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On a weak attractor of a class of PDEs with degenerate diffusion and chemotaxis

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Abstract. In this article we deal with a class of degenerate parabolic systems that encompasses two different effects: porous medium and chemotaxis. Such classes of equations arise in the mesoscale level modeling of biomass spreading mechanisms via chemotaxis. We prove estimates related to the existence of the global attractor under certain 'balance conditions' on the order of the porous medium degeneracy and the growth of the chemotactic function.

1. Introduction.

This study aims to consider the following model

$$\dot{M} = \nabla \cdot (|M|^{\alpha} \nabla M) - \nabla \cdot (|M|^{\gamma} \nabla \rho) + f(M, \rho) \quad \text{in } (0, \infty) \times \Omega, \tag{1.1}$$

$$\dot{\rho} = \Delta \rho - g(M, \rho) \quad \text{in } (0, \infty) \times \Omega, \tag{1.2}$$

$$M = 0, \quad \rho = 1 \quad \text{in } (0, \infty) \times \partial\Omega, \tag{1.3}$$

$$M(0, \cdot) = M_0, \quad \rho(0, \cdot) = \rho_0 \quad \text{in } \Omega, \tag{1.4}$$

where α and γ are given positive constants satisfying $(\alpha/2) + 1 \leq \gamma < \alpha$ (we call these conditions the *balance conditions*). Moreover, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain $(N = 1, 2, 3)$ and $M_0 \in L^{\infty}(\Omega)$, $\rho_0 \in W^{1,\infty}(\Omega)$. We assume that the functions f and g satisfy the following assumptions:
for all $M, \rho \in \mathbb{R}$ let

$$|f(M, \rho)| \leq F_1 (1 + |M|^{\xi})^{1/2} \quad \text{for some } \xi \in [0, \alpha - \gamma + 2),\ F_1 \in \mathbb{R}_0^+, \tag{1.5}$$

$$f(M, \rho) M \leq -F_2 M^2 + F_3 |M| \quad \text{for some } F_2 \in \mathbb{R}^+,\ F_3 \in \mathbb{R}_0^+, \tag{1.6}$$

$$g(M, \rho) = G_1 \rho + g_2(\rho) M \quad \text{for some } G_1 \in \mathbb{R}_0^+, \tag{1.7}$$

$$|g_2(\rho)| \leq G_3 \quad \text{for some } G_3 \in \mathbb{R}_0^+ \tag{1.8}$$

and, in order to ensure the uniqueness and the non-negativity of solutions for non-negative initial data, let

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$$\widetilde{f}(M,\rho) := f\big(M|M|^{(2/(2+\alpha))-1},\rho\big) - F_4 M|M|^{(2/(2+\alpha))-1},$$

$$\frac{\partial \widetilde{f}}{\partial M} \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}) \text{ for some } F_4 \in \mathbb{R}, \tag{1.9}$$

$$f \in W^{1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}), \quad g_2 \in W^{1,\infty}_{loc}(\mathbb{R}), \quad f(0,\rho)=0, \quad g_2(0)=0, \tag{1.10}$$

where

$$\mathbb{R}^+ = (0,+\infty), \quad \mathbb{R}^+_0 = [0,+\infty),$$
$$L^p_{loc}(Q) = \{u : Q \to \mathbb{R} : u \in L^p(K) \text{ for all compact sets } K \subset Q\},$$
$$W^{1,p}_{loc}(Q) = \{u : Q \to \mathbb{R} : u \in W^{1,p}(K) \text{ for all compact sets } K \subset Q\}$$

for $p \in [1,\infty]$, $Q \subset \mathbb{R}^m$. The following example of functions f and g satisfies the conditions (1.5)–(1.10):

Example 1.

$$f(M,\rho) = -M + \frac{M_+^{(2+\alpha)/2}}{M_+^{(2+\alpha)/2}+1}\arctan\rho,$$
$$g(M,\rho) = \rho + M\frac{\rho}{\rho+1},$$

where $M_+ = \max\{M,0\}$.

In the present paper, we treat weak solutions of the system (1.1)–(1.4). The definition is as follows:

Definition 1. For $T>0$, $\alpha>1$ and $\gamma>1$, a pair of functions (M,ρ) defined in $[0,T]\times\Omega$ is said to be a weak solution of (1.1)–(1.4) for $M_0 \in L^\infty(\Omega)$, $\rho_0 \in W^{1,\infty}(\Omega)$, if

(i) $M \in L^\infty([0,T]\times\Omega)$, $|M|^{\alpha/2}M \in L^2([0,T];H^1_0(\Omega))$, $\dot{M} \in L^2([0,T];H^{-1}(\Omega))$,
(ii) $\rho - 1 \in C([0,T];H^1_0(\Omega))$,
(iii) (M,ρ) satisfies the equation in the following sense:

$$\int_0^T (M,v)\dot{\varphi} - (|M|^\alpha\nabla M - |M|^\gamma\nabla\rho, \nabla v)\varphi + (f(M,\rho),v)\varphi\, ds = 0$$

for any $v \in H^1_0(\Omega)$, $\varphi \in C^\infty_0[0,T]$,

$$\begin{aligned}(\rho(t,x)-1) = &\int_\Omega G(t,x,y)(\rho_0(y)-1)dy\\ &- \int_0^t\int_\Omega G(t-s,x,y)g(M(s,y),\rho(s,y))dyds\end{aligned}$$

for a.e. $(t,x) \in [0,T] \times \Omega$, where G is a heat kernel in Ω with the homogeneous Dirichlet boundary condition and the initial conditions hold: $\rho(0) = \rho_0$ and, in $C_w([0,T]; L^2(\Omega))$-sense, $M(0) = M_0$.

NOTATION 1. For $p \in [1,\infty]\backslash\{2\}$, we write $\|\cdot\|_p$ in place of the $\|\cdot\|_{L^p(\Omega)}$-norm. $\|\cdot\|$ stands for $\|\cdot\|_{L^2(\Omega)}$-norm and (u,v) for $\int_\Omega u(x)v(x)\,dx$ or, more generally (in the case of distributional derivatives for instance), for $\langle u,v\rangle$.

REMARK 1. From $M \in L^\infty([0,T]; L^2(\Omega))$ and $\dot{M} \in L^2([0,T]; H^{-1}(\Omega))$ it follows (see [**2**]), that $M \in C_w([0,T]; L^2(\Omega))$ Recall that $C_w([0,T]; L^2(\Omega))$ denotes the space of functions $u : [0,T] \to L^2(\Omega)$ which are continuous with respect to the weak topology of $L^2(\Omega)$, therefore the initial condition for M makes sense.

REMARK 2. Note: we do not actually need the condition $\gamma \geq (\alpha/2) + 1$ for the dissipative estimate we want to obtain, but it was crucial for uniqueness of solutions (see [**4**], [**5**]).

This system of partial differential equations, models, for example, a population described in terms of its density M, which grows depending on a substrate with concentration ρ. The substrate is degraded by the abiotic decay. The spatial movement of the population is caused by two different effects. Firstly, the model includes a density dependent diffusion term. This non-linear diffusion effect becomes stronger as the population grows larger locally, following a power law as in the case of the porous medium equation. Secondly, the population moves towards regions with increased substrate availability, i.e. follows the chemical signal ρ. This effect is also controlled by the population density and its intensity increases as the local population density grows. Both effects of population mobility increase/diminish with the population, each following a power law. Thus, the model degenerates for $M = 0$. Finally, our model includes a 'source term': a non-linear reaction-interaction term f. As usual, it stays for the sink/source density (net number of particles created per unit time and per unit volume). At a high level of population density M the depth rate (caused by the exterior forces such as predation or intoxication) is no less than F_2.

The main focus of the present study is to prove a dissipative estimate for the problem (1.1)–(1.4). We emphasize the fact that the analysis of equations with a chemotaxis-type term even without degeneracy ($\alpha = 0$) is quite difficult, (examples, though for somewhat different biological models, can be found in [**7**], [**8**], [**10**], [**11**], [**12**], see also references therein) and in our degenerate case, we face significant difficulties. In order to overcome these difficulties we impose so-called 'balance conditions' between the order of porous-medium degeneracy and the growth order of the chemotaxis function: $(\alpha/2)+1 \leq \gamma < \alpha$. We showed in [**5**] (see also [**4**]) that our model is a well-posed one and that it exhibits no singular behavior. For each pair of starting values the solution is uniformly bounded in time and space. Recall that this is not the case for the models that contain the chemotaxis effect alone (the solution may blow up, see [**6**]). The condition $\alpha > \gamma$ (an improvement over the condition $\alpha \geq \gamma + 1$ imposed in [**4**]) reads: the density-dependent diffusion coefficient 'dominates' the intensity of response to the chemical signal as the population density grows. This, as we showed in [**5**], results in the uniform boundedness of M and ρ.

On the other hand, we also showed in [**5**] that even in the areas with low population density the porous medium effect is due to $(\alpha/2)+1 \leq \gamma$ strong enough to keep the population spreading without vanishing locally, which means that the support of $M(t,\cdot)$, the set $\overline{\{x \in \Omega \mid M(t,x) > 0\}}$, is not shrinking in t.

In [**5**] (see also [**4**]) we showed the time-global existence and boundedness of solution to our system. The main result of [**5**] can be summarized as follows:

THEOREM 1. *Let the functions f and g satisfy the assumptions* (1.5)–(1.10) *and let the given constants α and γ satisfy $\gamma \in [(\alpha/2)+1, \alpha)$. Then the initial boundary-value problem* (1.1)–(1.4) *has at most one non-negative solution* (*in the sense of Definition* 1) *for each pair of starting values $(M_0, \rho_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$. The solution is uniformly bounded in time in the phase space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$.*

However, the estimates derived there were not sufficient to show the existence of the attractor. In this paper we use the condition $\alpha > \gamma$ to establish a dissipative estimate for our model, which will be necessary to show the existence of the attractor.

Our main result reads:

THEOREM 2. *Let the functions f and g satisfy the assumptions* (1.5)–(1.10) *and let the given constants α and γ satisfy $\gamma \in [(\alpha/2)+1, \alpha)$. Then the following dissipative estimate holds for the initial boundary-value problem* (1.1)–(1.4) :

$$\begin{aligned} &\|M(t)\|_{L^\infty(\Omega)} + \|\rho(t)\|_{W^{1,\infty}(\Omega)} \\ &\quad \leq C_\infty \big(\|M_0\|_{L^\infty(\Omega)} + \|\rho_0\|_{W^{1,\infty}(\Omega)}\big)^{r_\infty} \cdot e^{-\omega_\infty t} + D_\infty \ \forall t \geq 0, \end{aligned} \tag{1.11}$$

where the positive constants $C_\infty, r_\infty, \omega_\infty, D_\infty$ depend only on α, γ, f and g and are independent of M_0, ρ_0 or t.

We will prove this theorem in Section 2. As a consequence of Theorems 1 and 2 we obtain the existence of the weak global attractor for (1.1)–(1.4): we prove in Section 3 the following

THEOREM 3. *Let the functions f and g satisfy the assumptions* (1.5)–(1.10) *and let the given constants α and γ satisfy $\gamma \in [(\alpha/2)+1, \alpha)$. Then the solutions of the problem* (1.1)–(1.4) *can be described by a semigroup $\{S(t)\}_{t\geq 0}$ that acts on the* (*Hausdorff*) *space $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$* ($L^\infty_{w-*}(\Omega)$ *denotes the space $L^\infty(\Omega)$ equipped with the weak$-*$ topology of $L^\infty(\Omega)$*) *and there exists the global attractor for $\{S(t)\}_{t\geq 0}$.*

2. Dissipative estimates (proof of Theorem 2).

In this section we derive several dissipative estimates in various phase spaces for the solutions of the problem (1.1)–(1.4), which in turn lead to a dissipative estimate in $L^\infty(\Omega)$ for both M and ρ.

We start with rewriting the equation (1.1) in the following way:

$$\dot{M} = \nabla \cdot \left(|M|^\gamma \nabla \left(\frac{1}{\alpha - \gamma + 1} M|M|^{(\alpha-\gamma+1)-1} - \rho \right) \right) + f(M, \rho). \tag{2.1}$$

In order to derive our first *a priori* estimate, we multiply this equation by $(1/(\alpha - \gamma + 1)M|M|^{(\alpha-\gamma+1)-1} - \rho)$ and integrate (formally) over Ω to get

$$\begin{aligned}
&\left(\dot{M}, \frac{1}{\alpha - \gamma + 1} M|M|^{(\alpha-\gamma+1)-1} - \rho \right) \\
&\quad = - \left(|M|^\gamma, \left| \nabla \left(\frac{1}{\alpha - \gamma + 1} M|M|^{(\alpha-\gamma+1)-1} - \rho \right) \right|^2 \right) \\
&\qquad + \left(f(M, \rho), \frac{1}{\alpha - \gamma + 1} M|M|^{(\alpha-\gamma+1)-1} - \rho \right) \\
&\quad \leq \left(f(M, \rho), \frac{1}{\alpha - \gamma + 1} M|M|^{(\alpha-\gamma+1)-1} - \rho \right) \\
\Leftrightarrow\ & \frac{d}{dt} \left(\frac{1}{(\alpha - \gamma + 1)(\alpha - \gamma + 2)} \left\| |M|^{\frac{\alpha-\gamma+2}{2}} \right\|^2 - (M, \rho) \right) \\
&\quad \leq \left(f(M, \rho), \frac{1}{\alpha - \gamma + 1} M|M|^{(\alpha-\gamma+1)-1} - \rho \right) - (\dot{\rho}, M)
\end{aligned} \tag{2.2}$$

and we multiply the equation (1.2) by $(\dot{\rho} + \rho - 1)$ in the same sense as above to get

$$\begin{aligned}
&\|\dot{\rho}\|^2 + \frac{1}{2} \frac{d}{dt} \|\rho - 1\|^2 = -\frac{1}{2} \frac{d}{dt} \|\nabla \rho\|^2 - \|\nabla \rho\|^2 - (g(M, \rho), \dot{\rho} + \rho - 1) \\
\Leftrightarrow\ & \frac{1}{2} \frac{d}{dt} \left(\|\nabla \rho\|^2 + \|\rho - 1\|^2 \right) = -\|\nabla \rho\|^2 - \|\dot{\rho}\|^2 - (g(M, \rho), \dot{\rho} + \rho - 1).
\end{aligned} \tag{2.3}$$

Adding the inequalities (2.2) and (2.3) together, we obtain

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{1}{(\alpha - \gamma + 1)(\alpha - \gamma + 2)} \left\| |M|^{\frac{\alpha-\gamma+2}{2}} \right\|^2 - (M, \rho) + \frac{1}{2} \|\nabla \rho\|^2 + \frac{1}{2} \|\rho - 1\|^2 \right) \\
&\quad \leq \left(f(M, \rho), \frac{1}{\alpha - \gamma + 1} M|M|^{(\alpha-\gamma+1)-1} - \rho \right) - \|\nabla \rho\|^2 - (\dot{\rho}, M) - \|\dot{\rho}\|^2 \\
&\qquad - (g(M, \rho), \dot{\rho} + \rho - 1).
\end{aligned} \tag{2.4}$$

We consider first the term containing $g(M, \rho) = G_1 \rho + g_2(\rho) M$. It holds:

$$\begin{aligned}
-(G_1 \rho, \dot{\rho} + \rho - 1) &= -\frac{1}{2} \frac{d}{dt} (G_1 \|\rho\|^2) - G_1 (\|\rho\|^2 - (1, \rho)) \\
&\leq -\frac{1}{2} \frac{d}{dt} (G_1 \|\rho\|^2) - (1 - \varepsilon) G_1 \|\rho\|^2 + \frac{1}{4\varepsilon} G_1 |\Omega|
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned} -(g_2(\rho)M, \dot\rho + \rho - 1) \ &\le\ \varepsilon\|\dot\rho\|^2 + \varepsilon\|\rho-1\|^2 + \frac{1}{2\varepsilon}\|g_2(\rho)M\|^2 \\ &\underset{(1.8)}{\le}\ \varepsilon\|\dot\rho\|^2 + \varepsilon\|\rho-1\|^2 + \frac{1}{2\varepsilon}G_3^2\|M\|^2, \end{aligned} \tag{2.6}$$

where $|\Omega|$ denotes the volume of Ω.

By combining (2.5) and (2.6) with the inequality

$$-(\dot\rho, M) - \|\dot\rho\|^2 \le \frac{1}{2}\|M\|^2 - \frac{1}{2}\|\dot\rho\|^2 \tag{2.7}$$

and by choosing $\varepsilon \le 1/2$ we have

$$\begin{aligned} &-(\dot\rho, M) - \|\dot\rho\|^2 - (g(M,\rho), \dot\rho + \rho - 1) \\ &\quad\le -\frac{1}{2}\frac{d}{dt}\big(G_1\|\rho\|^2\big) - (1-\varepsilon)G_1\|\rho\|^2 + \varepsilon\|\rho-1\|^2 + \frac{1}{4\varepsilon}G_1|\Omega| - \left(\frac{1}{2}-\varepsilon\right)\|\dot\rho\|^2 \\ &\qquad + \left(\frac{1}{2} + \frac{1}{2\varepsilon}G_3^2\right)\|M\|^2 \\ &\quad\underset{\varepsilon\le 1/2}{\le} -\frac{1}{2}\frac{d}{dt}\big(G_1\|\rho\|^2\big) - (1-\varepsilon)G_1\|\rho\|^2 + \varepsilon\|\rho-1\|^2 + \frac{1}{4\varepsilon}G_1|\Omega| \\ &\qquad + \left(\frac{1}{2} + \frac{1}{2\varepsilon}G_3^2\right)\|M\|^2. \end{aligned} \tag{2.8}$$

Further, we can estimate the terms with f from (2.4) in the following way:

$$\begin{aligned} \big(f(M,\rho), M|M|^{(\alpha-\gamma+1)-1}\big) \ &\underset{(1.6)}{\le}\ \big(-F_2M^2 + F_3|M|, |M|^{(\alpha-\gamma+1)-1}\big) \\ &=\ -F_2\big\||M|^{\frac{\alpha-\gamma+2}{2}}\big\|^2 + F_3\big\||M|^{\frac{\alpha-\gamma+1}{2}}\big\|^2, \end{aligned} \tag{2.9}$$

$$\begin{aligned} -(f(M,\rho),\rho) \ &\underset{(1.5)}{\le}\ \varepsilon\|\rho\|^2 + \frac{1}{4\varepsilon}F_1^2\big(|\Omega| + \big\||M|^{\frac{\xi}{2}}\big\|^2\big) \\ &\le\ 2\varepsilon\|\rho-1\|^2 + \left(2\varepsilon + \frac{1}{4\varepsilon}F_1^2\right)|\Omega| + \frac{1}{4\varepsilon}F_1^2\big\||M|^{\frac{\xi}{2}}\big\|^2. \end{aligned} \tag{2.10}$$

Using the inequalities (2.8)–(2.10) we conclude from (2.4):

$$\begin{aligned} &\frac{d}{dt}\Bigg(\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\big\||M|^{\frac{\alpha-\gamma+2}{2}}\big\|^2 - (M,\rho) \\ &\qquad + \frac{1}{2}\|\nabla\rho\|^2 + \frac{1}{2}\|\rho-1\|^2 + \frac{1}{2}G_1\|\rho\|^2\Bigg) \end{aligned}$$

$$\le -F_2\big\||M|^{\frac{\alpha-\gamma+2}{2}}\big\|^2 + F_3\big\||M|^{\frac{\alpha-\gamma+1}{2}}\big\|^2 + \frac{1}{4\varepsilon}F_1^2\big\||M|^{\frac{\xi}{2}}\big\|^2 + \left(\frac{1}{2}+\frac{1}{2\varepsilon}G_3^2\right)\|M\|^2$$

$$- \|\nabla\rho\|^2 - (1-\varepsilon)G_1\|\rho\|^2 + 3\varepsilon\|\rho-1\|^2 + \left(2\varepsilon + \frac{1}{4\varepsilon}G_1 + \frac{1}{4\varepsilon}F_1^2\right)|\Omega|. \qquad (2.11)$$

In order to shorten the formulas, we introduce a new variable:

$$\varphi := \frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\big\||M|^{\frac{\alpha-\gamma+2}{2}}\big\|^2 - (M,\rho)$$
$$+ \frac{1}{2}\|\nabla\rho\|^2 + \frac{1}{2}\|\rho-1\|^2 + \frac{1}{2}G_1\|\rho\|^2. \qquad (2.12)$$

$|M|^{(\alpha-\gamma+2)/2}$ is the leading M-power in the expression (2.12) due to the assumptions made on α, γ and ξ, and we also have the estimate

$$(M,\rho) \le \varepsilon\|\rho\|^2 + \frac{1}{4\varepsilon}\|M\|^2 \qquad (2.13)$$

for all $\varepsilon > 0$. Moreover, applying the Poincaré and the Hölder inequalities and adjusting the constant ε we can deduce from (2.11) the inequality

$$\dot{\varphi} \le -A_1\varphi + A_2 \qquad (2.14)$$

for some $A_1 \in \mathbb{R}^+$, $A_2 \in \mathbb{R}_0^+$ and finally obtain our first dissipative estimate: set for short

$$y_{\delta_0} := \|M\|_{\delta_0}^{\delta_0} + 1 + \|\nabla\rho\|^2, \qquad (2.15)$$
$$\delta_0 := \alpha - \gamma + 2 > 2,$$

it holds then with (2.14)

$$y_{\delta_0}(t) \le C_{y_{\delta_0}} y_{\delta_0}(0) e^{-\omega_{y_{\delta_0}} t} + D_{y_{\delta_0}} \qquad (2.16)$$

for some $C_{y_{\delta_0}}, \omega_{y_{\delta_0}}, D_{y_{\delta_0}}$ that dependent only upon the parameters of the problem.

NOTATION 2. For the sake of convenience, we assume that the constants B_i (will all appear below) for all indices i are only dependent upon the parameters of the problem (1.5)–(1.10), that is, upon the constants $\alpha, \gamma, F_2, F_3, G_1, G_3$ and the domain Ω, and **not** upon the initial data M_0, ρ_0, or t, or (unless stated otherwise) any other parameters.

In what follows we use (2.16) to obtain several intermediate dissipative estimates for M and ρ, which in turn lead to an L^∞-dissipative estimate. The following observation, which is an implication from the theory of abstract parabolic evolution equations (see [**13**]), will be helpful in further.

Having a $\delta \in (2,\infty)$ fixed, consider the unbounded operator

$$\Delta:\ L^{\delta}(\Omega)\to L^{\delta}(\Omega)$$

equipped with the domain

$$D(\Delta):=\big\{u\in W_0^{1,\delta}(\Omega)\cap W^{2,\delta}(\Omega)\big\}.$$

It is known (see [**13**]) that this operator generates an analytic semigroup $e^{t\Delta}$ and its spectrum lies entirely in $\{\lambda\in\mathbb{R}:\lambda\le-\beta\}$ for some $\beta>0$. As such it has the following properties:

$$(-\Delta)^{\mu}e^{t\Delta}=e^{t\Delta}(-\Delta)^{\mu}, \tag{2.17}$$

$$\|e^{t\Delta}(-\Delta)^{\mu}\|_{\delta}\le A^{\mu,\delta}e^{-\beta t}t^{-\mu} \tag{2.18}$$

for all $t>0$ and $\mu>0$ for some constants $A^{\mu,\delta}$ that depend only on μ,δ and the domain Ω. Now, the equation (1.2) can be rewritten in the following way:

$$\frac{d}{dt}(\rho-1)=\Delta(\rho-1)-g(M,\rho)$$

and can thus be regarded as an abstract parabolic evolution equation with respect to $\rho-1$. Therefore for all $t>0$ holds:

$$\rho(t)-1=e^{t\Delta}(\rho_0-1)-\int_0^t e^{(t-s)\Delta}g(M(s),\rho(s))ds \tag{2.19}$$

and applying the operator ∇ to both sides of (2.19) and making use of the property (2.17) we obtain

$$\nabla\rho(t)=e^{t\Delta}\nabla\rho_0-\int_0^t\nabla\big(e^{(t-s)\Delta}g(M(s),\rho(s))\big)ds. \tag{2.20}$$

The initial value ρ_0 is assumed to be sufficiently smooth, so that holds

$$\|\nabla\rho_0\|_{\delta}<\infty. \tag{2.21}$$

What remains is to estimate the L^{δ}-norm of the integral from (2.20) with help of (2.18) and the assumptions on g. Choosing $\mu\in(1/2,1)$ and $\hat{\delta}\ge1$ such that $W^{2\mu,\hat{\delta}}(\Omega)\subset W^{1,\delta}(\Omega)$ we arrive at the estimate

$$\begin{aligned}&\Big\|\int_0^t\nabla\big(e^{(t-s)\Delta}g(M(s),\rho(s))\big)ds\Big\|_{\delta}\\&\quad\le\int_0^t\big\|(-\Delta)^{\mu}\big(e^{(t-s)\Delta}g(M(s),\rho(s))\big)\big\|_{\hat{\delta}}ds\end{aligned}$$

$$\leq A^{\mu,\hat{\delta}} \int_0^t e^{-\beta(t-s)}(t-s)^{-\mu}\big(G_1\|\rho(s)\|_{\hat{\delta}} + G_3\|M(s)\|_{\hat{\delta}}\big)ds. \tag{2.22}$$

Altogether we obtain from (2.20)–(2.22) the following estimate:

$$\begin{aligned}\|\nabla\rho(t)\|_\delta \leq e^{-\beta t}\|\nabla\rho_0\|_\delta + A^{\mu,\hat{\delta}}(G_1+G_3)\\ \cdot \int_0^t e^{-\beta(t-s)}(t-s)^{-\mu}\big(\|\rho(s)\|_{\hat{\delta}} + \|M(s)\|_{\hat{\delta}}\big)ds.\end{aligned} \tag{2.23}$$

Leaving this result for a moment and returning to the equation (1.1) we multiply this equation by $M|M|^{\delta-1}$ for an arbitrary $\delta \geq \alpha - \gamma + 1$, so that all occurring powers remain non-negative, and (formally) integrate over Ω:

$$\big(\dot{M}, M|M|^{\delta-1}\big) = \big(\nabla\cdot(|M|^\alpha\nabla M) - \nabla\cdot(|M|^\gamma\nabla\rho) + f(M,\rho), M|M|^{\delta-1}\big).$$

It follows:

$$\begin{aligned}\frac{1}{\delta+1}\frac{d}{dt}\big\||M|^{\frac{\delta+1}{2}}\big\|^2 = & -\frac{4\delta}{(\alpha+\delta+1)^2}\big\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\big\|^2\\ & + \frac{2\delta}{\alpha+\delta+1}\big(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\big)\\ & + (f(M,\rho), M|M|^{\delta-1}).\end{aligned} \tag{2.24}$$

Denote $\vartheta(\delta) := (\gamma - (\alpha/2) + ((\delta-1)/2))/((\alpha+\delta+1)/2)$. Then $\vartheta(\delta) < 1$ holds due to the assumption $\alpha > \gamma$ we made. Applying Hölder's inequality we obtain:

$$\begin{aligned}\big(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\big) &= \big(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\vartheta(\delta)\frac{\alpha+\delta+1}{2}}\nabla\rho\big)\\ &\leq \|1\|_{\frac{6}{1-\theta(\delta)}}\big\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\big\|\big\||M|^{\frac{\alpha+\delta+1}{2}}\big\|_6^{\vartheta(\delta)}\|\nabla\rho\|_3\\ &\leq B_1\big\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\big\|^{1+\vartheta(\delta)}\|\nabla\rho\|_3.\end{aligned} \tag{2.25}$$

For the last inequality the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ has been used.

Further, we use once more the Hölder inequality and the assumptions on the function f and write:

$$(f(M,\rho), M|M|^{\delta-1}) \leq -F_2\big\||M|^{\frac{\delta+1}{2}}\big\|^2 + F_3\big\||M|^{\frac{\delta}{2}}\big\|^2 \tag{2.26}$$

$$\leq -F_2\big\||M|^{\frac{\delta+1}{2}}\big\|^2 + F_3\|1\|_{\delta+1}\Big(\big\||M|^{\frac{\delta+1}{2}}\big\|^2\Big)^{\frac{\delta}{\delta+1}}. \tag{2.27}$$

We can conclude from (2.24) using (2.25) and (2.27) that:

$$\begin{aligned}\frac{1}{\delta+1}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 \leq & -\frac{4\delta}{(\alpha+\delta+1)^2}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2\\ & + \frac{2\delta}{\alpha+\delta+1}B_1\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^{1+\vartheta(\delta)}\|\nabla\rho\|_3\\ & - F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2 + F_3\|1\|_{\delta+1}\left(\left\||M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}}.\end{aligned}$$

Since $1+\vartheta(\delta)<2$ it follows with the Young inequality:

$$\begin{aligned}\frac{1}{\delta+1}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 \leq & -F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2 + F_3\|1\|_{\delta+1}\left(\left\||M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}}\\ & + B_2(\delta)\|\nabla\rho\|_3^{\frac{2}{1-\vartheta(\delta)}},\end{aligned} \tag{2.28}$$

where

$$B_2(\delta)=\frac{1-\vartheta(\delta)}{2}\left(\frac{2\delta}{\alpha+\delta+1}B_1\right)^{\frac{2}{1-\vartheta(\delta)}}\left(\frac{4\delta}{(\alpha+\delta+1)^2}\frac{2}{1+\vartheta(\delta)}\right)^{-\frac{1+\vartheta(\delta)}{1-\vartheta(\delta)}},$$

therefore this constant depends only on δ and the parameters of the problem.

Next, we return to the equality (2.24) to repeat the whole procedure once more but this time being more precise about the estimates being made and using the regularity achieved up to this point. First, due to (2.26) and two obvious inequalities we have

$$\begin{aligned}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 = & -\frac{4\delta(\delta+1)}{(\alpha+\delta+1)^2}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2\\ & + \frac{2\delta(\delta+1)}{\alpha+\delta+1}\left(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\right)\\ & + (\delta+1)(f(M,\rho), M|M|^{\delta-1})\\ \leq & -B_3\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2\\ & + (\delta+1)B_4\|\nabla\rho\|_\infty\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|^{\vartheta(\delta)}\\ & - (\delta+1)F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2 + (\delta+1)B_5F_3\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|^{2\zeta}\end{aligned} \tag{2.29}$$

for $\delta\geq\alpha-\gamma+1$ with $\zeta=\delta/(\alpha+\delta+1)$.

Taking into account a special case of the interpolation inequality for Sobolev spaces (see [**1**]):

$$\|v\|\leq C_\kappa\|\nabla v\|^{3/5}\|v\|_1^{2/5},$$

where the constant C_κ depends only on the domain Ω, we obtain with the help of the Young inequality

$$\begin{aligned}
&(\delta+1)\|\nabla v\|\|v\|^{\vartheta(\delta)}\\
&\quad\le(\delta+1)C_\kappa^{\vartheta(\delta)}\|\nabla v\|^{1+\vartheta(\delta)(3/5)}\|v\|_1^{\vartheta(\delta)(2/5)}\\
&\quad\le C_\kappa^{\vartheta(\delta)}\left(\varepsilon\|\nabla v\|^2+B_6(\varepsilon)(\delta+1)^{\frac{2}{1-\vartheta(\delta)(3/5)}}\|v\|_1^{\frac{2\vartheta(\delta)(2/5)}{1-\vartheta(\delta)(3/5)}}\right)
\end{aligned}\tag{2.30}$$

and

$$\begin{aligned}
(\delta+1)F_3\|v\|^{2\zeta}&\le(\delta+1)F_3C_\kappa^{2\zeta}\|\nabla v\|^{2\zeta(3/5)}\|v\|_1^{2\zeta(2/5)}\\
&\le C_\kappa^{2\zeta}\left(\varepsilon\|\nabla v\|^2+B_7(\varepsilon)(F_3(\delta+1))^{\frac{1}{1-\zeta(3/5)}}\|v\|_1^{\frac{2\zeta(2/5)}{1-\zeta(3/5)}}\right),
\end{aligned}\tag{2.31}$$

where $B_6(\varepsilon)$ and $B_7(\varepsilon)$ depend only on ε and the parameters of the problem. With the Hölder inequality we also have

$$\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|_1\le\left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{q}{q-1}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q\tag{2.32}$$

for some $q\in(1,2)$ independent of δ. Combining (2.30)–(2.31) for $v:=|M|^{(\alpha+\delta+1)/2}$ with (2.32) and choosing ε small enough depending only on the parameters of the problem we can conclude from (2.29):

$$\begin{aligned}
\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2&\le B_8(\|\nabla\rho\|_\infty(\delta+1))^{\frac{2}{1-\vartheta(\delta)(3/5)}}\left(\left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{q}{q-1}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q\right)^{\frac{2\vartheta(\delta)(2/5)}{1-\vartheta(\delta)(3/5)}}\\
&\quad+B_8(F_3(\delta+1))^{\frac{1}{1-\zeta(3/5)}}\left(\left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{q}{q-1}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q\right)^{\frac{2\zeta(2/5)}{1-\zeta(3/5)}}\\
&\quad-F_2(\delta+1)\left\||M|^{\frac{\delta+1}{2}}\right\|^2
\end{aligned}$$

for $\delta\ge\alpha-\gamma+1$. Since $\vartheta(\delta),\zeta\in(0,1)$ it follows for all $\delta\ge\alpha-\gamma+2$:

$$\frac{d}{dt}\left(\|M\|_\delta^\delta+1\right)\le B_8\delta^5(\|\nabla\rho(s)\|_\infty+1)^5\|M\|_{\frac{\alpha}{2}\frac{q}{q-1}}^{\alpha}\left(\|M\|_{q\delta/2}^{q\delta/2}+1\right)^{2/q}-F_2\delta\left(\|M\|_\delta^\delta+1\right)$$

and once more we get an integral inequality for $\|M(t)\|_\delta^\delta+1$:

$$\begin{aligned}
\|M(t)\|_\delta^\delta+1\le B_8\int_0^t&e^{-\delta F_2(t-s)}\delta^5(\|\nabla\rho(s)\|_\infty+1)^5\|M(s)\|_{\frac{\alpha}{2}\frac{q}{q-1}}^{\alpha}\\
&\cdot\left(\|M(s)\|_{q\delta/2}^{q\delta/2}+1\right)^{2/q}ds+e^{-\delta F_2t}\left(\|M_0\|_\delta^\delta+1\right).
\end{aligned}\tag{2.33}$$

Now we are ready to derive more dissipative estimates for the problem (1.1)–(1.4). We will extensively use the following

LEMMA 1. *Let $z_1, z_2, z_3 : [0,+\infty) \to [0,+\infty)$ be such functions that*

$$\begin{aligned} &z_1(t) \le \psi_1(z_1(0))e^{-\omega_1 t} + D_1, \\ &z_2(t) \le \psi_2(z_2(0))e^{-\omega_2 t} + D_2, \\ &z_3(t) \le z_3(0)e^{-\omega_3 t} + \int_0^t e^{-\omega_3(t-s)} d_3(t,s) z_1(s)\, ds, \qquad (2.34) \\ &z_1(0),\ z_2(0),\ z_3(0) \ge 1, \end{aligned}$$

for some constants $\omega_1, \omega_2, \omega_3 > 0$ and $D_1, D_2 \ge 1$, some non-decreasing functions $\psi_1, \psi_2 : [1,+\infty) \to [1,+\infty)$ and some $d_3 \in L^\infty(\mathbb{R}_0^+, L_b^1(\mathbb{R}_0^+))$, where

$$L_b^1(\mathbb{R}_0^+) = \Big\{ u \in L_{loc}^1(\mathbb{R}_0^+) : \|u\|_{L_b^1(\mathbb{R}_0^+)} := \sup_{x_0 \in \mathbb{R}_0^+} \|u\|_{L^1([x_0, x_0+1])} < \infty \Big\}.$$

It holds:

1. $(z_1 + z_2)(t) \le (\psi_1 + \psi_2)((z_1 + z_2)(0)) e^{-\min\{\omega_1,\omega_2\} t} + D_1 + D_2.$
2. $z_1 z_2(t) \le 3D_1 D_2 \psi_1 \psi_2 (z_1 z_2(0)) e^{-\min\{\omega_1,\omega_2\} t} + D_1 D_2.$
3. $z_1^\sigma(t) \le \max\{1, 2^{\sigma-1}\}(\psi_1^\sigma(z_1(0)) e^{-\sigma\omega_1 t} + D_1^\sigma) \ \ \forall \sigma > 0.$
4. *For $\omega_1 \ne \omega_3$*

$$\begin{aligned} z_3(t) \le &\left(\psi_1(z_1(0)) \frac{1}{1 - e^{-|\omega_1 - \omega_3|}} e^{-\min\{\omega_1,\omega_3\} t} + D_1 \frac{1}{1 - e^{-\omega_3}} \right) \\ &\cdot \|d_3\|_{L^\infty(\mathbb{R}_0^+, L_b^1(\mathbb{R}_0^+))} + z_3(0) e^{-\omega_3 t} \qquad (2.35) \end{aligned}$$

and for $\omega_3 = \omega_1$

$$z_3(t) \le \left(\psi_1(z_1(0)) \lceil t \rceil e^{-\omega_1 t} + D_1 \frac{1}{1 - e^{-\omega_1}} \right) \|d_3\|_{L^\infty(\mathbb{R}_0^+, L_b^1(\mathbb{R}_0^+))} + z_3(0) e^{-\omega_1 t},$$

where $\lceil t \rceil$ is the ceiling function. For $\omega_1 < \omega_3$ we also have

$$z_3(t) \le z_3(0) e^{-\omega_3 t} + z_1(t) \int_0^t e^{-(\omega_3 - \omega_1)(t-s)} d_3(t,s) ds. \qquad (2.36)$$

(See Appendix A for some details regarding the proof of this lemma.)

Lemma 1 is very useful in our situation. It shows actually that the 'dissipative property' is persevered under standard operations (addition, multiplication, raising to a power and integration).

To shorten the formulas let us set

$$\begin{aligned} h_1 &:= \|\nabla\rho\|_3 + 1, \\ h_2 &:= \|\nabla\rho\|_\infty + 1, \end{aligned}$$

$$u_\delta := \|M\|_\delta^\delta + 1, \quad \delta \in [1, \infty).$$

Observe that particular powers of y_{δ_0} and h_1, h_2 and u_δ (for sufficiently large δ), u_7 and h_2 can be connected with one another by the inequalities of the type (2.34) in the same manner as z_1 and z_3 from Lemma 1 are. From the Lemma 1 we can conclude that all of them dissipate exponentially with t:

$$h_1(t) \leq C_{h_1}(h_1 + y_{\delta_0})^{r_{h_1}}(0)e^{-\omega_{h_1}t} + D_{h_1}, \tag{2.37}$$

$$h_2(t) \leq C_{h_2}(h_2 + u_7)^{r_{h_2}}(0)e^{-\omega_{h_2}t} + D_{h_2}, \tag{2.38}$$

$$u_\delta(t) \leq U\big(u_\delta(0) + C_{u_\delta}(h_1 + y_{\delta_0})^{r\delta}(0)\big)e^{-(F_2/2)\delta t} + D_{u_\delta} =: \widetilde{u}_\delta(t), \tag{2.39}$$

where the appearing coefficients depend on the parameters of the problem, and only the coefficients C_{u_δ} and D_{u_δ} depend on δ as well. We especially emphasize that r is independent from δ (it will be crucial for the existence of the uniform dissipative estimate). Indeed, from (2.23) and the definition of y_{δ_0} ($y_{\delta_0} > 1$, see (2.15)) we obtain:

$$\begin{aligned}
\|\nabla\rho(t)\|_3 &\leq e^{-\beta t}\|\nabla\rho_0\|_3 + A^{3/4,2}(G_1 + G_3) \\
&\quad \cdot \int_0^t e^{-\beta(t-s)}(t-s)^{-3/4}(\|\rho(s)\|_2 + \|M(s)\|_2)ds \\
&\leq e^{-\beta t}\|\nabla\rho_0\|_3 + C^{(1,2),2}A^{3/4,2}(G_1 + G_3) \\
&\quad \cdot \int_0^t e^{-\beta(t-s)}(t-s)^{-3/4}y_{\delta_0}(0)ds
\end{aligned} \tag{2.40}$$

since $\alpha - \gamma + 2 > 2$, $W^{2\cdot 3/4,2} \subset W^{1,3}$ and $W^{1,2} \subset L^2(\Omega)$ (with the embedding constant $C^{(1,2),2}$). Next, using (2.23) one more time, we obtain

$$\begin{aligned}
\|\nabla\rho(t)\|_\infty &\leq e^{-\beta t}\|\nabla\rho_0\|_\infty + A^{3/4,7}(G_1 + G_3) \\
&\quad \cdot \int_0^t e^{-\beta(t-s)}(t-s)^{-3/4}(\|\rho(s)\|_7 + \|M(s)\|_7)ds \\
&\leq e^{-\beta t}\|\nabla\rho_0\|_\infty + C^{(1,3),7}A^{3/4,7}(G_1 + G_3) \\
&\quad \cdot \int_0^t e^{-\beta(t-s)}(t-s)^{-3/4}(\|\nabla\rho(s)\|_3 + 1 + \|M(s)\|_7)ds
\end{aligned} \tag{2.41}$$

since $W^{2\cdot 3/4,7} \subset W^{1,\infty}$ and $W^{1,3}(\Omega) \subset L^7(\Omega)$ (with the embedding constant $C^{(1,3),7}$). The estimates for h_1 and h_2 now follow with (2.40)–(2.41) and Lemma 1 due to the fact that for the function $d(t,s) := (t-s)_+^{-3/4}$ the condition $\|d\|_{L^\infty(\mathbb{R}_0^+, L_b^1(\mathbb{R}_0^+))} < \infty$ is satisfied.

Let us now check the dissipative estimate (2.39). With (2.28) we have:

$$\frac{1}{\delta}\dot{u}_\delta \le -F_2 u_\delta + F_3|\Omega| u_\delta^{(\delta-1)/\delta} + B_2(\delta) h_1^{2/(1-\vartheta(\delta))}. \tag{2.42}$$

Recall that $\vartheta(\delta) = (\gamma - (\alpha/2) + ((\delta-2)/2))/((\alpha+\delta)/2)$ and consequently $2/(1-\vartheta(\delta)) = (\alpha+\delta)/(\alpha-\gamma+1) \le B_9\delta$ for some B_9 and $\delta \ge \delta_*$ sufficiently large. Now, the Young inequality yields:

$$u_\delta^{(\delta-1)/\delta} = (\varepsilon u_\delta)^{(\delta-1)/\delta}\varepsilon^{-(\delta-1)/\delta} \le \frac{\delta-1}{\delta}\varepsilon u_\delta + \frac{1}{\delta}\varepsilon^{-(\delta-1)},$$

therefore it follows from (2.42)

$$\begin{aligned}\dot{u}_\delta &\le -\delta\left(F_2 - \varepsilon F_3|\Omega|\frac{\delta-1}{\delta}\right)u_\delta + \varepsilon^{-(\delta-1)}F_3|\Omega| + \delta B_2(\delta)h_1^{B_9\delta}\\ &\le -\delta\frac{F_2}{2}u_\delta + \varepsilon^{-(\delta-1)}F_3|\Omega| + \delta B_2(\delta)h_1^{B_9\delta}\end{aligned}$$

for ε small (depends only on the parameters of the problem). Gronwall's lemma yields then

$$u_\delta(t) \le \int_0^t e^{-\delta(F_2/2)(t-s)}\big(\varepsilon^{-(\delta-1)}F_3|\Omega| + \delta B_2(\delta)h_1^{B_9\delta}(s)\big)ds + e^{-\delta(F_2/2)t}u_\delta(0). \tag{2.43}$$

The dissipate estimate (2.39) follows now with the estimate (2.35) of Lemma 1 and the dissipate estimate (2.37) for h_1.

Now, from the inequality (2.33) we can conclude

$$u_\delta(t) \le e^{-\delta F_2 t}u_\delta(0) + B_8\delta^5\int_0^t e^{-\delta F_2(t-s)}H_1(s)\widetilde{u}_{(q/2)\delta}^{2/q}(s)ds, \tag{2.44}$$

where

$$H_1(t) := h_2^5(t)\widetilde{u}_{(\alpha/2)(q/(q-1))}^{2(q-1)/q}(t).$$

Taking into account that $u_{(q/2)\delta}^{2/q}$ dissipates with $e^{-\delta(F_2/2)t}$ and that H_1 dissipates with an exponent independent of δ, we consecutively apply (2.36) to (2.44) and get

$$\begin{aligned}u_\delta(t) &\le e^{-\delta(F_2/2)t}u_\delta(0) + B_8\widetilde{u}_{(q/2)\delta}^{2/q}(t)\int_0^t e^{-\delta(F_2/2)(t-s)}\delta^5 H_1(s)ds\\ &\le e^{-\delta(F_2/2)t}u_\delta(0) + \frac{2}{F_2}B_8\delta^4 H_1(t)\widetilde{u}_{(q/2)\delta}^{2/q}(t)\end{aligned}$$

for $\delta \ge \delta_*$ sufficiently large. The bound δ_* depends only on the parameters of the problem. Therefore we may assume that

$$\widetilde{u}_\delta(t) = e^{-\delta(F_2/2)t} u_\delta(0) + B_{10}\delta^4 H_1(t)\widetilde{u}^{2/q}_{(q/2)\delta}(t). \tag{2.45}$$

Since

$$u_\delta(0) = \|M_0\|_\delta^\delta + 1 \le \|M_0\|_\infty^\delta |\Omega| + 1$$

we conclude from (2.45) that for

$$A_\delta(t) := \widetilde{u}_\delta(t)\left(\frac{e^{(F_2/2)t}}{\|M_0\|_\infty + 1}\right)^\delta + 1 \tag{2.46}$$

it holds

$$A_\delta(t) \le B_{11}\delta^4 H_1(t) A^{2/q}_{(q/2)\delta}(t).$$

One can show by induction then that

$$\begin{aligned} A^{(q/2)^n}_{(2/q)^n\delta_*}(t) \quad &\le \quad \big(B_{11}\delta_*^4 H_1(t)\big)^{\sum_{k=1}^n (q/2)^k} \left(\frac{q}{2}\right)^{4\sum_{k=1}^n k(q/2)^k} A_{\delta_*}(t) \\ &\underset{n\to\infty}{\longrightarrow} \big(B_{11}\delta_*^4 H_1(t)\big)^{(q/2)/(1-(q/2))} \left(\frac{q}{2}\right)^{2q(1/(1-(q/2)))^2} A_{\delta_*}(t) \\ &=: \quad H^{\delta_*}(t) A_{\delta_*}(t). \end{aligned}$$

Therefore we get

$$\limsup_{\delta\to\infty} A_\delta^{1/\delta}(t) \le H(t) A_{\delta_*}^{1/\delta_*}(t). \tag{2.47}$$

Combining (2.47) with (2.46) we finally arrive at an estimate for $\|M(t)\|_\infty$:

$$\begin{aligned} \|M(t)\|_\infty + 1 &= \lim_{\delta\to\infty} u_\delta^{1/\delta}(t) \\ &\le \limsup_{\delta\to\infty} \widetilde{u}_\delta^{1/\delta}(t) \\ &\le H(t)\big(\widetilde{u}_{\delta_*}^{1/\delta_*}(t) + (\|M_0\|_\infty + 1)e^{-(F_2/2)t}\big). \end{aligned} \tag{2.48}$$

Now, since the functions H and $\widetilde{u}_{\delta_*}$ dissipate exponentially (recall (2.38) and (2.39) and the definition of H and H_2), we apply Lemma 1 to (2.48) and conclude that $\|M\|_\infty$ dissipates exponentially as well. Moreover, there exists a dissipative estimate for $\|M\|_\infty$ of the form given in (1.11). This is a consequence of Lemma 1, the estimates (2.38) and (2.39) and the fact that we only used the initial data norms $\|M_0\|_\infty$ and $\|\nabla\rho_0\|_\infty$ thought the proof. All other parameters depended only upon the parameters of the problem (1.1)–(1.4). The dissipative estimate for $\|\nabla\rho\|_\infty + 1 = h_2$ is given in (2.38) and the Theorem 2 is thus proved.

3. Global attractor (proof of Theorem 3).

It is generally known that the long time behavior of an autonomous dynamical system can be described in terms of its global attractor $\mathcal{A}$. More precisely, assuming that the system is well-posed in a topological space $\mathcal{T}$, we can define a family of solving operators

$$S(t) : \mathcal{T} \mapsto \mathcal{T}, \quad t \geq 0$$

that acts on $\mathcal{T}$ mapping the initial data onto the solution at time t:

$$S(t)(u_0) := u(t), \quad t \geq 0,\ u_0 \in \mathcal{T}.$$

This family of operators satisfies

$$S(0) = \mathrm{id}_{\mathcal{T}},$$
$$S(t+s) = S(t) \circ S(s)\ \forall t, \quad s \geq 0,$$

where $\mathrm{id}_{\mathcal{T}}$ denotes the identity operator. We say that it forms a semigroup on the phase space $\mathcal{T}$. Recall now the general definition of the attractor for topological spaces (see [**3**]):

Definition 2. A set $\mathcal{A}$ is called the attractor for the family $\mathcal{B}$ in $\mathcal{T}$ if

(i) $\mathcal{A}$ is compact in $\mathcal{T}$ and $\mathcal{A}$ attracts $\mathcal{B}$, i.e., for every $B \in \mathcal{B}$ and every neighborhood V of $\mathcal{A}$ there exists a $T(B) > 0$ such that $S(t)B \subseteq V$ for all $t \geq T(B)$;
(ii) $\mathcal{A}$ is minimal compact set that attracts $\mathcal{B}$ with respect to $S(t)$, i.e., every compact attracting set of $\mathcal{B}$ contains $\mathcal{A}$.

Remark 3.

1. The family $\mathcal{B}$ is usually the family of bounded sets of the topological space $\mathcal{T}$. In this case the attractor $\mathcal{A}$ is called the global attractor.
2. The minimality condition can be replaced by the following invariance condition:

$$S(t)\mathcal{A} = \mathcal{A}\ \forall t \geq 0.$$

Our goal is now to apply the general theory to the problem (1.1)–(1.4). We showed in [**5**] that the problem (1.1)–(1.4) if considered as an equation with respect to (M, ρ) in the Banach space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ is well-posed: for each pair of initial values $(M_0, \rho_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ there exist a unique solution $(M(t), \rho(t))_{t \in \mathbb{R}_0}$ in terms of Definition 1. It appears reasonable to consider the space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ equipped with a weaker topology for the first component. Denote $L^\infty_{w-*}(\Omega)$ the space $L^\infty(\Omega)$ equipped with the weak$-*$ topology of $L^\infty(\Omega)$. We define the solving semigroup $S(t)$ of the problem (1.1)–(1.4) on the phase space $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$ as follows:

$$S(t)(M_0, \rho_0) := (M(t), \rho(t)) \text{ for all } t \geq 0.$$

The space $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$ is not metrizable due to the fact that the space $L^\infty_{w-*}(\Omega)$ is not metrizable. It is only a locally convex Hausdorff space. However, the dissipative estimate (1.11) provides the existence of a ball $B_* := B(0, 2D_\infty)$ centered in 0 with radius $2D_\infty$ which absorbs all bounded sets of the (normed) space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$. Since there is no difference between the spaces $L^\infty(\Omega)$ and $L^\infty_{w-*}(\Omega)$ with concern to boundedness, B_* remains to be an absorbing set in $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$ as well, which means: for all bounded sets $B \subseteq L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$ there exists a $T(B) > 0$ such that $S(t)B \subseteq B_*$ for all $t \geq T(B)$. Consequently, to show the existence of the global attractor (in the sense of Definition 2 and Remark 3) for the semigroup $S(t)$ it suffers to show the existence of the minimal compact set $\mathcal{A}$ that attracts bounded subsets of B_*.

Recall now a general criteria for existence of the global attractor (see [**2**], [**3**]):

THEOREM 4. *Let $S(t)$ be a continuous semigroup in a complete metric space $\mathcal{E}$ having a compact absorbing set $K \subseteq \mathcal{E}$. Then the semigroup $S(t)$ has a global attractor $\mathcal{A}$.*

Recall that $L^\infty(\Omega)$-balls are metrizable in $L^\infty_{w-*}(\Omega)$. Let

$$\{x_n\}_{n\in\mathbb{N}} \subseteq L^1(\Omega), \quad \|x_n\|_1 \leq 1 \ \forall n \in \mathbb{N} \tag{3.1}$$

be a set of functions that is separating for $L^\infty(\Omega)$. Then the function

$$d(x_*, y_*) := \sum_{n\in\mathbb{N}} 2^{-n} |(x_* - y_*, x_n)| \quad \forall f, g \in L^\infty(\Omega), \quad \|x_*\|_\infty, \|y_*\|_\infty \leq R$$

is an example of a suitable metric for a ball of radius R in $L^\infty(\Omega)$ which produces on it the topology equivalent to the topology of $L^\infty_{w-*}(\Omega)$ (see [**9**] for this and more details on the weak$-*$ topology). The relative topology of B_* is metrizable since $L^\infty(\Omega)$-balls are metrizable in the weak$-*$ topology, therefore it remains to show the existence of a compact absorbing set in B_* and the continuity of the semigroup operators $S(t)$ for all $t > 0$. The general criteria Theorem 4 would be then applicable to $S(t)$ in B_* equipped with the topology of $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$. The projection of B_* on the first component is a bounded norm closed set in $L^\infty(\Omega)$, therefore it is compact in $L^\infty_{w-*}(\Omega)$. Let us now show the existence of a compact absorbing set for the second component. Applying the operator Δ to both sides of the equation (1.2) we obtain

$$\Delta\rho(s) = \Delta e^{s\Delta}\rho_0 - \int_0^s \Delta e^{(s-\omega)\Delta} g(M(\omega), \rho(\omega))\, d\omega,$$

so that we get the following estimate in the L^6-norm:

$$\begin{aligned}\|\Delta\rho(s)\|_6 &\leq \left\| \Delta e^{s\Delta}\rho_0 - \int_0^s \Delta e^{(s-\omega)\Delta} g(M(\omega), \rho(\omega)) d\omega \right\|_6 \\ &\leq C_1 \frac{1}{\sqrt{s}} \|\nabla\rho_0\|_6 + C_1 \int_0^s \frac{1}{\sqrt{s-\omega}} \big\|\nabla g(M(\omega), \rho(\omega)) d\omega\big\|_6.\end{aligned}$$

Thus due to the assumptions on g there exists a nonnegative function $\Phi_\rho(s,x,y)$ which is nondecreasing with respect to s, x and y, independent of M_0 and ρ_0 and such that the following smoothing estimate holds:

$$\sqrt{s}\|\Delta\rho(s)\|_6 \leq \Phi_\rho\big(s, \|M_0\|_\infty, \|\nabla\rho_0\|_\infty\big). \tag{3.2}$$

With (3.2) and the compact embeddings

$$L^\infty(\Omega) \subset\subset L^\infty_{w-*}(\Omega),$$
$$W^{2,6}(\Omega) \subset\subset W^{1,\infty}(\Omega)$$

it follows that the set $S(T_*)B_*$ for some $T_* > 0$ such that $S(T_*)B_* \subseteq B_*$ is a relatively compact absorbing set for the semigroup $S(t)$ in B_* (T_* exists due to the fact that B_* is an absorbing set). It thus remains to show the continuity of the semigroup operators.

In [**5**] we derived local Lipschitz continuity for the solutions of (1.1)–(1.4) in the following sense: for all $T, R > 0$ it holds

$$\begin{aligned}&\big\|S(t)\big(M_0^{(1)}, \rho_0^{(1)}\big) - S(t)\big(M_0^{(2)}, \rho_0^{(2)}\big)\big\|_{H^{-1}(\Omega)\times L^2(\Omega)}\\ &\quad\leq L(t,R)\big\|\big(M_0^{(1)}, \rho_0^{(1)}\big) - \big(M_0^{(2)}, \rho_0^{(2)}\big)\big\|_{H^{-1}(\Omega)\times L^2(\Omega)}\end{aligned} \tag{3.3}$$

for all $\big\|(M_0^{(i)}, \rho_0^{(i)})\big\|_{L^\infty(\Omega)\times W^{1,\infty}(\Omega)} \leq R$, $i = 1, 2$ and some non-negative non-decreasing in both t and R function $L(t,R)$ independent of M, ρ and 0.

Recall that due to embedding theorems for Sobolev spaces we have

$$L^\infty(\Omega) \times W^{1,\infty}(\Omega) \subset H^{-1}(\Omega) \times L^2(\Omega). \tag{3.4}$$

The property (3.4) allows the interpretation

$$L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega) \subset H^{-1}(\Omega) \times L^2(\Omega). \tag{3.5}$$

Let $(M_0^{(n)}, \rho_0^{(n)})$ be a sequence of initial data convergent to some (M_0, ρ_0) in $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$. This sequence converges in $H^{-1}(\Omega) \times L^2(\Omega)$ to the same limit (M_0, ρ_0) due to the continuous embedding (3.5). From the property (3.3) we deduce that the sequences $\big(S(t)(M_0^{(n)}, \rho_0^{(n)})\big)$ converge to $S(t)(M_0, \rho_0)$ in $H^{-1}(\Omega) \times L^2(\Omega)$ for all $t \geq 0$. Let us further assume that for some $t \geq 0$ the sequence $\big(S(t)(M_0^{(n)}, \rho_0^{(n)})\big)$ is convergent in $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$. Due to the continuity of the embedding (3.5) the limit is $S(t)(M_0, \rho_0)$. Therefore we can conclude that the operators $S(t)$ if considered as mapping in $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$ are closed operators. Since any (non-linear) closed compact operator is completely continuous (that is, continuous and compact) we get the continuity of the operators $S(t)$ in $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$.

Applying the general Theorem 4 to $\mathcal{E} := B_*$ equipped with the topology of $L^\infty_{w-*}(\Omega)\times W^{1,\infty}(\Omega)$ and $K := S(T_*)B_*$ we deduce the existence of the global attractor $\mathcal{A}$ for the semigroup $\{S(t)\}_{t\geq 0}$ and the Theorem 3 is thus proved.

Appendix A. Proof of the auxiliary Lemma 1.

Consider first the differential inequality

$$\dot{y} \leq -\omega_y y + d_y y^{\zeta_y} \tag{A.1}$$

assuming that $y \geq 1$, $\zeta_y \in (0,1)$, $d_y \in L^1_b(\mathbb{R})$ so that with some computation the estimate

$$(y(t))^{1-\zeta_y} \leq (y(0))^{1-\zeta_y} e^{-\omega_y(1-\zeta_y)t} + (1-\zeta_y)\int_0^t e^{-\omega_y(1-\zeta_y)(t-s)} d_y(s)ds$$

follows.

Lemma 1. *Let $z_1, z_2, z_3 : [0,+\infty) \to [0,+\infty)$ be such functions that*

$$\begin{aligned}
&z_1(t) \leq \psi_1(z_1(0))e^{-\omega_1 t} + D_1,\\
&z_2(t) \leq \psi_2(z_2(0))e^{-\omega_2 t} + D_2,\\
&z_3(t) \leq z_3(0)e^{-\omega_3 t} + \int_0^t e^{-\omega_3(t-s)} d_3(t,s) z_1(s)ds, \qquad \text{(A.2)}\\
&z_1(0), z_2(0), z_3(0) \geq 1,
\end{aligned}$$

for some constants $\omega_1, \omega_2, \omega_3 > 0$ and $D_1, D_2 \geq 1$, some non-decreasing functions $\psi_1, \psi_2 : [1,+\infty) \to [1,+\infty)$ and some $d_3 \in L^\infty(\mathbb{R}^+_0, L^1_b(\mathbb{R}^+_0))$, where

$$L^1_b(\mathbb{R}^+_0) = \Big\{ u \in L^1_{loc}(\mathbb{R}^+_0) : \|u\|_{L^1_b(\mathbb{R}^+_0)} := \sup_{x_0 \in \mathbb{R}^+_0} \|u\|_{L^1([x_0,x_0+1])} < \infty \Big\}.$$

It holds:

1. $(z_1+z_2)(t) \leq (\psi_1+\psi_2)((z_1+z_2)(0))e^{-\min\{\omega_1,\omega_2\}t} + D_1 + D_2$.
2. $z_1 z_2(t) \leq 3D_1 D_2 \psi_1\psi_2(z_1 z_2(0))e^{-\min\{\omega_1,\omega_2\}t} + D_1 D_2$.
3. $z_1^\sigma(t) \leq \max\{1, 2^{\sigma-1}\}(\psi_1^\sigma(z_1(0))e^{-\sigma\omega_1 t} + D_1^\sigma) \quad \forall \sigma > 0$.
4. *For $\omega_1 \neq \omega_3$*

$$\begin{aligned}
z_3(t) \leq &\left(\psi_1(z_1(0))\frac{1}{1-e^{-|\omega_1-\omega_3|}}e^{-\min\{\omega_1,\omega_3\}t} + D_1\frac{1}{1-e^{-\omega_3}}\right)\\
&\cdot \|d_3\|_{L^\infty(\mathbb{R}^+_0, L^1_b(\mathbb{R}^+_0))} + z_3(0)e^{-\omega_3 t} \qquad \text{(A.3)}
\end{aligned}$$

and for $\omega_3 = \omega_1$

$$z_3(t) \leq \left(\psi_1(z_1(0))\lceil t\rceil e^{-\omega_1 t} + D_1\frac{1}{1-e^{-\omega_1}}\right)\|d_3\|_{L^\infty(\mathbb{R}^+_0, L^1_b(\mathbb{R}^+_0))} + z_3(0)e^{-\omega_1 t}.$$

For $\omega_1 < \omega_3$ we also have

$$z_3(t) \le z_3(0)e^{-\omega_3 t} + z_1(t)\int_0^t e^{-(\omega_3-\omega_1)(t-s)} d_3(t,s)ds. \tag{A.4}$$

Proof. We show only 4 for $\omega_1 \neq \omega_3$. Since

$$\begin{aligned}
&\int_0^t e^{-\omega_3(t-s)} e^{-\omega_1 s} d_3(t,s)ds \\
&= e^{-\min\{\omega_1,\omega_3\}t} \begin{cases} \int_0^t e^{-|\omega_1-\omega_3|(t-s)} d_3(t,s)ds & \text{if } \omega_1 < \omega_3 \\ \int_0^t e^{-|\omega_1-\omega_3|s} d_3(t,s)ds & \text{if } \omega_1 > \omega_3 \end{cases} \\
&\le \frac{1}{1-e^{-|\omega_1-\omega_3|}} e^{-\min\{\omega_1,\omega_3\}t} \|d_3\|_{L^\infty(\mathbb{R}_0^+, L_b^1(\mathbb{R}_0^+))}
\end{aligned} \tag{A.5}$$

we conclude from (A.2)

$$\begin{aligned}
&\int_0^t e^{-\omega_3(t-s)} d_3(t,s) z_1(s)ds \\
&\le \int_0^t e^{-\omega_3(t-s)} d_3(t,s)\big(\psi_1(z_1(s))e^{-\omega_1(t-s)} + D_1\big)ds \\
&\le \left(\psi_1(z_1(0))\frac{1}{1-e^{-|\omega_1-\omega_3|}} e^{-\min\{\omega_1,\omega_3\}t} + D_1\frac{1}{1-e^{-\omega_3}}\right)\|d_3\|_{L^\infty(\mathbb{R}_0^+, L_b^1(\mathbb{R}_0^+))}
\end{aligned}$$

and the statement follows. □

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