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Convergence of Aluthge iteration in semisimple Lie groups

This paper is dedicated to Professor Frank Uhlig on the occasion of his retirement from Auburn University in 2013

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Abstract. We extend the λ-Aluthge sequence convergence theorem of Antezana, Pujals and Stojanoff in the context of real noncompact connected semisimple Lie groups.

1. Introduction.

Given $0 < \lambda < 1$, the λ-Aluthge transform of $X \in \mathbb{C}_{n\times n}$ [**7**]:

$$\Delta_\lambda(X) := P^\lambda U P^{1-\lambda}$$

has been extensively studied, where $X = UP$ is the polar decomposition of X, that is, U is unitary and P is positive semidefinite. The notion can be extended to Hilbert space operators [**6**], [**7**]; see [**9**], [**10**], [**16**], [**18**], [**19**] for some recent research.

Define

$$\Delta_\lambda^m(X) := \Delta_\lambda(\Delta_\lambda^{m-1}(X)), \quad m \in \mathbb{N}$$

with $\Delta_\lambda^1(X) := \Delta_\lambda(X)$ and $\Delta_\lambda^0(X) := X$ so that we have the λ-Aluthge sequence $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$. It is known that $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ converges if $n = 2$ [**8**], if the eigenvalues of X have distinct moduli [**12**], and if X is diagonalizable [**2**], [**3**], [**4**]. Very recently Antezana, Pujals and Stojanoff [**5**] proved the following interesting result using ideas and techniques from dynamical systems and differential geometry.

Theorem 1 ([**5**, Theorem 6.1]). *Let $X \in \mathbb{C}_{n\times n}$ and $0 < \lambda < 1$.*

1. *The sequence $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ converges to a normal matrix $\Delta_\lambda^\infty(X) \in \mathbb{C}_{n\times n}$.*
2. *The function $\Delta_\lambda^\infty : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ defined by $X \mapsto \Delta_\lambda^\infty(X)$ is continuous.*

The convergence problem for $\mathbb{C}_{n\times n}$ is reduced to $\mathrm{GL}_n(\mathbb{C})$ [**1**] and can be further reduced to $\mathrm{SL}_n(\mathbb{C})$ since $\Delta_\lambda(cX) = c\Delta_\lambda(X)$, $c \in \mathbb{C}$.

Not much is known about the limit $\Delta_\lambda^\infty(X)$. For $X \in \mathrm{SL}_2(\mathbb{C})$ with equal eigenvalue moduli [**17**],

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$$\Delta_\lambda^\infty(X) = \frac{\operatorname{tr} X}{2} I_2 + \frac{\sqrt{4 - \operatorname{tr} X^2}}{2\sqrt{\operatorname{tr}(XX^*) + 2\det X - \operatorname{tr} X^2}}(X - X^*).$$

Our goal is to extend Theorem 1 to Lie groups with the right properties.

2. Main Results.

Let G be a real noncompact connected semisimple Lie group, and let $\mathfrak{g}$ be its Lie algebra, with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a fixed Cartan decomposition of $\mathfrak{g}$. Let $K \subset G$ be the connected subgroup with Lie algebra $\mathfrak{k}$. Set $P := \exp \mathfrak{p}$. The Cartan decomposition [**14**, p. 362] asserts that the map

$$K \times \mathfrak{p} \to G, \qquad (k, X) \mapsto k \exp X \tag{2.1}$$

is a diffeomorphism [**14**, p. 362]. In particular $G = KP$ and every element $g \in G$ can be uniquely written as $g = kp$, where $k \in K$, $p \in P$. Given $0 < \lambda < 1$, the λ-Aluthge transform of $\Delta_\lambda : G \to G$ is defined as

$$\Delta_\lambda(g) := p^\lambda k p^{1-\lambda},$$

where $p^\lambda := \exp(\lambda X) \in P$ if $p = \exp X$ for some $X \in \mathfrak{p}$. The map $(0,1) \times G \to G$ defined by $(\lambda, g) \mapsto \Delta_\lambda(g)$ is smooth; thus $\Delta_\lambda : G \to G$ is smooth [**13**]. We define

$$\Delta_\lambda^m(g) := \Delta_\lambda(\Delta_\lambda^{m-1}(g)),$$

with $\Delta_\lambda^1(g) := \Delta_\lambda(g)$ and $\Delta_\lambda^0(g) := g$ so that we have the *generalized λ-Aluthge sequence* $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$. Clearly $\Delta_\lambda(g) = p^\lambda g (p^\lambda)^{-1}$ so that all members of the Aluthge sequence are in the same conjugacy class.

An element $g \in G$ is said to be *normal* if $kp = pk$, where $g = kp$ ($k \in K$ and $p \in P$) is the Cartan decomposition of g. It is known that the center Z of G is contained in K [**11**, p. 252]. So $g \in G$ is normal if and only if zg is normal for all $z \in Z$.

Equip $\mathfrak{g}$ once and for all with an inner product [**14**, p. 360] such that the operator $\operatorname{Ad} k \in \mathrm{GL}(\mathfrak{g})$ on $\mathfrak{g}$ is orthogonal for all $k \in K$, and $\operatorname{Ad} p \in \mathrm{GL}(\mathfrak{g})$ is positive definite for all $p \in P$. Notice that $\operatorname{Ad} G = (\operatorname{Ad} K)(\operatorname{Ad} P)$ is the polar decomposition of $\operatorname{Ad} G \subset \mathrm{GL}(\mathfrak{g})$.

LEMMA 2. (1) *The element $g \in G$ is normal if and only if $\operatorname{Ad} g \in \mathrm{GL}(\mathfrak{g})$ is normal.* (2) *Let $0 < \lambda < 1$. An element $g \in G$ is normal if and only if g is invariant under Δ_λ.*

PROOF. (1) One implication is trivial. For the other implication, consider $g = kp$ such that $\operatorname{Ad} g$ is normal, i.e., $\operatorname{Ad}(kp) = \operatorname{Ad}(pk)$. Since $\ker \operatorname{Ad} = Z \subset K$ [**11**, p. 130], $kp = pkz$ where $z \in Z$, i.e., $kpk^{-1} = zp$. Now $kpk^{-1} \in P$ because $\mathfrak{p}$ is invariant under $\operatorname{Ad} k$ for all $k \in K$. By the uniqueness of Cartan decomposition, $z = 1$ and $kpk^{-1} = p$, i.e., $kp = pk$.

(2) Suppose that $g = kp$ is normal, where $k \in K$, $p = \exp X \in P$ and $X \in \mathfrak{p}$, i.e., $kp = pk$. Then $kpk^{-1} = p$ so that $\exp(\operatorname{Ad}(k)X) = \exp X$. Since $\operatorname{Ad}(k)\mathfrak{p} = \mathfrak{p}$ and the map (2.1) is a diffeomorphism, we have $\operatorname{Ad}(k)X = X$. Thus $\operatorname{Ad}(k)(tX) = tX$ for all $t \in \mathbb{R}$ so

that $kp^tk^{-1} = p^t$, i.e., $kp^t = p^tk$. As a result $\Delta_\lambda(g) = p^\lambda kp^{1-\lambda} = g$. Conversely if $g = kp$ is invariant under Δ_λ, then $p^\lambda kp^{1-\lambda} = kp$, i.e., $p^\lambda k = kp^\lambda$. So $\exp(\mathrm{Ad}(k)\lambda X) = \exp(\lambda X)$ where $\exp X = p$. Using the diffeomorphism (2.1) $\mathrm{Ad}(k)\lambda X = \lambda X$ so that $\mathrm{Ad}(k)X = X$ and thus $kp = pk$. $\square$

LEMMA 3. *Let G be a noncompact connected semisimple Lie group and $g \in G$, and let $\varphi : G \to G$ be a diffeomorphism such that $\varphi(cg) = c\varphi(g)$ for each $c \in Z$, where Z is the center of G. If $\{\mathrm{Ad}\,\varphi^m(g)\}_{m\in\mathbb{N}}$ converges to L so that $\mathrm{Ad}^{-1}(L)$ contains some fixed point ℓ of φ, then $\{\varphi^m(g)\}_{m\in\mathbb{N}}$ converges to an element $\varphi^\infty(g) \in G$.*

PROOF. If $\{\mathrm{Ad}\,\varphi^m(g)\}_{m\in\mathbb{N}}$ converges, then the limit L is of the form $\mathrm{Ad}\,\ell$ for some $\ell \in G$ since $\mathrm{Ad}\,G$ is closed in $\mathrm{GL}(\mathfrak{g})$ [**11**, p. 132]. We may assume that ℓ is a fixed point of φ. Since (G, Ad) is a covering group of $\mathrm{Ad}\,G$ [**11**, p. 272], there is a (local) homeomorphism, induced by Ad, between neighborhoods U of ℓ and $\mathrm{Ad}\,U$ of $\mathrm{Ad}\,\ell$. Thus there is a sequence $\{g_m\}_{m\in\mathbb{N}} \subset U$ converging to ℓ and $\mathrm{Ad}\,g_m = \mathrm{Ad}\,\varphi^m(g)$. Since $\ker \mathrm{Ad} = Z$, there is a sequence $\{z_m\}_{m\in\mathbb{N}} \in Z$ such that $g_m = z_m\varphi^m(g)$, and

$$\lim_{m\to\infty} z_m\varphi^m(g) = \ell. \tag{2.2}$$

Apply φ on (2.2) to have

$$\lim_{m\to\infty} z_m\varphi^{m+1}(g) = \varphi(\ell) = \ell.$$

Hence

$$\lim_{m\to\infty} z_{m+1}z_m^{-1} = 1,$$

where $1 \in G$ denotes the identity element. The converging sequence $\{z_{m+1}z_m^{-1}\}_{m\in\mathbb{N}}$ is contained in the center Z which is discrete [**11**, p. 116]. So $z_{m+1} = z_m = z$ (say) for sufficiently large $m \in \mathbb{N}$. Hence $\{\varphi^m(g)\}_{m\in\mathbb{N}}$ converges to $\varphi^\infty(g) := \ell z^{-1}$. $\square$

Our main result is

THEOREM 4. *Let G be a real connected noncompact semisimple Lie group, and let $g \in G$. Let $0 < \lambda < 1$.*

(1) *The λ-Aluthge sequence $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$ converges to a normal $\Delta_\lambda^\infty(g) \in G$.*
(2) *The map $\Delta_\lambda^\infty : G \to G$ defined by $g \mapsto \Delta_\lambda^\infty(g)$ is continuous.*

PROOF. (1) By [**13**],

$$\mathrm{Ad}(\Delta_\lambda^m(g)) = \Delta_\lambda^m(\mathrm{Ad}(g)), \qquad m \in \mathbb{N}, \tag{2.3}$$

where Δ_λ on the left is the Aluthge transform of $g \in G$ with respect to the Cartan decomposition $G = KP$ and that on the right is the matrix Aluthge transform of $\mathrm{Ad}(g) \in \mathrm{Ad}\,G \subset \mathrm{GL}(\mathfrak{g})$ with respect to the polar decomposition. By Theorem 1 $\{\Delta_\lambda^m(\mathrm{Ad}(g))\}_{m\in\mathbb{N}}$

converges to a normal $\operatorname{Ad}\ell$ for some $\ell \in G$ since $\operatorname{Ad} G$ is closed in $\mathrm{GL}(\mathfrak{g})$ [**11**, p. 132]; so does $\{\operatorname{Ad}(\Delta_\lambda^m(g))\}_{m\in\mathbb{N}}$. Since ℓ is normal by Lemma 2, ℓ is fixed by Δ_λ. Moreover central elements factor out of Δ_λ so that Lemma 3 applies immediately, i.e., $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$ converges to the normal $\Delta_\lambda^\infty(g) := \ell z^{-1} \in G$.

(2) By 2.3,

$$\begin{aligned}\operatorname{Ad}(\Delta_\lambda^\infty(g)) &= \operatorname{Ad}\left(\lim_{m\to\infty}\Delta_\lambda^m(g)\right) = \lim_{m\to\infty}\operatorname{Ad}(\Delta_\lambda^m(g))\\ &= \lim_{m\to\infty}\Delta_\lambda^m(\operatorname{Ad}(g)) = \Delta_\lambda^\infty(\operatorname{Ad}(g)).\end{aligned}$$

So

$$\operatorname{Ad}\circ\Delta_\lambda^\infty = \Delta_\lambda^\infty\circ\operatorname{Ad}. \tag{2.4}$$

The $\Delta_\lambda^\infty : \mathrm{GL}(\mathfrak{g}) \to \mathrm{GL}(\mathfrak{g})$ on the right of (2.4) is continuous by Theorem 1(b), thus $\operatorname{Ad}\circ\Delta_\lambda^\infty$ is continuous. Since $\operatorname{Ad} G \cong G/Z$ [**11**, p. 129], $\operatorname{Ad} : G \to \operatorname{Ad} G$ on the left of (2.4) is an open map [**11**, p. 123], [**15**, p. 97]. Hence $\Delta_\lambda^\infty : G \to G$ is continuous. □

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