J. Math. Soc. Japan
Vol. 66, No. 4 (2014) pp. 1073–1089
doi: 10.2969/jmsj/06641073

Algebraic Montgomery-Yang problem: the log del Pezzo surface case

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(Received June 9, 2012)
(Revised Sep. 20, 2012)

Abstract. We prove that a log del Pezzo surface of Picard number one contains at most 3 singular points if its smooth locus is simply connected. This establishes the algebraic Montgomery-Yang problem for log del Pezzo surfaces.

1. Introduction.

The present paper is a continuation of two papers [**HK2**] and [**HK3**] on the conjecture called algebraic Montgomery-Yang problem.

Conjecture 1.1 ([**K**]). (Algebraic Montgomery-Yang Problem). *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities, i.e., a normal projective surface with quotient singularities such that $b_2(S) = 1$. Assume that $S^0 := S \backslash Sing(S)$ is simply connected. Then S contains at most 3 singular points.*

In previous papers [**HK2**] and [**HK3**], we have confirmed the conjecture when S contains at least one non-cyclic singularity or S is not rational.

In this paper we confirm the conjecture when $-K_S$ is ample, or equivalently when S is a log del Pezzo surface. By [**HK2**], we may assume that S has cyclic singularities only.

Theorem 1.2. *Let S be a log del Pezzo surface of Picard number one with cyclic singularities only. If $H_1(S^0, \mathbb{Z}) = 0$, then S contains at most 3 singular points.*

The condition $H_1(S^0, \mathbb{Z}) = 0$ is weaker than the condition $\pi_1(S^0) = 1$. In fact, there are log del Pezzo surfaces S of Picard number one with $H_1(S^0, \mathbb{Z}) = 0$ but $\pi_1(S^0) \neq 1$. Such surfaces have been classified in [**HK2**], under the assumption that the number of singularities is at least 4 and at least one of the singularities is non-cyclic.

The main ingredient of the proof is the classification theory of log del Pezzo surfaces of Picard number one developed by Zhang [**Z**], Gurjar and Zhang [**GZ**], Belousov [**Be**] together with the formulas developed in [**HK3**] for the intersection numbers of divisors on the minimal resolution.

Conjecture 1.1 is now reduced to the case where S is a rational surface with cyclic singularities such that K_S is ample. We do not know any example of a rational surface with 4 cyclic singularities such that K_S is ample. However, there are infinitely many

2010 *Mathematics Subject Classification.* Primary 14J26; Secondary 14J17, 14J45.
Key Words and Phrases. Montgomery-Yang problem, log del Pezzo surface, quotient singularity.

examples with smaller number of singularities ([**KM**], [**K**] and [**HK4**]).

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

2. Algebraic surfaces with cyclic singularities.

2.1.

A singularity p of a normal surface S is called a *cyclic singularity* if the germ is locally analytically isomorphic to $(\mathbb{C}^2/G, O)$ for some nontrivial finite cyclic subgroup G of $GL_2(\mathbb{C})$ without quasi-reflections. Such subgroups are completely classified by Brieskorn ([**Br**]).

For a cyclic singularity of type $\frac{1}{q}(1, q_1)$, one can associate a Hirzebruch-Jung continued fraction

$$[n_1, n_2, \dots, n_l] = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}} = \frac{q}{q_1}.$$

Let $\mathcal{H}$ be the set of all Hirzebruch-Jung continued fractions $[n_1, n_2, \dots, n_l]$,

$$\mathcal{H} = \bigcup_{l \geq 1} \{[n_1, n_2, \dots, n_l] \mid \text{all } n_j \text{ are integers} \geq 2\}.$$

We will use the following notation in this paper.

Notation 2.1. Fix $w = [n_1, n_2, \dots, n_l] \in \mathcal{H}$ and an integer $0 \leq s \leq l+1$.

(1) The *length* of w, denoted by $l(w)$, is the number of entries of w. We will write simply l for $l(w)$ if there is no confusion.
(2) Let q be the order of the cyclic singularity corresponding to w, i.e.,

$$q = |w| = |[n_1, n_2, \dots, n_l]| := |\det(M(-n_1, \dots, -n_l))|$$

where

$$M(-n_1, \dots, -n_l) = \begin{pmatrix} -n_1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -n_2 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & -n_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n_{l-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & -n_l \end{pmatrix}$$

is the intersection matrix corresponding to the singularity $[n_1, n_2, \dots, n_l]$.
(3) $u_s := |[n_1, n_2, \dots, n_{s-1}]|$ $(2 \leq s \leq l+1)$, $\quad u_0 = 0,\ u_1 = 1$.
(4) $v_s := |[n_{s+1}, n_{s+2}, \dots, n_l]|$ $(0 \leq s \leq l-1)$, $\quad v_l = 1,\ v_{l+1} = 0$.

Now let S be a normal projective surface with cyclic singularities and

$$f : S' \to S$$

be a minimal resolution of S. Since cyclic singularities are log-terminal singularities, one can write

$$K_{S'} \underset{num}{\equiv} f^*K_S - \sum_{p\in Sing(S)} \mathcal{D}_p,$$

where $\mathcal{D}_p = \sum(a_j A_j)$ is an effective $\mathbb{Q}$-divisor with $0 \le a_j < 1$ supported on $f^{-1}(p) = \cup A_j$ for each singular point p. Intersecting the formula with $\mathcal{D}_p$, we get

$$K_S^2 = K_{S'}^2 - \sum_p \mathcal{D}_p^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'}.$$

When p is a cyclic singularity of order q, the coefficients of $\mathcal{D}_p$ can be expressed in terms of v_j and u_j (see Notation 2.1) as follows.

Lemma 2.2 ([**HK3**, Lemma 3.1]). *Let p be a cyclic singular point of S. Assume that $f^{-1}(p)$ has l components $A_1, \ldots, A_l$ with $A_i^2 = -n_i$ forming a string of smooth rational curves* $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$. *Then*

(1) $\mathcal{D}_p K_{S'} = -\mathcal{D}_p^2 = \displaystyle\sum_{j=1}^{l}\left(1 - \frac{v_j + u_j}{q}\right)(n_j - 2),$

(2) $\mathcal{D}_p^2 = 2l - \displaystyle\sum_{j=1}^{l} n_j + 2 - \frac{q_1 + q_l + 2}{q}.$

In particular, if $l = 1$, then $\mathcal{D}_p^2 = -\dfrac{(n_1-2)^2}{n_1}$.

2.2.
The torsion-free part of the second cohomology group,

$$H^2(S', \mathbb{Z})_{free} := H^2(S', \mathbb{Z})/(torsion)$$

has a lattice structure which is unimodular. For a cyclic singular point $p \in S$, let

$$R_p \subset H^2(S', \mathbb{Z})_{free}$$

be the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the components of $f^{-1}(p)$. Then it is a negative definite lattice. Let

$$R = \bigoplus_{p\in Sing(S)} R_p \subset H^2(S', \mathbb{Z})_{free}$$

be the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the exceptional curves of $f : S' \to S$. Here, the order $|G_p|$ of the local fundamental group is equal to the absolute value $|\det(R_p)|$ of the determinant of the intersection matrix of R_p.

The following will be also useful in our proof.

Lemma 2.3 ([**HK2**, Lemma 2.5]). *Let S be a log del Pezzo surface of Picard number one with cyclic singularities such that $H_1(S^0, \mathbb{Z}) = 0$. Let $f : S' \to S$ be a minimal resolution. Then*

(1) *$H^2(S', \mathbb{Z})$ is torsion free, i.e., $H^2(S', \mathbb{Z}) = H^2(S', \mathbb{Z})_{free}$,*
(2) *R is a primitive sublattice of the unimodular lattice $H^2(S', \mathbb{Z})$,*
(3) *the orders $|G_p| = |\det(R_p)|$ of the local fundamental groups are pairwise relatively prime,*
(4) *$D := |\det(R + \langle K_{S'} \rangle)| = |\det(R)| K_S^2$ and is a nonzero square number.*

The intersection numbers $EK_{S'}$ and E^2 can be expressed in terms of the intersection numbers $EA_{j,p}$ of E and the exceptional curves $A_{j,p}$. See ([**HK3**, Section 4]) for a more general description.

Proposition 2.4 ([**HK3**, Proposition 4.2]). *Let S' be a minimal resolution of a log del Pezzo surface of Picard number one with cyclic singularities, and E be a divisor on it. Then, for some positive integer m depending on E, the following hold true.*

(1) $EK_{S'} = -\dfrac{m}{\sqrt{D}} K_S^2 - \displaystyle\sum_p \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) EA_{j,p}.$

(2) *If $EA_{j,p} = 0$ for $j \neq s_p, t_p$ for some s_p and t_p with $1 \le s_p < t_p \le l_p$, then*

$$E^2 = \frac{m^2}{D} K_S^2 - \sum_p \left(\frac{v_{s_p} u_{s_p}}{q_p} (EA_{s_p})^2 + \frac{v_{t_p} u_{t_p}}{q_p} (EA_{t_p})^2 + \frac{2 v_{t_p} u_{s_p}}{q_p} (EA_{s_p})(EA_{t_p}) \right).$$

Theorem 2.5 ([**HK1**, Theorem 1.1], [**Be**, Theorem 1.2]). *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities. If S is rational, then it contains at most 4 singular points.*

Proof. This is the result of Belousov ([**Be**, Theorem 1.2]) if $-K_S$ is ample, and is one of our previous results ([**HK1**, Theorem 1.1]) if K_S is nef. $\square$

3. Log del Pezzo surfaces of Picard number one.

Throughout this section, S denotes a log del Pezzo surface of Picard number one. Let

$$f : S' \to S$$

be its minimal resolution. We denote by

$$\mathcal{F} := f^{-1}(Sing(S))$$

the reduced exceptional divisor of f.

We review the work of Zhang [**Z**], Gurjar and Zhang [**GZ**] and Belousov [**Be**] on log del Pezzo surfaces of Picard number one. Assume that S does not contain any non-cyclic singularities, even though most of the results in this section hold for general case.

LEMMA 3.1. *$B^2 \geq -1$ for any irreducible curve $B \subset S'$ not contracted by $f : S' \to S$.*

PROOF. This is well-known (cf. [**HK2**, Lemma 2.1]). □

The following lemma is given in Lemma 4.1 in [**Z**], and can also be easily derived from the inequality of Proposition 2.4 (1).

LEMMA 3.2 ([**Z**, Lemma 4.1]). *Let E be a (-1)-curve on S'. Let $A_1, \ldots, A_r$ exhaust all irreducible components of $\mathcal{F}$ such that $EA_i > 0$. Suppose that $A_1^2 \geq A_2^2 \geq \cdots \geq A_r^2$. Then the r-tuple $(-A_1^2, \ldots, -A_r^2)$ is one of the following*:

$$(2, \ldots, 2, n), n \geq 2,\ (2, \ldots, 2, 3, 3),\ (2, \ldots, 2, 3, 4),\ (2, \ldots, 2, 3, 5).$$

An irreducible curve C on S' is called a *minimal curve* if $C.(-f^*K_S)$ attains the minimal positive value.

LEMMA 3.3 ([**Be**, Lemma 3.2, Lemma 4.1]). *A minimal curve C is a smooth rational curve.*

LEMMA 3.4. *Let C be a minimal curve. Suppose that $|C + \mathcal{F} + K_{S'}| \neq \emptyset$. Then there is a unique decomposition $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$ such that*

(1) *$\mathcal{F}'$ consists of (-2)-curves not meeting $C + \mathcal{F}''$,*
(2) *$C + \mathcal{F}'' + K_{S'} \sim 0$,*
(3) *$\mathcal{F}'' = f^{-1}(p)$ for some singular point p unless $\mathcal{F}'' = 0$.*

Furthermore, if $\mathcal{F}'' \neq 0$, then $C\mathcal{F}'' = C\mathcal{F} = 2$ and one of the following holds:

(1) *$\mathcal{F}''$ consists of one irreducible component, which C meets in a single point with multiplicity 2 or in two points,*
(2) *$\mathcal{F}''$ consists of two irreducible components, whose intersection point C passes through,*
(3) *$\mathcal{F}''$ consists of at least two irreducible components, and C meets the two end components of $\mathcal{F}''$.*

PROOF. The result can be easily derived from either [**GZ**, Lemma 3.2, Remark 3.4], or [**Be**, Lemma 3.1, Lemma 3.2]. □

LEMMA 3.5 ([**GZ**, Proposition 3.6]). *Let C be a minimal curve. Suppose that $|C + \mathcal{F} + K_{S'}| = \emptyset$. Then either C is a (-1)-curve, $S \cong \mathbb{P}^2$, or S is the Hirzebruch surface with the minimal section contracted.*

LEMMA 3.6 ([**Be**, Lemma 4.1]). *Suppose that S' contains a minimal curve C with $C^2 = -1$. Suppose that $|C + \mathcal{F} + K_{S'}| = \emptyset$. Then $C\mathcal{F}' \leq 1$ for any connected component*

$\mathcal{F}'$ of $\mathcal{F}$.

Lemma 3.7 ([**Z**, Lemma 4.4]). *Suppose that S' contains a minimal curve C with $C^2 = -1$. Suppose that $|C + \mathcal{F} + K_{S'}| = \emptyset$, and that C meets exactly two components F_1, F_2 of $\mathcal{F}$. Then either $F_1^2 = -2$ or $F_2^2 = -2$.*

The following lemma was proved in ([**Z**, Proof of Lemma 5.3]).

Lemma 3.8. *With the same assumption as in Lemma 3.7, assume further that $F_1^2 = F_2^2 = -2$. If F_1 is not an end component, then one of the following two cases holds*:

(1) *There exists another minimal (-1)-curve C' such that $|C' + \mathcal{F} + K_{S'}| \neq \emptyset$.*
(2) *$F_2 = f^{-1}(p_i)$ for some singular point p_i.*

Lemma 3.9. *Suppose that S' contains a minimal curve C with $C^2 = -1$. Suppose that $|C + \mathcal{F} + K_{S'}| = \emptyset$, and that C meets three components F_1, F_2, F_3 of $\mathcal{F}$ and possibly more. Define*

$$G := 2C + F_1 + F_2 + F_3 + K_{S'}.$$

Then either $G \sim 0$ or $G \sim \Gamma$ for some (-1)-curve Γ such that $C\Gamma = F_i\Gamma = 0$ for $i = 1, 2, 3$. Furthermore, the following hold true.

(1) *In the first case, there are 3 singular points p_1, p_2, p_3 such that $f^{-1}(p_i) = F_i$, and C meets no component of $\mathcal{F} - (F_1 + F_2 + F_3)$.*
(2) *In the second case,*
    (a) *$L = 2 - (F_1^2 + F_2^2 + F_3^2)$, where L is the number of irreducible components of $\mathcal{F}$,*
    (b) *each curve in $\mathcal{F} - F_1 - F_2 - F_3$ is a (-2)- or a (-3)-curve and there are at most two (-3)-curves in $\mathcal{F} - F_1 - F_2 - F_3$,*
    (c) *each connected component of $\mathcal{F}$ contains at most one $(-n)$-curve with $n \geq 3$.*

Proof. The main assertion is exactly ([**Z**, Lemma 2.3]).

(1) Let F_i be an irreducible component of $f^{-1}(p_i)$. Suppose that $f^{-1}(p_i)$ has at least 2 irreducible components. Then there is an irreducible component I of $f^{-1}(p_i)$ such that $IF_i = 1$. By Lemma 3.6, $IC = 0$, hence

$$0 = IG = I.(2C + F_1 + F_2 + F_3 + K_{S'}) = IF_i + IK_{S'} = 1 - I^2 - 2.$$

Thus $I^2 = -1$, a contradiction.
Suppose that C meets a component J of $\mathcal{F} - (F_1 + F_2 + F_3)$. Then

$$0 = JG = J.(2C + F_1 + F_2 + F_3 + K_{S'}) = 2 + JK_{S'},$$

so $J^2 = 0$, a contradiction.

(2) By ([**GZ**, Remark 6.4]), we may assume that $f^{-1}(p_i)$ has at least 2 irreducible components for $i = 1, 2$ or 3. Alternatively, by using Proposition 2.4, one can also derive

a contradiction for the case when $f^{-1}(p_i)$ consists of only one irreducible component for each $i = 1, 2$ and 3, but it needs lengthy computation.

Now (2-b) and (2-c) directly follows from ([**GZ**, Lemma 6.6]).

(2-a) We note that

$$G^2 = (2C + F_1 + F_2 + F_3 + K_{S'})^2 = 1 - L - \left(F_1^2 + F_2^2 + F_3^2\right)$$

where L denotes the number of irreducible components of $\mathcal{F}$. Since $G^2 = \Gamma^2 = -1$, we have $L = 2 - (F_1^2 + F_2^2 + F_3^2)$. □

The following lemma was proved in ([**Z**, Proof of Lemma 5.2]).

Lemma 3.10. *With the same assumption as in Lemma* 3.9*, assume further that* $2C+F_1+F_2+F_3+K_{S'} \sim \Gamma$ *for some* (-1)*-curve* Γ*, and that at least two of* F_1, F_2, F_3 *are* (-2)*-curves. Then there exists another minimal* (-1)*-curve* C' *such that* $|C'+\mathcal{F}+K_{S'}| \neq \emptyset$.

The first reduction results shown in [**HK3**] can be reformulated, in the case of log del Pezzo surfaces, as follows:

Lemma 3.11 ([**HK3**, Lemma 5.2, Lemma 5.3, Lemma 5.4, Lemma 5.6]). *Let* S *be a log del Pezzo surface of Picard number one containing exactly* 4 *cyclic singular points* p_1, p_2, p_3, p_4 *of orders* (q_1, q_2, q_3, q_4)*. Let* E *be a* (-1)*-curve on* S'*. Then* $E.\mathcal{F} \geq 2$*, and one of the following cases occurs.*

(1) *The orders are* $(2, 3, 5, q)$ *where* $q \geq 7$ *and* $\gcd(q, 30) = 1$*. Moreover, the order* 3 *singularity must be of type* $\frac{1}{3}(1,1)$*. In this case,* $E.\mathcal{F} = 2$ *if and only if* $E.f^{-1}(p_i) = 0$ *for* $i = 1, 2, 3$ *and* $E.f^{-1}(p_4) = 2$*.*
(2) *The orders are either* $(2, 3, 7, q)$ *where* $11 \leq q \leq 41$ *or* $(2, 3, 11, 13)$*. Moreover, the singularity type of* S *is precisely one of the* 24 *cases in Table* 1*. In this case, if* $E.\mathcal{F} = 2$*, then* E *does not meet an end component of* $f^{-1}(p_i)$ *for any* $i = 1, 2, 3, 4$*.*

4. Proof of Theorem 1.2.

Throughout this section, S denotes a log del Pezzo surface of Picard number one such that $H_1(S^0, \mathbb{Z}) = 0$. Then S contains at most 4 singular points by Theorem 2.5. Suppose that S contains 4 cyclic singular points p_1, p_2, p_3, p_4. By Lemma 3.11, it remains to consider the following cases:

- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,1) + \frac{1}{q}(1,q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,2) + \frac{1}{q}(1,q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- $A_1 + \frac{1}{3}(1,1) + A_4 + \frac{1}{q}(1,q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- the 24 cases in Table 1.

Let

$$\mathcal{F} = f^{-1}(Sing(S))$$

Table 1.

No.	Type of R	orders	K_S^2
1	$A_1 + A_2 + [7] + [13]$	$(2,3,7,13)$	$\frac{1536}{91}$
2	$A_1 + A_2 + [7] + [3,2,2,2,2,2,2,2,2]$	$(2,3,7,19)$	$\frac{6}{133}$
3	$A_1 + A_2 + [7] + [5,4]$	$(2,3,7,19)$	$\frac{1350}{133}$
4	$A_1 + A_2 + [7] + [3,4,2]$	$(2,3,7,19)$	$\frac{1014}{133}$
5	$A_1 + A_2 + [4,2] + [2,2,4,2,2,2]$	$(2,3,7,31)$	$\frac{150}{217}$
6	$A_1 + A_2 + [4,2] + [6,2,2,2,2,2]$	$(2,3,7,31)$	$\frac{486}{217}$
7	$A_1 + [3] + [3,2,2] + [4,2,2,2,3]$	$(2,3,7,29)$	$\frac{968}{609}$
8	$A_1 + A_2 + [3,2,2] + [7,2,2,2]$	$(2,3,7,25)$	$\frac{24}{7}$
9	$A_1 + A_2 + [7] + [2,2,3,2,2,2,2,2,2]$	$(2,3,7,31)$	$\frac{54}{217}$
10	$A_1 + [3] + [4,2] + [3,3,2,2,3]$	$(2,3,7,41)$	$\frac{2888}{861}$
11	$A_1 + A_2 + [3,2,2] + [7,2,2,2,2,2]$	$(2,3,7,37)$	$\frac{384}{259}$
12	$A_1 + A_2 + [4,2] + [11,2,2]$	$(2,3,7,31)$	$\frac{2166}{217}$
13	$A_1 + [3] + A_6 + [2,6,2,2]$	$(2,3,7,29)$	$\frac{56}{87}$
14	$A_1 + [3] + [3,2,2] + [4,3]$	$(2,3,7,11)$	$\frac{1058}{231}$
15	$A_1 + [3] + [3,2,2] + [3,2,2,2,2]$	$(2,3,7,11)$	$\frac{50}{231}$
16	$A_1 + [3] + [3,2,2] + [4,2,2,3]$	$(2,3,7,23)$	$\frac{1250}{483}$
17	$A_1 + [3] + [3,2,2] + [6,5]$	$(2,3,7,29)$	$\frac{5000}{609}$
18	$A_1 + A_2 + [3,2,2] + [3,5,2]$	$(2,3,7,25)$	$\frac{24}{7}$
19	$A_1 + A_2 + [3,2,2] + [13,2]$	$(2,3,7,25)$	$\frac{1944}{175}$
20	$A_1 + A_2 + [4,2] + [4,2,2,2]$	$(2,3,7,13)$	$\frac{216}{91}$
21	$A_1 + A_2 + [4,2] + [5,2,2]$	$(2,3,7,13)$	$\frac{384}{91}$
22	$A_1 + A_2 + [4,2] + [4,2,2,2,2,2]$	$(2,3,7,19)$	$\frac{54}{133}$
23	$A_1 + [3] + [3,2,2,2,2] + [4,2,2,2]$	$(2,3,11,13)$	$\frac{8}{429}$
24	$A_1 + [3] + [3,2,2,2,2] + [5,2,2]$	$(2,3,11,13)$	$\frac{800}{429}$

be the reduced exceptional divisor of the minimal resolution $f : S' \to S$, and L be the number of irreducible components of $\mathcal{F}$. Let C be a (fixed) minimal curve on S'.

4.1. Step 1. $|C + \mathcal{F} + K_{S'}| = \emptyset$.

Proof. Suppose that $|C + \mathcal{F} + K_{S'}| \neq \emptyset$. By Lemma 3.4 (1) and (3), we see that S contains at least 3 rational double points.

In the case of $(2,3,5,q)$, by Lemma 3.11 (1) we see that S contains at least 3 rational double points, only if the singularities are of type $A_1 + [3] + A_4 + A_{q-1}$. In this case, by Lemma 2.2,

$$L = q + 5 \quad \text{and} \quad K_S^2 = 9 - (q+5) + \frac{1}{3} < 0,$$

a contradiction.

We also see that each of the 24 cases from Table 1 contains at most 2 rational double points. □

4.2. Step 2.

(1) C is a (-1)-curve.

(2) $C\mathcal{F} = 3$, and C meets three distinct components F_1, F_2, F_3 of $\mathcal{F}$.

PROOF. (1) It immediately follows from Lemma 3.5 since S contains 4 singularities.

(2) By Lemma 3.6, $C\mathcal{F} \leq 4$. Since $C^2 = -1 < 0$ and the lattice R is negative definite, $C\mathcal{F} \geq 1$.

Assume that $C\mathcal{F} = 1$. Blowing up the intersection point, then contracting the proper transform of C and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane with 5 quotient singularities, which may not be a log del Pezzo surface, i.e., whose canonical class may be nef. Even this case contradicts Theorem 2.5.

Assume that $C\mathcal{F} = 4$. By Lemma 3.6, C meets four components F_1, F_2, F_3, F_4 of $\mathcal{F}$, where $F_i \subset f^{-1}(p_i)$. Then $G \sim \Gamma$ by Lemma 3.9 (1). By Lemma 3.2, at least two of F_1, F_2, F_3, F_4 have self-intersection -2. Thus, by Lemma 3.10, there exists another minimal (-1)-curve C' such that $|C' + \mathcal{F} + K_{S'}| \neq \emptyset$. This is impossible by Step 1.

Assume that $C\mathcal{F} = 2$.

(a) Suppose that the case $(2, 3, 5, q)$ occurs for some $q \geq 7$ with $\gcd(q, 30) = 1$. By Lemma 3.11 (1), $C.f^{-1}(p_4) = 2$. But, by Lemma 3.6, $C.f^{-1}(p_4) \leq 1$, a contradiction.

(b) Now suppose that one of the 24 cases of Table 1 occurs. By Lemma 3.6, there are two components F_1 and F_2 of $\mathcal{F}$ with $CF_1 = CF_2 = 1$. By Lemma 3.7, we may assume that $F_1^2 = -2$. Moreover, by Lemma 3.11 (2), C does not meet an end component of $f^{-1}(p_i)$ for any i, i.e., both F_1 and F_2 are middle components. Thus $F_2^2 \neq -2$ by Lemma 3.8 and Step 1. After contracting the (-1)-curve C, by contracting the proper transforms of all irreducible components of $\mathcal{F} - F_1$, we obtain a $\mathbb{Q}$-homology projective plane with 5 quotient singularities, again contradicting Theorem 2.5. □

4.3. Step 3.

$2C + F_1 + F_2 + F_3 + K_{S'} \sim \Gamma$ for some (-1)-curve Γ.

PROOF. Suppose that

$$2C + F_1 + F_2 + F_3 + K_{S'} \sim 0.$$

Then, by Lemma 3.9 (1), each F_i is equal to the inverse image of a singular point of S. By Table 1 and Lemma 3.11, only the following cases satisfy this condition:

$$\begin{aligned} &A_1 + A_2 + [7] + [13] \qquad \text{(Case 1, Table 1)},\\ &A_1 + [3] + [2,2,2,2] + [q],\\ &A_1 + [3] + [3,2] + [q],\\ &A_1 + [3] + [5] + \frac{1}{q}(1, q_1). \end{aligned}$$

Thus,

$$\big(- F_1^2, -F_2^2, -F_3^2\big) = (2,7,13), (2,3,q), (2,5,q), (3,5,q), (2,3,5).$$

Then Lemma 3.2 rules out the first four possibilities, since $q \geq 7$.

In the last case $(-F_1^2, -F_2^2, -F_3^2) = (2,3,5)$, $F_i = f^{-1}(p_i)$ for $i = 1,2,3$. In this case we consider the sublattice

$$\langle C, F_1, F_2, F_3\rangle \subset H^2(S', \mathbb{Z})$$

generated by C, F_1, F_2, F_3. It is of rank 4 and has

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -5 \end{pmatrix}$$

as its intersection matrix. It has determinant -1, hence the orthogonal complement of $\langle C, F_1, F_2, F_3\rangle$ in $H^2(S', \mathbb{Z})$ is unimodular. The orthogonal complement is an over-lattice of the lattice R_{p_4} generated by the components of $f^{-1}(p_4)$. Since R_{p_4} is a primitive sublattice of $H^2(S', \mathbb{Z})$, it must be unimodular, hence $q = 1$, a contradiction. □

4.4. Step 4.
If one of the cases $(2,3,5,q)$, $q \geq 7$, $\gcd(q, 30) = 1$, occurs, then $C.f^{-1}(p_4) = 1$.

PROOF. Suppose that the case $(2,3,5,q)$ occurs for some $q \geq 7$ with $\gcd(q, 30) = 1$. By Lemma 3.11 (1), p_2 is of type [3].

By Lemma 3.6, $C.f^{-1}(p_i) \leq 1$ for $i = 1,2,3,4$.

Suppose on the contrary that $C.f^{-1}(p_4) = 0$.

Then,

$$C.f^{-1}(p_1) = C.f^{-1}(p_2) = C.f^{-1}(p_3) = 1.$$

Let $F_i \subset f^{-1}(p_i)$ be the component with $CF_i = 1$ for $i = 1,2,3$.

Assume that p_3 is of type [5]. Then $(-F_1^2, -F_2^2, -F_3^2) = (2,3,5)$ and the sublattice $\langle C, F_1, F_2, F_3\rangle \subset H^2(S', \mathbb{Z})$ has determinant -1, leading to the same contradiction as above, since the orthogonal complement of $\langle C, F_1, F_2, F_3\rangle$ in $H^2(S', \mathbb{Z})$ is R_{p_4}.

Assume that p_3 is of type [2, 3]. Then $(-F_1^2, -F_2^2, -F_3^2) = (2,3,2)$ or $(2,3,3)$. Let $f^{-1}(p_3) = F_3 + F_3'$. If $F_3^2 = -2$, then

$$\big|\det\langle C, F_1, F_2, F_3, F_3'\rangle\big| = 13,$$

and by Lemma 3.9 (2-a) $L = 2 + 2 + 3 + 2 = 9$, so $l = 5$. The orthogonal complement of $\langle C, F_1, F_2, F_3, F_3'\rangle$ in $H^2(S', \mathbb{Z})$ is R_{p_4}, hence

$$|\det(R_{p_4})| = q = 13.$$

This leads to a contradiction since there is no continued fraction of length 5 with $q = 13$. If $F_3^2 = -3$, then

$$\big|\det\langle C, F_1, F_2, F_3, F_3'\rangle\big| = 7,$$

hence $|\det(R_{p_4})| = q = 7$. By Lemma 3.9 (2), $L = 2+2+3+3 = 10$, so $l = 6$. Thus p_4 is of type A_6. But, then

$$K_S^2 = 9 - L - \mathcal{D}_{p_2}^2 - \mathcal{D}_{p_3}^2 = -1 + \frac{1}{3} + \frac{2}{5} < 0,$$

a contradiction.

Assume that p_3 is of type $A_4 = [2,2,2,2]$. Then $(-F_1^2, -F_2^2, -F_3^2) = (2,3,2)$. Let $f^{-1}(p_3) = H_1 + H_2 + H_3 + H_4$. If F_3 is an end component of $f^{-1}(p_3)$, say H_1, then

$$\big|\det\langle C, F_1, F_2, H_1, H_2, H_3, H_4\rangle\big| = 19,$$

and by Lemma 3.9 (2-a) $L = 2+2+3+2 = 9$, so $l = 3$. Thus $|\det(R_{p_4})| = q = 19$ and $\operatorname{rank}(R_{p_4}) = 3$. Among all Hirzebruch-Jung continued fractions of order 19, only two, $[7,2,2]$ and $[3,4,2]$, have length 3. In each of these two cases, $f^{-1}(p_4)$ contains an irreducible component with self-intersection ≤ -4. Since $f^{-1}(p_4) \subset \mathcal{F} - F_1 - F_2 - F_3$, we have a contradiction by Lemma 3.9 (2-b). If F_3 is a middle component of $f^{-1}(p_3)$, say H_2, then

$$\big|\det\langle C, F_1, F_2, H_1, H_2, H_3, H_4\rangle\big| = 31,$$

and by Lemma 3.9 (2-a) $L = 2+2+3+2 = 9$, so $l = 3$. Thus $q = 31$ and p_4 is of type $[11,2,2]$, $[3,6,2]$, or $[5,2,4]$. In each of these three cases, $f^{-1}(p_4)$ contains an irreducible component with self-intersection ≤ -4, a contradiction by Lemma 3.9 (2-b). This proves that $C.f^{-1}(p_4) = 1$. □

4.5. Step 5.

None of the cases $(2,3,5,q)$, $q \geq 7$, $\gcd(q,30) = 1$, occurs.

PROOF. Suppose that the case $(2,3,5,q)$ occurs for some $q \geq 7$ with $\gcd(q,30) = 1$.
By Lemma 3.11 (1), p_2 is of type $[3]$.
By Step 2, $C\mathcal{F} = 3$ and C meets the three components F_1, F_2, F_3 of $\mathcal{F}$.
By Step 3,

$$2C + F_1 + F_2 + F_3 + K_{S'} \sim \Gamma$$

for some (-1)-curve Γ.

By Step 4, we may assume that $F_3 \subset f^{-1}(p_4)$.

Let

$$f^{-1}(p_4) = \underset{D_1}{\overset{-n_1}{\circ}} - \underset{D_2}{\overset{-n_2}{\circ}} - \cdots - \underset{D_l}{\overset{-n_l}{\circ}}$$

and $F_3 = D_j$ for some $1 \le j \le l$. Note first that by Lemma 3.9 (2-b), $n_k \le 3$ for all $k \ne j$.

Assume that p_3 is of type [5]. By Lemma 3.9 (2-b), C must meet $f^{-1}(p_3)$, so we may assume that $F_2 = f^{-1}(p_3)$. Since $F_1 = f^{-1}(p_1)$ or $F_1 = f^{-1}(p_2)$, by Lemma 3.2,

$$\big(-F_1^2, -F_2^2, -F_3^2\big) = (2,5,2), (3,5,2), (2,5,3).$$

By Lemma 3.9 (2-a), we have

$$(L, n_j) = (11, 2), (12, 2), (12, 3),$$

hence

$$(l, n_j) = (8, 2), (9, 2), (9, 3).$$

By Lemma 3.9 (2-b) and (2-c),

$$\begin{aligned}[n_1, \ldots, n_l] = {} & [3,2,2,2,2,2,2,2], [2,2,2,2,2,2,2,2];\\ & [3,2,2,2,2,2,2,2,2], [2,2,2,2,2,2,2,2,2]\end{aligned}$$

up to permutation of $n_1, \ldots, n_l$. As you can see in Table 2, none of these 11 cases satisfies the following three conditions:

- (#1) $\gcd(q, 30) = 1$,
- (#2) $K_S^2 > 0$,
- (#3) $D = |\det(R)| K_S^2$ is a positive square integer.

Table 2.

Type of p_4	q	$\gcd(q,30)$	K^2	$\sqrt{D}$
A_8	9	$\ne 1$	–	–
$[3,2,2,2,2,2,2,2]$	17	1	$\frac{154}{255}$	$2\sqrt{77}$
$[2,3,2,2,2,2,2,2]$	23	1	$\frac{256}{345}$	$16\sqrt{2}$
$[2,2,3,2,2,2,2,2]$	27	$\ne 1$	–	–
$[2,2,2,3,2,2,2,2]$	29	1	$\frac{358}{435}$	$2\sqrt{179}$
A_9	10	$\ne 1$	–	–
$[3,2,2,2,2,2,2,2,2]$	19	1	$-\frac{112}{285}$	–
$[2,3,2,2,2,2,2,2,2]$	26	$\ne 1$	–	–
$[2,2,3,2,2,2,2,2,2]$	31	1	$-\frac{88}{465}$	–
$[2,2,2,3,2,2,2,2,2]$	34	$\ne 1$	–	–
$[2,2,2,2,3,2,2,2,2]$	35	$\ne 1$	–	–

Assume that p_3 is of type $[2,3]$. Then, by Lemma 3.2,

$$\big(-F_1^2, -F_2^2, -F_3^2\big) = (2,3,n_j),\ n_j \le 5, \quad \text{or } (3,3,2), \quad \text{or } (2,2,n_j).$$

The last case can be ruled out by Lemma 3.10 and Step 1. Now, by Lemma 3.9 (2), we have

$$(l, n_j) = (5,2), (6,3), (7,4), (8,5), (6,2),$$

and

$$[n_1, \ldots, n_l] = [3,2,2,2,2], [2,2,2,2,2]; [3,2,2,2,2,2]; \\ [4,2,2,2,2,2,2]; [5,2,2,2,2,2,2,2]; [2,2,2,2,2,2],$$

up to permutation of $n_1, \ldots, n_l$. None of the 16 cases satisfies the three conditions (#1), (#2), (#3). Table 3 summarizes the computation.

Table 3.

Type of p_4	q	$\gcd(q,30)$	K^2	$\sqrt{D}$
A_5	6	$\neq 1$	–	–
$[3,2,2,2,2]$	11	1	$\frac{196}{165}$	$14\sqrt{2}$
$[2,3,2,2,2]$	14	$\neq 1$	–	–
$[2,2,3,2,2]$	15	$\neq 1$	–	–
A_6	7	1	$-\frac{4}{15}$	–
$[3,2,2,2,2,2]$	13	1	$\frac{38}{195}$	$2\sqrt{19}$
$[2,3,2,2,2,2]$	17	1	$\frac{82}{255}$	$2\sqrt{41}$
$[2,2,3,2,2,2]$	19	1	$\frac{104}{285}$	$4\sqrt{13}$
$[4,2,2,2,2,2,2]$	22	$\neq 1$	–	–
$[2,4,2,2,2,2,2]$	32	$\neq 1$	–	–
$[2,2,4,2,2,2,2]$	38	$\neq 1$	–	–
$[2,2,2,4,2,2,2]$	40	$\neq 1$	–	–
$[5,2,2,2,2,2,2,2]$	33	$\neq 1$	–	–
$[2,5,2,2,2,2,2,2]$	51	$\neq 1$	–	–
$[2,2,5,2,2,2,2,2]$	63	$\neq 1$	–	–
$[2,2,2,5,2,2,2,2]$	69	$\neq 1$	–	–

Assume that p_3 is of type $[2,2,2,2]$. Then, by Lemma 3.2,

$$\big(-F_1^2, -F_2^2, -F_3^2\big) = (2,3,n_j),\ n_j \le 5, \quad \text{or } (2,2,n_j).$$

The last case can be ruled out by Lemma 3.10 and Step 1. Now, by Lemma 3.9 (2), we have

$$(l, n_j) = (3, 2), (4, 3), (5, 4), (6, 5),$$

and

$$[n_1, \ldots, n_l] = [3, 2, 2], [2, 2, 2]; [3, 2, 2, 2]; [4, 2, 2, 2, 2]; [5, 2, 2, 2, 2, 2],$$

up to permutation of $n_1, \ldots, n_l$. None of the 11 cases satisfies the three conditions (#1), (#2), (#3). Table 4 summarizes the computation.

Table 4.

Type of p_4	q	$\gcd(q, 30)$	K^2	$\sqrt{D}$
A_3	4	$\neq 1$	–	–
[3, 2, 2]	7	1	$\frac{16}{21}$	$4\sqrt{10}$
[2, 3, 2]	8	$\neq 1$	–	–
[3, 2, 2, 2]	9	$\neq 1$	–	–
[2, 3, 2, 2]	11	1	$-\frac{4}{33}$	–
[4, 2, 2, 2, 2]	16	$\neq 1$	–	–
[2, 4, 2, 2, 2]	22	$\neq 1$	–	–
[2, 2, 4, 2, 2]	24	$\neq 1$	–	–
[5, 2, 2, 2, 2, 2]	25	$\neq 1$	–	–
[2, 5, 2, 2, 2, 2]	37	1	$-\frac{26}{111}$	–
[2, 2, 5, 2, 2, 2]	43	1	$-\frac{20}{129}$	–

□

Next, we will show that none of the cases $(2, 3, 7, q)$, $11 \leq q \leq 41$, $\gcd(q, 42) = 1$, and $(2, 3, 11, 13)$ occurs. To do this, it is enough to consider the 24 cases of Table 1.

4.6. Step 6.

None of the 24 cases of Table 1 occurs.

Proof. By Step 2, $C\mathcal{F} = 3$ in each of the 24 cases of Table 1.

Each of Cases (1), (2), (3), (4), (6), (8), (9), (11), (12), (13), (17), and (19), contains an irreducible component F' with self-intersection ≤ -6. Lemma 3.9 (2-b) implies that C meets F'. Thus C meets two components of $\mathcal{F}$ with self-intersection -2 by Lemma 3.2. Thus we get a contradiction for those cases by Lemma 3.10 and Step 1.

By Lemma 3.9 (2-c), we get a contradiction immediately for Cases (7), (10), (14), (16), (18), since each of these cases contains a connected component of $\mathcal{F}$ with at least two irreducible components of self-intersection ≤ -3.

By Lemma 3.2 and Lemma 3.9 (2-b), we get a contradiction immediately for Cases (5), (20), (21), (22), since each of these cases contains at least two irreducible components with self-intersection ≤ -4.

We need to rule out the remaining three cases: (15), (23), (24).

Consider Case (24). Note that $L = 10$ in this case. On the other hand, by Lemma 3.9 (2-b), C must meet the component having self-intersection number -5. Thus, we may assume that $F_3^2 = -5$. Since $F_1^2 \leq -2$, $F_2^2 \leq -2$, Lemma 3.9 (2-a) gives $L = 2 - (F_1^2 + F_2^2 + F_3^2) \geq 2 + 2 + 2 + 5 = 11$, a contradiction.

Case (15): Let

$$\begin{array}{ccccccccccccccccccc} -2 & & -3 & & -3 & & -2 & & -2 & & -3 & & -2 & & -2 & & -2 & & -2 \\ \circ & & \circ & & \circ & - & \circ & - & \circ & & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ A & & B & & C_1 & & C_2 & & C_3 & & D_1 & & D_2 & & D_3 & & D_4 & & D_5 \end{array}$$

be the exceptional curves. In this case, $K_S^2 = 50/231$, $\sqrt{D} = 10$.

Since $L = 10 = 2 - (F_1^2 + F_2^2 + F_3^2)$, C meets only two of B, C_1, D_1.

If $CC_1 = CD_1 = 1$, then $CA = 1$. Applying Proposition 2.4 (1) to C and looking at Table 5, we get

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{3}{7} - \frac{5}{11} = \frac{9}{77},$$

thus $m = 27/5$, not an integer, a contradiction.

Table 5.

	[2]	[3]	[3, 2, 2]			[3, 2, 2, 2, 2]				
j	1	1	1	2	3	1	2	3	4	5
$1 - \frac{v_j + u_j}{q}$	0	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{5}{11}$	$\frac{4}{11}$	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$

If $CB = CC_1 = CA = 1$, then Γ only meets C_2 and D_1, a contradiction to Lemma 3.11 (2).

If $CB = CC_1 = CD_j = 1$ for some j, then Proposition 2.4 (1) gives

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{3}{7} - \left(1 - \frac{v_j + u_j}{q}\right) > 0,$$

hence $j = 4, 5$. If $j = 4$, then

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{3}{7} - \frac{2}{11} = \frac{13}{231},$$

thus $m = 13/5$, a contradiction. If $j = 5$, then

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{3}{7} - \frac{1}{11} = \frac{34}{231},$$

thus $m = 34/5$, a contradiction.

If $CB = CD_1 = CA = 1$, then

$$\frac{m}{\sqrt{D}}K_S^2 = 1 - \frac{1}{3} - \frac{5}{11} = \frac{7}{33},$$

thus $m = 49/5$, a contradiction.

If $CB = CD_1 = CC_2 = 1$, then

$$\frac{m}{\sqrt{D}}K_S^2 = 1 - \frac{1}{3} - \frac{2}{7} - \frac{5}{11} = -\frac{17}{231} < 0,$$

a contradiction.
If $CB = CD_1 = CC_3 = 1$, then

$$\frac{m}{\sqrt{D}}K_S^2 = 1 - \frac{1}{3} - \frac{1}{7} - \frac{5}{11} = \frac{16}{231},$$

thus $m = 16/5$, a contradiction.

Case (23): Let

$$\overset{-2}{\underset{A}{\circ}} \quad \overset{-3}{\underset{B}{\circ}} \quad \overset{-3}{\underset{C_1}{\circ}} - \overset{-2}{\underset{C_2}{\circ}} - \overset{-2}{\underset{C_3}{\circ}} - \overset{-2}{\underset{C_4}{\circ}} - \overset{-2}{\underset{C_5}{\circ}} \quad \overset{-4}{\underset{D_1}{\circ}} - \overset{-2}{\underset{D_2}{\circ}} - \overset{-2}{\underset{D_3}{\circ}} - \overset{-2}{\underset{D_4}{\circ}}$$

be the exceptional curves. Since C meets D_1 and $L = 11$, C must meet only one of B and C_1.

If $CB = CA = 1$, then Γ meets exactly two irreducible components C_1, D_2 with multiplicity 1, a contradiction to Lemma 3.11 (2).

If $CB = CC_j = 1$ for some $j \geq 2$, then Table 6 gives

$$\frac{m}{\sqrt{D}}K_S^2 \leq 1 - \frac{1}{3} - \frac{1}{11} - \frac{8}{13} < 0,$$

a contradiction.

If $CC_1 = 1$, then $CA = 1$ and Proposition 2.4 (1) together with Table 6 gives

$$\frac{m}{\sqrt{D}}K_S^2 = 1 - 0 - \frac{5}{11} - \frac{8}{13} < 0,$$

a contradiction.

Table 6.

	[2]	[3]	[3, 2, 2, 2, 2]					[4, 2, 2, 2]			
j	1	1	1	2	3	4	5	1	2	3	4
$1 - \frac{v_j+u_j}{q}$	0	$\frac{1}{3}$	$\frac{5}{11}$	$\frac{4}{11}$	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{8}{13}$	$\frac{6}{13}$	$\frac{4}{13}$	$\frac{2}{13}$

□

This completes the proof of Theorem 1.2.

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