcal{V}_n} h_2$.

Now, we shall inductively construct maps $g_n : A \setminus B \to X \setminus f(B)$, $n \in \omega$, a tower of finite subsets $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \tau$ such that

(1) $g_n|_{A_n}$ is an embedding into $\mathbb{R}^{\Gamma_n} \cap (X \setminus f(B))$,
(2) $g_{n+1}|_{A_n} = g_n|_{A_n}$ and
(3) $g_{n+1} \simeq_{\mathcal{V}_n} g_n$,

where $g_0 = f'$ and $A_0 = \emptyset$. Assume that g_j and Γ_j have been obtained for all $j < n$. Let $\lambda_n < 1$ be a Lebesgue number for $\mathcal{V}'_n$ with respect to $g_{n-1}(A_n)$. By the same argument as the proof of Theorem 4 in [**24**], since $g_{n-1}(A_n)$ is compact, we can choose a finite subset $\Gamma'_n \subset \tau$ so that $\Gamma_{n-1} \subset \Gamma'_n$ and $p_{\Gamma'_n} g_{n-1}|_{A_n}$ is $\lambda_n/4$-close to $g_{n-1}|_{A_n}$. Let Γ''_n be a finite subset of $\tau \setminus \Gamma'_n$ of cardinality $= 2\dim(A_n) + 2$. Then $\Delta_n = X \cap \mathbb{R}^{\Gamma''_n}$ is a simplex of $\Delta(\tau)$ of dimension $= \operatorname{card}(\Gamma''_n) - 1 = 2\dim(A_n) + 1$, so we have an embedding $q_n : A_n \to \Delta_n \subset X$. Let $\Gamma_n = \Gamma'_n \cup \Gamma''_n$, so $\mathbb{R}^{\Gamma_n} = \mathbb{R}^{\Gamma'_n} \oplus \mathbb{R}^{\Gamma''_n}$. Taking a map $k_n : A_n \to \boldsymbol{I}$ with $k_n^{-1}(0) = A_{n-1}$, we can define the map $g'_n : A_n \to X \cap \mathbb{R}^{\Gamma_n}$ as follows:

$$g'_n(x) = \frac{p_{\Gamma'_n} g_{n-1}(x) + \lambda_n k_n(x) q_n(x)/4}{\|p_{\Gamma'_n} g_{n-1}(x) + \lambda_n k_n(x) q_n(x)/4\|_1}.$$

It follows from $k_n(A_{n-1}) = 0$ that $g'_n|_{A_{n-1}} = g_{n-1}|_{A_{n-1}}$.

Now, we shall show that g'_n is an embedding. For the sake of convenience, let

$$g''_n(x) = p_{\Gamma'_n} g_{n-1}(x) + \frac{\lambda_n}{4} k_n(x) q_n(x) \in \mathbb{R}^{\Gamma_n}.$$

Suppose $x, y \in A_n$ such that $g'_n(x) = g'_n(y)$, so

$$\frac{\lambda_n k_n(x)}{4\|g''_n(x)\|_1} q_n(x) = p_{\Gamma''_n} g'_n(x) = p_{\Gamma''_n} g'_n(y) = \frac{\lambda_n k_n(y)}{4\|g''_n(y)\|_1} q_n(y).$$

In case $k_n(x) = 0$, we get $k_n(y) = 0$. Hence

$$g_{n-1}(x) = g'_n(x) = g'_n(y) = g_{n-1}(y),$$

which implies $x = y$. In case $k_n(x) > 0$, we have $k_n(y) > 0$. In addition, $\|q_n(x)\|_1 = \|q_n(y)\|_1 = 1$. It follows that $\lambda_n k_n(x)/(4\|g''_n(x)\|_1) = \lambda_n k_n(y)/(4\|g''_n(y)\|_1)$, hence $q_n(x) = q_n(y)$, which means that $x = y$ because q_n is an embedding into Δ_n. Consequently, g'_n is an embedding.

Next, we show that $g'_n \simeq_{\mathcal{V}_n} g_{n-1}|A_n$. Given any $x \in A_n$,

$$\begin{aligned}\|g_{n-1}(x) - g''_n(x)\|_1 &\le \|g_{n-1}(x) - p_{\Gamma'_n} g_{n-1}(x)\|_1 + \frac{\lambda_n}{4} k_n(x) \|q_n(x)\|_1 \\ &< \frac{\lambda_n}{4} + \frac{\lambda_n}{4} = \frac{\lambda_n}{2} \text{ and} \\ 1 - \frac{\lambda_n}{2} \le \|g_{n-1}(x)\|_1 - \left\|g_{n-1}(x) - g''_n(x)\right\|_1 &\le \left\|g''_n(x)\right\|_1 \\ &\le \|g_{n-1}(x)\|_1 + \left\|g_{n-1}(x) - g''_n(x)\right\|_1 \le 1 + \frac{\lambda_n}{2}.\end{aligned}$$

Observe that

$$\left\|g''_{n-1}(x) - g'_n(x)\right\|_1 = \left|1 - \frac{1}{\|g''_n(x)\|_1}\right| \left\|g''_n(x)\right\|_1 = \left|\|g''_n(x)\|_1 - 1\right| \le \frac{\lambda_n}{2}.$$

Therefore, we have

$$\begin{aligned}\left\|g_{n-1}(x) - g'_n(x)\right\|_1 &\le \left\|g_{n-1}(x) - g''_n(x)\right\|_1 + \left\|g''_n(x) - g'_n(x)\right\|_1 \\ &< \frac{\lambda_n}{2} + \frac{\lambda_n}{2} = \lambda_n.\end{aligned}$$

As λ_n is a Lebesgue number for $\mathcal{V}'_n$ with respect to $g_{n-1}(A_n)$, we have $g'_n \sim_{\mathcal{V}'_n} g_{n-1}|A_n$, so it follows from the definition of $\mathcal{V}'_n$ that $g'_n \simeq_{\mathcal{V}_n} g_{n-1}|A_n$. Note that $g'_n \sim_{\mathcal{V}} f'|_{A_n}$, so $g'(A_n) \subset \mathrm{st}(f'(A_n), \mathcal{V}) \subset X \setminus f(B)$. The Homotopy Extension Theorem allows us to find the desired map $g_n : A \setminus B \to X \setminus f(B)$ so that

(1) $g_n|_{A_n} = g'_n$ is an embedding into $\mathbb{R}^{\Gamma_n} \cap (X \setminus f(B))$,
(2) $g_n|_{A_{n-1}} = g'_n|_{A_{n-1}} = g_{n-1}|_{A_{n-1}}$ and
(3) $g_n \simeq_{\mathcal{V}_n} g_{n-1}$.

This completes the inductive step.

After completing the inductive construction, due to (2), (3) and mesh $\mathcal{V}_n < 2^{-n}$ for all $n \in \omega$ the sequence $\{g_n\}_{n\in\omega}$ converges to the map $g : A \setminus B \to X \setminus f(B)$. It follows that $g|_{A_n} = g_n|_{A_n}$ for all $n \in \omega$, so $g \sim_{\mathcal{V}} f'|_{A\setminus B}$. The definition of $\mathcal{V}$ extends g to the desired map $\tilde{g} : A \to X$ by $\tilde{g}|_B = f|_B$. Indeed, the restriction $g|_{A_n} = g_n|_{A_n}$ is an embedding into $X \setminus f(B)$ by (1), hence $\tilde{g}$ is also an embedding. It is easy to see that $\tilde{g} \sim_{\mathcal{U}} f$. In conclusion, $\tilde{g}$ is the desired embedding. □

We have seen that the metric polyhedron $|\Delta(\tau)|_m$ satisfies the conditions of Main Theorem. The combination of Main Theorem and Theorem 6 of [**13**] implies Theorem A.

7. A Proof of Theorem B.

This section is devoted to proving Theorem B. We use an admissible metric d on $J(\tau)^{\mathbb{N}}$ as follows:

$$d(x,y)=\sum_{i\in\mathbb{N}}2^{-i}\|x(i)-y(i)\|_1 \text{ for every } x,y\in J(\tau)^{\mathbb{N}}.$$

Let $\mathrm{pr}_i : J(\tau)^{\mathbb{N}} \to J(\tau)$ be the projection onto the ith coordinate. Moreover, for $x, y \in \ell_1(\tau)$, the line segment between x and y is denoted by $\langle x, y\rangle$, that is,

$$\langle x,y\rangle=\{(1-t)x+ty \mid t\in \boldsymbol{I}\}.$$

First, we shall show that $J(\tau)^{\mathbb{N}}_f$ satisfies the condition (1) of Main Theorem.

Proposition 7.1. *The space $J(\tau)^{\mathbb{N}}_f$ is an AR of weight $=\tau$ and a countable union of closed subsets which are discrete unions of f.d. compact metrizable spaces.*

Proof. First, we verify that $J(\tau)^{\mathbb{N}}_f$ is an AR of weight $=\tau$. It is clear that $\mathrm{w}(J(\tau)^{\mathbb{N}}_f)=\tau$. The hedgehog $J(\tau)$ is an AR, so is $J(\tau)^{\mathbb{N}}$. The space $J(\tau)^{\mathbb{N}}_f$ is homotopy dense in $J(\tau)^{\mathbb{N}}$. Indeed, we can take a contraction $\phi : J(\tau)\times \boldsymbol{I} \to J(\tau)$ such that $\phi_0=\mathrm{id}_{J(\tau)}$ and $\phi_1(J(\tau))=\{\mathbf{0}\}$. Then, the homotopy $h : J(\tau)^{\mathbb{N}}\times \boldsymbol{I}\to J(\tau)^{\mathbb{N}}$ is defined as follows: $h(x,0)=x$ and

$$h(x,t)=\big(\mathrm{pr}_1(x),\dots,\mathrm{pr}_{i-1}(x),\phi(\mathrm{pr}_i(x),2^it-1),\mathbf{0},\mathbf{0},\dots\big),$$
$$\text{for each } x\in J(\tau)^{\mathbb{N}} \text{ and } 2^{-i}\le t\le 2^{-i+1}.$$

It follows that $h_0=\mathrm{id}_{J(\tau)}$ and $h((0,1])\subset J(\tau)^{\mathbb{N}}_f$, hence $J(\tau)^{\mathbb{N}}_f$ is homotopy dense in $J(\tau)^{\mathbb{N}}$. As is well known, a homotopy dense subset of an AR is also an AR. Therefore $J(\tau)^{\mathbb{N}}_f$ is an AR.

Next, we shall show that $J(\tau)^{\mathbb{N}}_f$ is a countable union of closed subsets which are discrete unions of f.d. compact metrizable spaces. Let $\mathrm{Fin}(\mathbb{N})$ be the all non-empty finite subsets of $\mathbb{N}$. For each $M\in\mathrm{Fin}(\mathbb{N})$, $n\in\omega$ and each function $\psi_M : M\to\tau$, we define the f.d. compact subset of $J(\tau)^{\mathbb{N}}_f$ as follows:

$$A^{\psi_M}_{(M,n)}=\left\{x\in J(\tau)^{\mathbb{N}}\,\middle|\,\begin{array}{ll} x(i)\in\langle 2^{-n}\boldsymbol{e}_{\psi_M(i)},\boldsymbol{e}_{\psi_M(i)}\rangle, & i\in M, \text{ and}\\ x(i)=\mathbf{0}, & \textit{otherwise}\end{array}\right\},$$

which is homeomorphic to the cube $\boldsymbol{I}^{\mathrm{card}(M)}$. Let

$$\mathcal{A}_{(M,n)}=\big\{A^{\psi_M}_{(M,n)} \mid \psi_M : M\to\tau\big\} \text{ for each } M\in\mathrm{Fin}(\mathbb{N}) \text{ and } n\in\omega.$$

Fix a point $x\in J(\tau)^{\mathbb{N}}_f\setminus\{\mathbf{0}\}$, so we have

$$M=\{i\in\mathbb{N}\mid x(i)\neq\mathbf{0}\}\in\mathrm{Fin}(\mathbb{N}).$$

Define the function $\psi_M : M\to\tau$ as follows:

$$\psi_M(i)=\gamma\in\tau \text{ if } x(i)(\gamma)>0 \text{ for each } i\in M.$$

Taking $n \in \omega$ so that $2^{-n} \leq \min_{i \in M} \|x(i)\|_1$, we can easily see that $x \in A^{\psi_M}_{(M,n)}$. It follows that

$$J(\tau)^{\mathbb{N}}_f = \{\mathbf{0}\} \cup \Big(\bigcup_{M \in \mathrm{Fin}(\mathbb{N}), n \in \omega} \Big(\bigcup \mathcal{A}_{(M,n)} \Big) \Big).$$

Moreover, $\mathcal{A}_{(M,n)}$ is discrete in $J(\tau)^{\mathbb{N}}_f$ for each $M \in \mathrm{Fin}(\mathbb{N})$ and $n \in \omega$. Indeed, let $x \in J(\tau)^{\mathbb{N}}_f$. When $x(i) = \mathbf{0}$ for some $i \in M$, we have $B_d(x, 2^{-n}) \cap A^{\psi_M}_{(M,n)} = \emptyset$ for every $\psi_M : M \to \tau$. When $x(i) \neq \mathbf{0}$ for all $i \in M$, as is observed, we can take the unique function $\psi_M : M \to E$ such that $x(i) \in \langle \mathbf{0}, \boldsymbol{e}_{\psi_M(i)} \rangle \setminus \{\mathbf{0}\}$. Then, define $\delta = \min_{i \in M} \|x(i)\|_1$, so $B_d(x, \delta) \cap A^{\psi'_M}_{(M,n)} = \emptyset$ for every $\psi'_M : M \to \tau$ with $\psi'_M \neq \psi_M$. Thus, the proof is complete. □

The condition (2) of Main Theorem holds with respect to $J(\tau)^{\mathbb{N}}_f$.

Proposition 7.2. *For all $n \in \omega$, the space $J(\tau)^{\mathbb{N}}_f$ has the τ-discrete n-cells property.*

Proof. For simplicity, let X be $J(\tau)^{\mathbb{N}}_f$. Due to Lemma 4.2, it suffices to show that X has τ-LFAP($\boldsymbol{I}^n$) for each $n \in \omega$. Suppose that $f : D = \bigoplus_{\gamma \in \tau} D_\gamma \to X$ is a map, where $D_\gamma = \boldsymbol{I}^n$ for all $\gamma \in \tau$. We shall construct a map $g : D \to X$ for each $\alpha : X \to (0,1)$ such that $d(g(x), f(x)) < \alpha f(x)$ for every $x \in D$ and $\{g(D_\gamma) \mid \gamma \in \tau\}$ is locally finite in X. Let

$$\Gamma_i = \Big\{ \gamma \in \tau \mid 2^{-i} < \min_{x \in D_\gamma} \alpha f(x) \leq 2^{-i+1} \Big\} \text{ for each } i \in \mathbb{N}.$$

Then $\tau = \bigoplus_{i \in \mathbb{N}} \Gamma_i$. Now, we define the desired map $g : D \to X$ as follows:

$$g(x) = (\mathrm{pr}_1 f(x), \ldots, \mathrm{pr}_i f(x), \boldsymbol{e}_\gamma, \mathbf{0}, \mathbf{0}, \ldots) \text{ for each } x \in D_\gamma, \gamma \in \Gamma_i \text{ and } i \in \mathbb{N}.$$

It follows that if $x \in D_\gamma$, $\gamma \in \Gamma_i$ and $i \in \mathbb{N}$, then

$$d(g(x), f(x)) \leq 2^{-i} < \min_{x' \in D_\gamma} \alpha f(x') \leq \alpha f(x),$$

that is, $d(g(x), f(x)) < \alpha f(x)$ for every $x \in D$. Moreover, $\{g(D_\gamma) \mid \gamma \in \Gamma_i\}$ is discrete in X for each $i \in \mathbb{N}$.

To verify the local finiteness of $\{g(D_\gamma) \mid \gamma \in \tau\}$, fix a point $x \in X$ arbitrarily. In case $x(i) \notin \{\boldsymbol{e}_\gamma \mid \gamma \in \tau\}$ for all $i \in \mathbb{N}$, we see

$$(B_d(x, \delta) \cap X) \cap g(D) = \emptyset \text{ for some } \delta > 0.$$

Indeed, $\|x(i)\|_1 < 1$ for all $i \in \mathbb{N}$ and there is $j_0 \in \mathbb{N}$ such that $x(j) = \mathbf{0}$ for every $j > j_0$ because $x \in X$. Then, we can define $\delta = 1 - \max_{i \leq j_0} \|x(i)\|_1 > 0$. In case $x(i_0) \in \{\boldsymbol{e}_\gamma \mid \gamma \in \tau\}$ for some $i_0 \in \mathbb{N}$, there exists $j_0 > i_0$ such that $x(j) = \mathbf{0}$ for all

$j \geq j_0$. Then, we have

$$(B_d(x,1) \cap X) \cap g(D_\gamma) = \emptyset \text{ for each } \gamma \in \Gamma_j, j \geq j_0 - 1.$$

On the other hand, since $\{g(D_\gamma) \mid \gamma \in \Gamma_i\}$ is discrete in X for each $i < j_0 - 1$, we can find an open neighborhood U_i of x in X so that

$$\operatorname{card}\{\gamma \in \Gamma_i \mid U_i \cap g(D_\gamma) \neq \emptyset\} \leq 1.$$

Taking an open neighborhood $U = \left(\bigcap_{i<j_0-1} U_i\right) \cap B_d(x,1)$ of x in X, we have

$$\operatorname{card}\{\gamma \in \tau \mid U \cap g(D_\gamma) \neq \emptyset\} < j_0 - 1 < \infty,$$

that is, $\{g(D_\gamma) \mid \gamma \in \tau\}$ is locally finite in X. Thus, the proof is complete. □

The following proposition implies the condition (4) of Main Theorem.

PROPOSITION 7.3. *Every compact subset of $J(\tau)^{\mathbb{N}}_f$ is a strong Z-set.*

PROOF. For the sake of convenience, $J(\tau)^{\mathbb{N}}_f$ is denoted by X. Let C be a compact subset of X and $\alpha : X \to (0,1)$ be a map. Due to Proposition 2.3, it is sufficient to show that there exists a map $g : X \to X$ such that $d(g(x), x) \leq \alpha(x)$ for all $x \in X$ and $\operatorname{cl} g(X) \subset X \setminus C$. Since τ is infinite and C is compact, there is an element $\gamma(i) \in \tau$ for each $i \in \mathbb{N}$ such that $\operatorname{pr}_i(C) \cap \langle \boldsymbol{e}_{\gamma(i)}/2, \boldsymbol{e}_{\gamma(i)}\rangle = \emptyset$. Then, we can take a contraction $\phi_i : J(\tau) \times \boldsymbol{I} \to J(\tau)$ for each $i \in \mathbb{N}$ such that $(\phi_i)_0 = \operatorname{id}_{J(\tau)}$ and $(\phi_i)_1(J(\tau)) = \{\boldsymbol{e}_{\gamma(i)}\}$.

Now, we define the map $g : X \to X$ as follows:

$$\begin{aligned} g(x) = \big(&\operatorname{pr}_1(x), \ldots, \operatorname{pr}_{i+1}(x), \\ &\phi_{i+2}(\operatorname{pr}_{i+2}(x), 2^i\alpha(x) - 1), \boldsymbol{e}_{\gamma(i+3)}, (2 - 2^i\alpha(x))\boldsymbol{e}_{\gamma(i+4)}, \mathbf{0}, \mathbf{0}, \ldots\big) \\ &\qquad \text{if } 2^{-i} \leq \alpha(x) \leq 2^{-i+1}. \end{aligned}$$

Then, we have

$$d(g(x), x) \leq \sum_{j \geq i+2} 2^{-j+1} = 2^{-i} \leq \alpha(x).$$

Taking any point $x \in \bigcup_{y\in C} B_d(y, 1/4)$, we have $x(i) \neq \boldsymbol{e}_{\gamma(i)}$ for each $i \in \mathbb{N}$. Indeed, we can choose $y \in C$ so that $d(x, y) < 1/4$. Then, we get

$$\left\|x(i) - \boldsymbol{e}_{\gamma(i)}\right\|_1 \geq \left\|y(i) - \boldsymbol{e}_{\gamma(i)}\right\|_1 - \|x(i) - y(i)\|_1 > \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0,$$

hence $x(i) \neq \boldsymbol{e}_{\gamma(i)}$. Therefore $x \in X \setminus g(X)$. It follows that $\bigcup_{y\in C} B_d(y, 1/4) \subset X \setminus g(X)$, so

$$\operatorname{cl} g(X) \subset X \setminus \bigcup_{y \in C} B_d(y, 1/4) \subset X \setminus C.$$

Consequently, g is the desired map. $\square$

The following proposition, which is proved by the same argument as Proposition 6.7, is no other than the condition (3) on $J(\tau)^{\mathbb{N}}_f$.

Proposition 7.4. *The space $J(\tau)^{\mathbb{N}}_f$ is strongly universal for $\mathfrak{M}^f_0$.*

Proof. For the sake of convenience, let X be $J(\tau)^{\mathbb{N}}_f$ and

$$X_m = \{x \in X \mid x(i) = \mathbf{0} \text{ for all } i > m\} \subset X \text{ for each } m \in \mathbb{N}.$$

Suppose that $A \in \mathfrak{M}^f_0$, B is a closed subset of A, $f : A \to X$ such that $f|_B$ is a Z-embedding and $\mathcal{U} \in \operatorname{cov}(X)$. Due to Proposition 7.3, it is sufficient to construct an embedding $\tilde{g} : A \to X$ such that $\tilde{g} \sim_{\mathcal{U}} f$ and $\tilde{g}|_B = f|_B$. We have $A \backslash B = \bigcup_{n \in \mathbb{N}} A_n$, where $A_1 \subset A_2 \subset \cdots$ are closed subsets of A, and $\mathcal{U} \succ^{\star} \mathcal{U}' \in \operatorname{cov}(X)$. By the same argument of Proposition 6.7, we can find a map $f' : A \to X$ such that $f' \sim_{\mathcal{U}'} f$, $f'|_B = f|_B$ and $f'(A \setminus B) \subset X \setminus f(B)$. Let $\lambda > 1$ be a Lebesgue number for $\mathcal{U}'$ with respect to $f'(A)$. Moreover, we can take open covers

$$\mathcal{V} \succ^{\star} \mathcal{V}_0 \succ^{\star} \mathcal{V}_1 \succ^{\star} \cdots \in \operatorname{cov}(X \setminus f(B)) \text{ and } \mathcal{V}_n \succ \mathcal{V}'_n \in \operatorname{cov}(X \setminus f(B)), n \in \omega,$$

with $\operatorname{mesh} \mathcal{V} < \lambda$ and $\operatorname{mesh} \mathcal{V}_n < 2^{-n}$ for every $n \in \omega$ so as to satisfy the following conditions:

($*$) For a map $h : (f')^{-1}(X \setminus f(B)) = A \backslash B \to X \setminus f(B)$, if $h \sim_{\mathcal{V}} f'|_{A \backslash B}$, then h extends to the map $\tilde{h} : A \to X$ by $\tilde{h}|_B = f|_B$.

($**$) Given a space Y and maps $h_1, h_2 : Y \to X \setminus f(B)$, if $h_1 \sim_{\mathcal{V}'_n} h_2$, then $h_1 \simeq_{\mathcal{V}_n} h_2$.

By induction, we shall construct maps $g_n : A \backslash B \to X \setminus f(B)$, $n \in \omega$, and a sequence of natural numbers $1 = m(0) < m(1) < \cdots$ such that

(1) $g_n|_{A_n}$ is an embedding into $X_{m(n)} \setminus f(B)$,
(2) $g_{n+1}|_{A_n} = g_n|_{A_n}$ and
(3) $g_{n+1} \simeq_{\mathcal{V}_n} g_n$,

where $g_0 = \mathrm{id}_A$ and $A_0 = \emptyset$. Similar to Proposition 6.7, the sequence $\{g_n\}_{n \in \omega}$ converges to the map $g : A \backslash B \to X \setminus f(B)$ such that $g|_{A_n} = g_n|_{A_n}$ and g is extended to the desired embedding $\tilde{g} : A \to X$ by $\tilde{g}|_B = f|_B$. Therefore, it remains to complete the induction.

Assume that g_j and $m(j)$ have been obtained for all $j < n$. Let $\lambda_n < 1$ be a Lebesgue number for $\mathcal{V}'_n$ with respect to $g_{n-1}(A_n)$. Then, there is a number $m(n)' \geq m(n-1)$ such that $\sum_{i > m(n)'} 2^{-i+1} < \lambda_n$. Let $m(n) = m(n)' + 2\dim(A) + 2$. Fix an unit vector $\boldsymbol{e}$ of $\ell_1(\tau)$. Remark that the segment $\langle \boldsymbol{e}/2, \boldsymbol{e} \rangle$ is contained in $J(\tau)$. By the finite dimensionality of A_n, there exists an embedding $q_n : A_n \to \langle \boldsymbol{e}/2, \boldsymbol{e} \rangle^{2\dim(A)+1}$. Taking a map $k_n : A_n \to \boldsymbol{I}$ with $k_n^{-1}(0) = A_{n-1}$, we can define the map $g'_n : A_n \to X_{m(n)} \setminus f(B)$ as follows:

$$\operatorname{pr}_i g_n'(x) = \begin{cases} \operatorname{pr}_i g_{n-1}(x) & \text{if } i \leq m(n)', \\ k_n(x) p_{i-m'_{(n)}} q_n(x) & \text{if } m(n)' < i < m(n), \\ k_n(x)\boldsymbol{e} & \text{if } i = m(n), \\ \mathbf{0} & \text{if } m(n) < i, \end{cases}$$

where $p_j : \langle \boldsymbol{e}/2, \boldsymbol{e} \rangle^{2\dim(A)+1} \to \langle \boldsymbol{e}/2, \boldsymbol{e} \rangle$ is the projection onto the jth coordinate, $j = 1, \ldots, 2\dim(A)+1$. Then g_n' is an embedding. Indeed, take two distinct points $x, y \in A_n$ arbitrarily. In case $x, y \in A_{n-1}$, we have $k_n(x) = k_n(y) = 0$, so

$$g_n'(x) = g_{n-1}(x) \neq g_{n-1}(y) = g_n'(y)$$

since $g_{n-1}|_{A_{n-1}}$ is an embedding. In case $x \in A_n \setminus A_{n-1}$ and $y \in A_{n-1}$, we get $k_n(x) > 0 = k_n(y)$, hence

$$\operatorname{pr}_{m(n)} g_n'(x) = k_n(x)\boldsymbol{e} \neq \mathbf{0} = \operatorname{pr}_{m(n)} g_n'(y),$$

that is, $g_n'(x) \neq g_n'(y)$. In case $x, y \in A_n \setminus A_{n-1}$, we have $k_n(x), k_n(y) > 0$. When $k_n(x) \neq k_n(y)$, we see

$$\operatorname{pr}_{m(n)} g_n'(x) = k_n(x)\boldsymbol{e} \neq k_n(y)\boldsymbol{e} = \operatorname{pr}_{m(n)} g_n'(y),$$

so $g_n'(x) \neq g_n'(y)$. When $k_n(x) = k_n(y)$, there is $m(n)' < i < m(n)$ such that

$$\operatorname{pr}_i g_n'(x) = k_n(x) \operatorname{pr}_i q_n(x) \neq k_n(y) \operatorname{pr}_i q_n(y) = \operatorname{pr}_i g_n'(y)$$

because q_n is an embedding. Therefore $g_n'(x) \neq g_n'(y)$. Moreover, $g_n'|_{A_{n-1}} = g_{n-1}|_{A_{n-1}}$ because $g_{n-1}(A_{n-1}) \subset X_{m_{(n-1)}}$ and $k_n(A_{n-1}) = 0$. For every $x \in A_n$, we have

$$\begin{aligned} d(g_n'(x), g_{n-1}(x)) &= \sum_{i \in \mathbb{N}} 2^{-i} \big\| \operatorname{pr}_i g_n'(x) - \operatorname{pr}_i g_{n-1}(x) \big\|_1 \\ &\leq \sum_{i \leq m(n)'} 2^{-i} \big\| \operatorname{pr}_i g_n'(x) - \operatorname{pr}_i g_{n-1}(x) \big\|_1 + \sum_{i > m(n)'} 2^{-i+1} \\ &= \sum_{i > m(n)'} 2^{-i+1} < \lambda_n, \end{aligned}$$

hence $g_n' \sim_{\mathcal{V}_n'} g_{n-1}|_{A_n}$. By $(**)$, $g_n' \simeq_{\mathcal{V}_n} g_{n-1}|_{A_n}$. Applying the Homotopy Extension Theorem to g_n', we can obtain an extension $g_n : A \setminus B \to X \setminus f(B)$, which is desired, of g_n' such that $g_n \simeq_{\mathcal{V}_n} g_{n-1}$. □

Combining Main Theorem with Theorem 6 of [**13**] establishes Theorem B.

8. A Proof of Theorem C.

In this section, we shall prove Theorem C. Let $\mathcal{C}$ be a topological and closed hereditary class of spaces. We denote the collection of closed subspaces in a space X which belong to $\mathcal{C}$ by $\mathcal{C}(X)$. A subspace Y of X is said to be *weakly $\mathcal{C}(X)$-absorptive*[7] if for each $A \in \mathcal{C}(X)$, each closed subset B of A contained in Y and each $\mathcal{U} \in \operatorname{cov}(X)$, there exists an embedding $f : A \to Y$ such that $f \sim_{\mathcal{U}} \mathrm{id}_A$ and $f|_B = \mathrm{id}_B$. A space Y has a *$\mathcal{C}$-complex structure* $\{\mathcal{A}_n\}_{n\in\omega}$ if each $\mathcal{A}_n$ is a subcollection of $\mathcal{C}(Y)$ with the following properties:

(1) $Y = \bigcup_{n\in\omega}(\bigcup \mathcal{A}_n)$,
(2) $A_n = \bigcup_{i=0}^{n}(\bigcup \mathcal{A}_i)$ is closed in Y for each $n \in \omega$ and
(3) for each $n \in \omega$, there exists a pairwise disjoint open cover $\mathcal{U}_n$ of $A_n \setminus A_{n-1}$ in Y such that $U \cap A_n \setminus A_{n-1} \in \{A \setminus A_{n-1} \mid A \in \mathcal{A}_n\}$ for each $U \in \mathcal{U}_n$, where $A_{-1} = \emptyset$.

J. E. West established the following characterization of $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pairs in 1970, see Theorem 6 of [**24**].

Theorem 8.1. *Let τ be an infinite cardinal. For spaces $Y \subset X$, the pair (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, Y is weakly $\mathfrak{M}_0^f(X)$-absorptive and has an $\mathfrak{M}_0^f$-complex structure.*

Due to Theorem 6 of [**13**] and Theorem 1 of [**24**], we can classify $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pairs according to homotopy type.

Theorem 8.2. *Let τ be an infinite cardinal. Suppose that (X, Y) and (X', Y') are $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pairs. If X and X' (or Y and Y') have the same homotopy type, then (X, Y) and (X', Y') are homeomorphic.*

Remark 5. While it is not mentioned in [**24**], the similar characterization of $(\ell_2(\tau) \times \boldsymbol{Q}, \ell_2^f(\tau) \times \boldsymbol{Q})$-manifold pairs can be established as follows:

> The pair (X, Y) is an $(\ell_2(\tau) \times \boldsymbol{Q}, \ell_2^f(\tau) \times \boldsymbol{Q})$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold,[8] Y is weakly $\mathfrak{M}_0(X)$-absorptive and has an $\mathfrak{M}_0$-complex structure.

In addition, Theorem 8.2 is valid for $(\ell_2(\tau) \times \boldsymbol{Q}, \ell_2^f(\tau) \times \boldsymbol{Q})$-manifold pairs.

Although the complex structure is defined by imitating the simplicial complex structure, it is complicated. The following proposition is useful to verify that for a topological and closed hereditary class $\mathcal{C}$ a metrizable space X has a $\mathcal{C}$-complex structure.

Proposition 8.3. *For a topological and closed hereditary class $\mathcal{C}$, a metrizable space X is a countable union of closed sets which are discrete unions of members of $\mathcal{C}$ if and only if X has a $\mathcal{C}$-complex structure.*

Proof. First, we show the "only if" part. Let $X = \bigcup_{n\in\omega}(\bigcup \mathcal{A}_n)$, where $\mathcal{A}_n$ is a discrete collection of X whose members are in $\mathcal{C}$ and the union $\bigcup \mathcal{A}_n$ is closed in X

[7]This notion is introduced in Theorem 6 of [**24**].
[8]Remark that $\ell_2(\tau) \times \boldsymbol{Q}$ is homeomorphic to $\ell_2(\tau)$.

for each $n \in \omega$. Note that $\mathcal{A}_n \subset \mathcal{C}(X)$ for all $n \in \omega$. Then $A_n = \bigcup_{i=0}^n(\bigcup \mathcal{A}_i)$ is closed in X for every $n \in \omega$. Since each $\mathcal{A}_n$ is discrete in X, there exists a pairwise disjoint collection $\mathcal{U}_n = \{U(A) \mid A \in \mathcal{A}_n\}$ of open subsets of X such that $A \subset U(A)$ for each $A \in \mathcal{A}_n$. Observe that $U(A) \cap (A_n \setminus A_{n-1}) = A \setminus A_{n-1}$ for each $A \in \mathcal{A}_n$ and $n \in \omega$, where $A_{-1} = \emptyset$. Consequently, the collections $\{\mathcal{A}_n\}_{n\in\omega}$ is a $\mathcal{C}$-complex structure of X.

Next, we prove the "if" part. Let $\{\mathcal{A}_n\}_{n\in\omega}$ be a $\mathcal{C}$-complex structure of X. Then, for each $n \in \omega$ there exists a pairwise disjoint collection $\mathcal{U}_n$ of open subsets of X satisfying the following condition:

> each $\mathcal{U}_n$ covers $A_n \setminus A_{n-1}$ so that $U \cap A_n \setminus A_{n-1} \in \{A \setminus A_{n-1} \mid A \in \mathcal{A}_n\}$ for every $U \in \mathcal{U}_n$, where $A_{-1} = \emptyset$.

For every $U \in \mathcal{U}_n$ and $n \in \omega$, we can choose $A \in \mathcal{A}_n$ so that $U \cap A_n \setminus A_{n-1} = A \setminus A_{n-1}$, which is open in A, so an F_σ-set in A. Hence, we can write $U \cap A_n \setminus A_{n-1} = \bigcup_{m\in\omega} A^m_{(n,U)}$, where each $A^m_{(n,U)}$ is closed in A, so closed in X. It is easy to see that $\mathcal{A}_{(n,m)} = \{A^m_{(n,U)} \mid U \in \mathcal{U}_n\}$ is discrete in X and the union $\bigcup \mathcal{A}_{(n,m)}$ is closed in X for all $n, m \in \omega$. Moreover, $X = \bigcup_{n,m\in\omega}(\bigcup \mathcal{A}_{(n,m)})$. Indeed, for each $x \in X$, choose $n \in \omega$ such that $x \in A_n \setminus A_{n-1}$. Since $\mathcal{U}_n$ covers $A_n \setminus A_{n-1}$, there is $U \in \mathcal{U}_n$ such that $x \in U \cap A_n \setminus A_{n-1} = \bigcup_{m\in\omega} A^m_{(n,U)}$, which implies that $x \in A^m_{(n,U)} \subset \bigcup \mathcal{A}_{(n,m)}$ for some $m \in \omega$. Thus, X is a countable union of closed sets which are discrete unions of members of $\mathcal{C}$. □

PROPOSITION 8.4. *Let $\mathcal{C}$ be a topological and closed hereditary subclass of $\mathfrak{M}$. Suppose that a homotopy dense subset Y of a metrizable space X satisfies the following conditions*:

(∗) *Y is strongly universal for $\mathcal{C}$.*
(∗∗) *Every closed subset $C \in \mathcal{C}(Y)$ is a Z-set in Y.*

Then Y is weakly $\mathcal{C}(X)$-absorptive.

PROOF. Fix $A \in \mathcal{C}(X)$, a closed subset B of A contained in Y and $\mathcal{U} \in \operatorname{cov}(X)$. Take $\mathcal{V} \in \operatorname{cov}(X)$ so that $\mathcal{V} \prec^\star \mathcal{U}$. Since Y is homotopy dense in X, we can find a homotopy $h : X \times \boldsymbol{I} \to X$ such that $h_0 = \mathrm{id}_X$ and $h(X \times (0, 1]) \subset Y$. Then, we have a map $k : A \to \boldsymbol{I}$ such that $k^{-1}(0) = B$ and $\{\{x\} \times [0, k(x)] \mid x \in A\} \prec h^{-1}(\mathcal{V})$. Define a map $f : A \to Y \subset X$ by $f(x) = h(x, k(x))$ for each $x \in A$, so $f \sim_{\mathcal{V}} \mathrm{id}_A$ and $f|_B = h_0|_B = \mathrm{id}_B$. On the other hand, since $\mathcal{C}$ is closed hereditary, it follows from (∗∗) that B is a Z-set in Y, hence the restriction $f|_B$ is a Z-embedding into Y. Then, applying the strong universality of Y to f, we can obtain a Z-embedding $g : A \to Y$ such that $g \sim_{\mathcal{V}|_Y} f$ and $g|_B = f|_B = \mathrm{id}_B$. Observe that $g \sim_{\mathcal{U}} \mathrm{id}_A$. Consequently, Y is weakly $\mathcal{C}(X)$-absorptive. □

A subset $A \subset X$ is said to be *locally homotopy negligible* in a space X if for each $n \in \omega$, $x \in X$ and open neighborhood U of x, there exists a neighborhood V of x such that given a map $f : (\boldsymbol{I}^n, \mathrm{bd}\, \boldsymbol{I}^n) \to (V, V \setminus A)$, there is a homotopy $h : (\boldsymbol{I}^n, \mathrm{bd}\, \boldsymbol{I}^n) \times \boldsymbol{I} \to (U, U \setminus A)$ with $h_0 = f$ and $h_1(\boldsymbol{I}^n) \subset U \setminus A$, where $\mathrm{bd}\, \boldsymbol{I}^n$ is the boundary of $\boldsymbol{I}^n$. It is easy to see that for every infinite cardinal τ, the subset $\ell_2(\tau) \setminus \ell_2^f(\tau)$ is locally homotopy negligible in $\ell_2(\tau)$. Now, we shall demonstrate Theorem C.

Proof of Theorem C. First, we prove the "only if" part. Since $\ell_2(\tau) \setminus \ell_2^f(\tau)$ is locally homotopy negligible in $\ell_2(\tau)$, it follows from Remark 2.2 of [**21**] that $U \setminus \ell_2^f(\tau)$ is locally homotopy negligible in U for every open subset $U \subset \ell_2(\tau)$. This means that $X \setminus Y$ is locally homotopy negligible in X, recall that (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair. Thus, Y is homotopy dense in X by Theorem 2.4 of [**21**].

Next, we show the "if" part. Since Y is an $\ell_2^f(\tau)$-manifold, it follows from (1) of Main Theorem and Proposition 8.3 that Y has an $\mathfrak{M}_0^f$-complex structure. Moreover, (3) and (4) of Main Theorem imply the conditions $(*)$ and $(**)$ in Proposition 8.4 for the class $\mathfrak{M}_0^f$, respectively. Because Y is homotopy dense in X, we have that Y is weakly $\mathfrak{M}_0^f(X)$-absorptive by Proposition 8.4. Then, we can apply Proposition 8.1 to the pair (X, Y), so (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair. □

Remark 6. Combining Theorem 5.3, Propositions 8.3 and 8.4 with Remark 5, we can obtain another characterization of $(\ell_2(\tau) \times \boldsymbol{Q}, \ell_2^f(\tau) \times \boldsymbol{Q})$-manifold pairs as follows:

> For spaces $Y \subset X$, the pair (X, Y) is an $(\ell_2(\tau) \times \boldsymbol{Q}, \ell_2^f(\tau) \times \boldsymbol{Q})$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, Y is an $(\ell_2^f(\tau) \times \boldsymbol{Q})$-manifold and Y is homotopy dense in X.

Due to Theorems A, B, C and 8.2, in order to prove Corollaries A and B we may verify that $\mathrm{cl}_{\ell_1(\tau)} |\Delta(\tau)|$ (respectively, $J(\tau)^{\mathbb{N}}$) is homeomorphic to $\ell_2(\tau)$ and $|\Delta(\tau)|_m$ (respectively, $J(\tau)^{\mathbb{N}}_f$) is homotopy dense in $\mathrm{cl}_{\ell_1(\tau)} |\Delta(\tau)|$ (respectively, $J(\tau)^{\mathbb{N}}$).

Proof of Corollary A. The closed convex subset $\mathrm{cl}_{\ell_1(\tau)} |\Delta(\tau)| \subset \ell_1(\tau)$ is homeomorphic to $\ell_2(\tau)$ due to Theorem 2 of [**3**]. Moreover, by Corollary 6.8.5 of [**18**], the convex subset $|\Delta(\tau)|_m \subset \ell_1(\tau)$ is a uniform AR. Since the uniform AR $|\Delta(\tau)|_m$ is dense in $\mathrm{cl}_{\ell_1(\tau)} |\Delta(\tau)|$, it follows from Theorem 2 of [**17**] that $|\Delta(\tau)|_m$ is homotopy dense in $\mathrm{cl}_{\ell_1(\tau)} |\Delta(\tau)|$. □

Proof of Corollary B. The countable product space $J(\tau)^{\mathbb{N}}$ is homeomorphic to $\ell_2(\tau)$, see Remark of Theorem 5.1 in [**22**]. Furthermore, as is seen in the proof of Proposition 7.1, the space $J(\tau)^{\mathbb{N}}_f$ is homotopy dense in $J(\tau)^{\mathbb{N}}$. □

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