J. Math. Soc. Japan
Vol. 66, No. 2 (2014) pp. 523–534
doi: 10.2969/jmsj/06620523

Hausdorff continuous sections

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(Received Oct. 17, 2011)
(Revised June 14, 2012)

Abstract. It is shown that a space X is strongly paracompact if and only if for every complete metric space (Y,ρ), every l.s.c. mapping from X into the nonempty closed subsets of Y has a separable-valued Hausdorff continuous section. Several applications are demonstrated as well.

1. Introduction.

All spaces in this paper are assumed to be Hausdorff topological spaces. For a space Y, let 2^Y be the power set of Y; $\mathscr{F}(Y)$ be the set of all nonempty closed subsets of Y; and $\mathscr{C}(Y)$ — that of all compact members of $\mathscr{F}(Y)$. For a set-valued mapping $\varphi : X \to 2^Y$ and subsets $A \subset X$ and $B \subset Y$, let

$$\varphi[A] = \bigcup\{\varphi(x) : x \in A\}, \text{ and}$$
$$\varphi^{-1}[B] = \{x \in X : \varphi(x) \cap B \neq \emptyset\}.$$

A mapping $\varphi : X \to 2^Y$ is *lower semi-continuous*, or l.s.c., if the set $\varphi^{-1}[U]$ is open in X for every open $U \subset Y$; and φ is *upper semi-continuous*, or u.s.c., if the set $\varphi^{\#}[U] = X \setminus \varphi^{-1}[Y \setminus U] = \{x \in X : \varphi(x) \subset U\}$ is open in X for every open $U \subset Y$. For convenience, we say that φ is *usco* if it is u.s.c. and nonempty-compact-valued; and that φ is *continuous* if it is both l.s.c. and u.s.c. A mapping $\varphi : X \to 2^Y$ is a *multi-selection* (or, a *set-valued selection*) for $\Phi : X \to 2^Y$ if $\varphi(x) \subset \Phi(x)$ for every $x \in X$; and $\varphi : X \to 2^Y$ is a *section* for $\Phi : X \to 2^Y$ if $\varphi(x) \cap \Phi(x) \neq \emptyset$ for every $x \in X$. If φ is a section for Φ, then both φ and Φ must be nonempty-valued. Of course, every nonempty-valued multi-selection for Φ is also a section for Φ.

For $\varepsilon > 0$ and a subset F of a metric space (Z, d), let $B^d_\varepsilon(F)$ be the open ε-neighbourhood of F with respect to d, i.e. $B^d_\varepsilon(F) = \{z \in Z : d(z, F) < \varepsilon\}$. A subset $S \subset Z$ is *totally ε-bounded* (see, [**9**]) if there exists a finite subset $F \subset S$, with $S \subset B^d_\varepsilon(F)$. It is well known that if d is a complete metric on Z, then a subset $K \subset Z$ is compact if and only if it is closed and totally bounded with respect to d (i.e., totally ε-bounded for every $\varepsilon > 0$). Finally, recall that a metric d on Z is *non-Archimedean*, or an *ultrametric*, [**6**], [**10**] if

$$d(x, y) \le \max\{d(x, z), d(z, y)\}, \quad \text{for every } x, y, z \in Z.$$

2010 *Mathematics Subject Classification.* Primary 54C60, 54C65; Secondary 54B20, 54D20.

Key Words and Phrases. strong paracompactness, set-valued mapping, lower semi-continuous, upper semi-continuous, Hausdorff continuous, section, tree, branch.

A metrizable space Z is called *non-Archimedean* if it has a non-Archimedean metric compatible with its topology. Often, a metric space (Z, d) is called *ultrametric* if d is an ultrametric.

There is a natural relationship between covering properties of topological spaces and multi-selections of set-valued mappings. A starting point for the present paper is a recent result in this regard. Recall that a cover $\mathscr{U}$ of X is *star-finite* if the set $\{W \in \mathscr{U} : W \cap U \neq \emptyset\}$ is finite for every $U \in \mathscr{U}$. A space X is *strongly paracompact* (called, also, *hypocompact*) if every open cover of X has a star-finite open refinement. Every strongly paracompact space is paracompact, but the converse is not necessarily true, see [**7**, Exercises 5.3.F and 6.1.E]. The following theorem was proved in [**9**].

Theorem 1.1 ([**9**]). *For a space X, the following are equivalent*:

(a) *X is strongly paracompact.*
(b) *If (Y, ρ) is a complete metric space and $\Phi : X \to \mathscr{F}(Y)$ is an l.s.c. mapping, then there exists a* (*complete*) *ultrametric space (Z, d), an usco mapping $\psi : X \to \mathscr{C}(Z)$ and a uniformly continuous map $g : Z \to Y$ such that $g \circ \psi : X \to \mathscr{C}(Y)$ is a multi-selection for Φ and the set $\psi[\psi^{-1}[S]]$ is totally ε-bounded whenever $S \subset Z$ is totally ε-bounded for some $\varepsilon > 0$.*

Let $\mathscr{W}$ be a collection of subsets of a set X. If $U, V \in \mathscr{W}$, then a finite sequence $W_1, W_2, \ldots, W_k$ of elements of $\mathscr{W}$ is called a *chain* from U to V if $U = W_1$, $V = W_k$ and $W_i \cap W_{i+1} \neq \emptyset$ for every $i = 1, \ldots, k-1$. A subset $\mathscr{P} \subset \mathscr{W}$ is called *connected* if every pair of elements of $\mathscr{P}$ is connected by a chain. The components of $\mathscr{W}$ are defined as the maximal connected subsets of $\mathscr{W}$. A space X is called *super-paracompact* (Pasynkov, see [**15**]) if every open cover of X has an open refinement each component of which is finite. In contrast to Theorem 1.1, components of covers were associated with sections of set-valued mappings. The following theorem was proved in [**2**].

Theorem 1.2 ([**2**]). *A space X is super-paracompact if and only if for every completely metrizable space Y, every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ has a continuous section $\varphi : X \to \mathscr{C}(Y)$.*

Turning to the main purpose of this paper, let us remark that strong paracompactness can also be expressed in terms of components of covers. Namely, according to [**7**, Lemma 5.3.9 and Theorem 5.3.10] (see, also, [**1**, Theorem 2.3]), a space X is strongly paracompact iff every open cover of X admits an open refinement with countable components. So, it is natural to expect that strongly paracompact spaces possess a similar characterisation in terms of sections. Indeed, the following theorem will be proved in this paper.

Theorem 1.3. *A space X is strongly paracompact if and only if for every complete metric space (Y, ρ), every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ has a ρ-continuous section $\varphi : X \to \mathscr{L}(Y)$.*

Here, $\mathscr{L}(Y) = \{S \in \mathscr{F}(Y) : S \text{ is Lindelöf}\}$, while a mapping $\varphi : X \to \mathscr{F}(Y)$ is called *ρ-continuous* (sometimes, also, *Hausdorff continuous*) if for every $\varepsilon > 0$, every

$x \in X$ has a neighbourhood U such that

$$\varphi(x) \subset B^\rho_\varepsilon(\varphi(z)) \text{ and } \varphi(z) \subset B^\rho_\varepsilon(\varphi(x)), \quad \text{for every } z \in U.$$

Let us remark, that if $\varphi : X \to \mathscr{C}(Y)$ is continuous and Y is metrizable, then φ is ρ-continuous for every compatible metric ρ on Y. That is, Theorem 1.3 is a natural continuation of Theorem 1.2.

A word should be said also for the paper itself. Theorem 1.3 is proved in Section 3, see Corollary 3.5; the preparation for this proof is done in the next section. The last Section 4 contains several applications of Theorem 1.3 and the technique developed in this paper, see Corollaries 4.2, 4.3 and 4.4, also Theorems 4.5 and 4.6.

2. Additive sieves on complete metric spaces.

A partially ordered set $(T, \preceq)$ is a *tree* if $\{s \in T : s \preceq t\}$ is well-ordered for every $t \in T$. For a tree $(T, \preceq)$, we use $T(0)$ to denote the set of the minimal elements of T. Given an ordinal α, if $T(\beta)$ is defined for every $\beta < \alpha$, then $T(\alpha)$ denotes the minimal elements of $T \setminus \bigcup\{T(\beta) : \beta < \alpha\}$. The set $T(\alpha)$ is called the α^{th}*-level* of T, while the *height* of T is the least ordinal α such that $T = \bigcup\{T(\beta) : \beta < \alpha\}$. We say that $(T, \preceq)$ is an α*-tree* if its height is α. A maximal linearly ordered subset of a tree $(T, \preceq)$ is called a *branch*, and $\mathscr{B}(T)$ is used to denote the set of all branches of T. A tree $(T, \preceq)$ is *pruned* if each element of T has a successor in T, i.e. if for every $s \in T$ there exists $t \in T$, with $s \prec t$. In these terms, an ω-tree $(T, \preceq)$ is pruned if each branch $\beta \in \mathscr{B}(T)$ is infinite.

Following Nyikos [**17**], for a tree $(T, \preceq)$ and $t \in T$, let

$$\mathscr{O}(t) = \{\beta \in \mathscr{B}(T) : t \in \beta\}. \tag{2.1}$$

If $(T, \preceq)$ is a pruned ω-tree, then the family $\{\mathscr{O}(t) : t \in T\}$ is a base for a completely metrizable non-Archimedean topology on $\mathscr{B}(T)$. We will refer to this topology as the *branch topology*, and to the resulting topological space as the *branch space*. Throughout this paper, $\mathscr{B}(T)$ will be always endowed with the branch topology when it comes to consider it as a topological space. It is well known that the branch space $\mathscr{B}(T)$ is compact if and only if all levels of $(T, \preceq)$ are finite.

For a tree $(T, \preceq)$ and $t \in T$, the *node* of t in T is the subset $\mathbf{node}(t) \subset T$ of all immediate successors of t. For convenience, let $\mathbf{node}(\emptyset) = T(0)$. Given a set Y and a pruned ω-tree $(T, \preceq)$, a mapping $\mathscr{S} : T \to 2^Y$ is a *sieve* on Y if $Y = \mathscr{S}[\mathbf{node}(\emptyset)]$ and $\mathscr{S}(t) = \mathscr{S}[\mathbf{node}(t)]$ for every $t \in T$.

To every mapping $\mathscr{S} : T \to 2^Y$ defined on a tree $(T, \preceq)$, we associate another one $\Omega_\mathscr{S} : \mathscr{B}(T) \to 2^Y$, called the *polar mapping*, by letting

$$\Omega_\mathscr{S}(\beta) = \bigcap\{\mathscr{S}(t) : t \in \beta\}, \quad \beta \in \mathscr{B}(T). \tag{2.2}$$

The value $\Omega_\mathscr{S}(\beta)$ for a branch $\beta \in \mathscr{B}(T)$ is called the *polar* of β by $\mathscr{S}$. We will also associate the mapping $\overline{\mathscr{S}} : T \to 2^Y$, defined by $\overline{\mathscr{S}}(t) = \overline{\mathscr{S}(t)}$, $t \in T$, which will play the role of a *pointwise closure* of $\mathscr{S}$.

A sieve $\mathscr{S} : T \to 2^Y$ on a space Y is *locally finite* (*discrete*, etc.) provided that each cover $\{\mathscr{S}(t) : t \in T(n)\}$, $n < \omega$, is locally finite (discrete, etc.). Suppose that (Y, ρ) is a complete metric space. According to [**13**, Lemma 2.2], Y has a nonempty-open-valued locally finite sieve $\mathscr{M} : T \to 2^Y$ with $\mathbf{diam}_\rho(\mathscr{M}(t)) < 2^{-n}$ for every $t \in T(n)$ and $n < \omega$. By Cantor's intersection theorem, each polar $\Omega_{\overline{\mathscr{M}}}(\beta) = \bigcap_{t\in\beta} \overline{\mathscr{M}}(t)$, $\beta \in \mathscr{B}(T)$, will be a singleton, hence $\Omega_{\overline{\mathscr{M}}} : \mathscr{B}(T) \to 2^Y$ will be singleton-valued. Since every singleton-valued mapping has the same graph as a single-valued one (representing the same relation), we will make no difference between such mappings. Thus, in this case, the polar mapping $\Omega_{\overline{\mathscr{M}}} : \mathscr{B}(T) \to 2^Y$ is a usual map. The following proposition now follows immediately by (2.1) and (2.2).

PROPOSITION 2.1. *Let (Y, ρ) be a complete metric space, and let $\mathscr{M} : T \to 2^Y$ be a nonempty-open-valued sieve on Y such that $\mathbf{diam}_\rho(\mathscr{M}(t)) < 2^{-n}$ for every $t \in T(n)$ and $n < \omega$. Then, the polar mapping $\Omega_{\overline{\mathscr{M}}} : \mathscr{B}(T) \to \mathscr{C}(Y)$ is singleton-valued and $\rho\big(\Omega_{\overline{\mathscr{M}}}(\gamma), \Omega_{\overline{\mathscr{M}}}(\eta)\big) \leq 2^{-n}$, whenever $\gamma, \eta \in \mathscr{O}(t)$, $t \in T(n)$ and $n < \omega$. In particular, $\Omega_{\overline{\mathscr{M}}}$ is continuous.*

We now turn to the main purpose of this section.

For a space Z, the *degree of compactness* of Z (see, [**5**]) is defined as the least cardinal number $\boldsymbol{k}(Z)$ such that every open cover of Z has an open refinement of cardinality $< \boldsymbol{k}(Z)$. In these terms, a space Z is compact iff $\boldsymbol{k}(Z) \leq \omega$; and Z is Lindelöf iff $\boldsymbol{k}(Z) \leq \omega_1$. Let τ be an infinite cardinal number. For a space Y and a set D, let

$$\mathscr{F}_\tau(Y) = \{S \in \mathscr{F}(Y) : \boldsymbol{k}(S) \leq \tau\},$$
$$[D]^{<\tau} = \{S \subset D : 1 \leq |S| < \tau\}.$$

Then, $\mathscr{C}(Y) = \mathscr{F}_\omega(Y)$ and $\mathscr{L}(Y) = \mathscr{F}_{\omega_1}(Y)$, see [**16**]. If D is endowed with the discrete topology, we also have that $[D]^{<\tau} = \mathscr{F}_\tau(D)$.

Given a pruned ω-tree $(T, \preceq)$, let

$$\mathbb{H}_\tau[T] = \bigcup\{[T(n)]^{<\tau} : n < \omega\}. \tag{2.3}$$

Define a relation $\preceq$ on $\mathbb{H}_\tau[T]$ by letting for $\sigma, \mu \in \mathbb{H}_\tau[T]$ that $\sigma \prec \mu$ if

$$\mu \subset \bigcup\{\mathbf{node}(s) : s \in \sigma\} \quad \text{and} \quad \mu \cap \mathbf{node}(s) \neq \emptyset,\ s \in \sigma. \tag{2.4}$$

Next, extend this relation to a partial order on $\mathbb{H}_\tau[T]$ by making it transitive. Thus, we get a pruned ω-tree $(\mathbb{H}_\tau[T], \preceq)$ because so is T. Let us observe that, with respect to this partial order, for each branch $\boldsymbol{\beta} \in \mathscr{B}(\mathbb{H}_\tau[T])$, $\bigcup \boldsymbol{\beta}$ is a pruned subtree of T such that $\big(\bigcup \boldsymbol{\beta}\big) \cap T(n) \neq \emptyset$, $n < \omega$. In particular, we have that

$$\mathscr{B}\left(\bigcup \boldsymbol{\beta}\right) \subset \mathscr{B}(T), \quad \boldsymbol{\beta} \in \mathscr{B}(\mathbb{H}_\tau[T]). \tag{2.5}$$

We may regard the tree $(\mathbb{H}_\tau[T], \preceq)$ as some kind of "Vietoris-like" hypertree of T, in fact the order defined in (2.4) is reassembling the definition of the Vietoris topology. We are going to prove the following theorem.

THEOREM 2.2. *Let (Y,ρ) be a complete metric space, and $\mathscr{M} : T \to 2^Y$ be a nonempty-open-valued locally finite sieve on Y such that $\mathbf{diam}_\rho(\mathscr{M}(t)) < 2^{-n}$, for every $t \in T(n)$ and $n < \omega$. Whenever τ is an infinite regular cardinal number, define another sieve $\mathscr{R} : \mathbb{H}_\tau[T] \to 2^Y$ on Y by $\mathscr{R}(\sigma) = \mathscr{M}[\sigma]$, for every $\sigma \in \mathbb{H}_\tau[T]$. Then, the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(\mathbb{H}_\tau[T]) \to 2^Y$ is $\mathscr{F}_\tau(Y)$-valued and ρ-continuous.*

PROOF. First of all, let us observe that

$$\Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta}) = \Omega_{\overline{\mathscr{M}}}\Big[\mathscr{B}\Big(\bigcup \boldsymbol{\beta}\Big)\Big], \quad \text{for every } \boldsymbol{\beta} \in \mathscr{B}(\mathbb{H}_\tau[T]). \tag{2.6}$$

Indeed, by (2.5), $\mathscr{B}\big(\bigcup \boldsymbol{\beta}\big) \subset \mathscr{B}(T)$ and the inclusion $\Omega_{\overline{\mathscr{M}}}\big[\mathscr{B}\big(\bigcup \boldsymbol{\beta}\big)\big] \subset \Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta})$ now follows by the definition of $\mathscr{R}$. To see the converse, take a point $y \in \Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta})$ and consider the subtree $K(y) = \big\{t \in \bigcup \boldsymbol{\beta} : y \in \overline{\mathscr{M}}(t)\big\}$ of $\bigcup \boldsymbol{\beta}$. Since $\mathscr{M}$ is locally finite, each $K(y) \cap T(n)$, $n < \omega$, is nonempty and finite. Hence, by Köning's lemma (see, Lemma 5.7 in Chapter II of [**11**]), the subtree $K(y)$ contains an infinite branch $\gamma \in \mathscr{B}(K(y)) \subset \mathscr{B}\big(\bigcup \boldsymbol{\beta}\big) \subset \mathscr{B}(T)$. Therefore, $y \in \Omega_{\overline{\mathscr{M}}}(\gamma) \subset \Omega_{\overline{\mathscr{M}}}\big[\mathscr{B}\big(\bigcup \boldsymbol{\beta}\big)\big]$.

To show that $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(\mathbb{H}_\tau[T]) \to \mathscr{F}_\tau(Y)$, take a branch $\boldsymbol{\beta} \in \mathscr{B}(\mathbb{H}_\tau[T])$, and consider the corresponding subtree $\bigcup \boldsymbol{\beta} \subset T$. If $\tau = \omega$, then each lever of $\bigcup \boldsymbol{\beta}$ is finite (by (2.3)) and, as mentioned before, the branch space $\mathscr{B}\big(\bigcup \boldsymbol{\beta}\big)$ is compact. Consequently, by (2.6) and Proposition 2.1,

$$\Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta}) = \Omega_{\overline{\mathscr{M}}}\Big[\mathscr{B}\Big(\bigcup \boldsymbol{\beta}\Big)\Big] \in \mathscr{C}(Y) = \mathscr{F}_\omega(Y).$$

In case $\tau > \omega$, it follows by (2.1) that $w\big(\mathscr{B}\big(\bigcup \boldsymbol{\beta}\big)\big) < \tau$ because each level of $\bigcup \boldsymbol{\beta}$ has a cardinality $< \tau$ (by (2.3)) while τ has an uncountable cofinality. Then, by (2.6), $\Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta}) \in \mathscr{F}_\tau(Y)$ because $\Omega_{\overline{\mathscr{M}}}$ is singleton-valued and continuous, see Proposition 2.1.

To show finally that the polar mapping $\Omega_{\overline{\mathscr{R}}}$ is ρ-continuous, take $\varepsilon > 0$, $m < \omega$ with $2^{-m} < \varepsilon$, and $\sigma \in \mathbb{H}_\tau[T](m)$. Also, let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathscr{O}(\sigma) \subset \mathscr{B}(\mathbb{H}_\tau[T])$ and $y \in \Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta})$. By (2.6), there exists a branch $\gamma \in \mathscr{B}(\bigcup \boldsymbol{\beta}) \subset \mathscr{B}(T)$ such that $\Omega_{\overline{\mathscr{M}}}(\gamma) = \{y\}$. Since $\sigma \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ because $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathscr{O}(\sigma)$, there now exist $t \in \gamma \cap \sigma$ and $\eta \in \mathscr{B}(\bigcup \boldsymbol{\alpha})$ with $t \in \eta$. Then, $\gamma, \eta \in \mathscr{O}(t)$ and, by Proposition 2.1, we have $\rho\big(\Omega_{\overline{\mathscr{M}}}(\gamma), \Omega_{\overline{\mathscr{M}}}(\eta)\big) \leq 2^{-m} < \varepsilon$. According to (2.6), this implies that $\Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta}) \subset B^\rho_\varepsilon\big(\Omega_{\overline{\mathscr{R}}}(\boldsymbol{\alpha})\big)$. By the same argument as above, we can show that $\Omega_{\overline{\mathscr{R}}}(\boldsymbol{\alpha}) \subset B^\rho_\varepsilon\big(\Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta})\big)$. Thus, $\Omega_{\overline{\mathscr{R}}}$ is ρ-continuous, see (2.1). $\square$

3. Refining additive sieves and pseudo-sections.

Given a cardinal number τ, to each family $\mathscr{U}$ of subsets of a set X we will associate another family $\mathscr{U}^\tau$ defined by $\mathscr{U}^\tau = \big\{\bigcup \mathscr{W} : \mathscr{W} \in [\mathscr{U}]^{<\tau}\big\}$. In what follows, it will be convenient to each space X to associate a topological invariant $\mathbf{sp}(X)$ defined as the least cardinal number τ with the property that for every open cover $\mathscr{U}$, the cover $\mathscr{U}^\tau$ has a pairwise disjoint open refinement. As the reader may guess, $\mathbf{sp}(X)$ indicates the degree of strong paracompactness of X. Indeed, according to [**3**, Proposition 2.3] (see, also, [**4**, Theorem 2.2]) and [**1**, Theorem 2.3], we have the following basic example illustrating the relationship of $\mathbf{sp}(X)$ with super-paracompactness and strong paracompactness.

PROPOSITION 3.1. *For a space X, the following holds*:

(a) $\mathbf{sp}(X) \leq \omega$ *if and only if X is super-paracompact.*
(b) $\mathbf{sp}(X) \leq \omega_1$ *if and only if X is strongly paracompact.*

The cardinal invariant $\mathbf{sp}(X)$ allows to construct special refinements of the "hyper"-sieves considered in the previous section.

PROPOSITION 3.2. *Let X be a space such that $\mathbf{sp}(X) \leq \tau$ for an infinite regular cardinal τ, and let $\mathscr{S} : T \to 2^X$ be an open-valued sieve on X. Define a sieve $\mathscr{P} : \mathbb{H}_\tau[T] \to 2^X$ by $\mathscr{P}(\sigma) = \mathscr{S}[\sigma]$, for every $\sigma \in \mathbb{H}_\tau[T]$. Then, there exists a discrete open-valued sieve $\mathscr{L} : \mathbb{H}_\tau[T] \to 2^X$ which is a multi-selection for $\mathscr{P}$, i.e. $\mathscr{L}(\sigma) \subset \mathscr{P}(\sigma)$ for every $\sigma \in \mathbb{H}_\tau[T]$.*

PROOF. Consider the cover $\mathscr{U}_0 = \{\mathscr{P}(\sigma) : \sigma \in \mathbb{H}_\tau[T](0)\}$ of X, and observe that $\mathscr{U}_0^\tau = \mathscr{U}_0$ because τ is regular. Since $\mathbf{sp}(X) \leq \tau$, X has a pairwise disjoint open cover $\{\mathscr{L}(\sigma) : \sigma \in \mathbb{H}_\tau[T](0)\}$ with $\mathscr{L}(\sigma) \subset \mathscr{P}(\sigma)$, $\sigma \in \mathbb{H}_\tau[T](0)$. Take an element $\sigma \in \mathbb{H}_\tau[T](0)$, and consider the cover $\mathscr{U}_\sigma = \{\mathscr{P}(\eta) \cap \mathscr{L}(\sigma) : \eta \in \mathbf{node}(\sigma)\}$ of $\mathscr{L}(\sigma)$. Just like before, we have that $\mathscr{U}_\sigma^\tau = \mathscr{U}_\sigma$. Since $\mathscr{L}(\sigma)$ is a clopen subset of X, we also have that $\mathbf{sp}(\mathscr{L}(\sigma)) \leq \tau$. Hence, $\mathscr{L}(\sigma)$ has a pairwise disjoint open cover $\{\mathscr{L}(\eta) : \eta \in \mathbf{node}(\sigma)\}$ such that $\mathscr{L}(\eta) \subset \mathscr{P}(\eta) \cap \mathscr{L}(\sigma)$ for $\eta \in \mathbf{node}(\sigma)$. We may proceed by induction on the levels $\mathbb{H}_\tau[T](n)$, $n < \omega$, of the tree $\mathbb{H}_\tau[T]$ to finalise the construction of the sieve $\mathscr{L} : \mathbb{H}_\tau[T] \to 2^X$. □

We now turn to the main result of this section. In this result, and what follows, for a metric space (Y, ρ) and subsets $B, C \subset Y$, let

$$\boldsymbol{D}_\rho(B, C) = \inf\{\rho(y, z) : y \in B \text{ and } z \in C\}.$$

THEOREM 3.3. *For a space X and an infinite regular cardinal number τ, the following are equivalent*:

(a) $\mathbf{sp}(X) \leq \tau$.
(b) *Whenever (Y, ρ) is a complete metric space and $\Phi : X \to \mathscr{F}(Y)$ is an l.s.c. mapping, there exists a non-Archimedean completely metrizable space Z, a continuous map $g : X \to Z$ and a ρ-continuous mapping $\psi : Z \to \mathscr{F}_\tau(Y)$ such that $\boldsymbol{D}_\rho(\psi(g(x)), \Phi(x)) = 0$ for every $x \in X$.*
(c) *Whenever (Y, ρ) is a complete metric space and $\Phi : X \to \mathscr{F}(Y)$ is an l.s.c. mapping, there exists a ρ-continuous mapping $\varphi : X \to \mathscr{F}_\tau(Y)$ such that $\boldsymbol{D}_\rho(\varphi(x), \Phi(x)) = 0$ for every $x \in X$.*

PROOF. (a) ⇒ (b). Let $\mathbf{sp}(X) \leq \tau$, (Y, ρ) be a complete metric space, and let $\Phi : X \to \mathscr{F}(Y)$ be an l.s.c. mapping. Take a nonempty-open-valued locally finite sieve $\mathscr{M} : T \to 2^Y$ on Y such that $\mathbf{diam}_\rho(\mathscr{M}(t)) < 2^{-n}$ for every $t \in T(n)$ and $n < \omega$, see Section 2. Next, let $\mathscr{R} : \mathbb{H}_\tau[T] \to 2^Y$ be defined as in Theorem 2.2, i.e. by $\mathscr{R}(\sigma) = \mathscr{M}[\sigma]$,

$\sigma \in \mathbb{H}_\tau[T]$. Define an open-valued sieve $\mathscr{S} : T \to 2^X$ on X by $\mathscr{S}(t) = \Phi^{-1}[\mathscr{M}(t)]$, $t \in T$. Finally, define $\mathscr{P} : \mathbb{H}_\tau[T] \to 2^X$ by $\mathscr{P}(\sigma) = \mathscr{S}[\sigma]$, $\sigma \in \mathbb{H}_\tau[T]$, and then observe that

$$\mathscr{P}(\sigma) = \Phi^{-1}[\mathscr{R}(\sigma)], \quad \sigma \in \mathbb{H}_\tau[T]. \tag{3.1}$$

By Proposition 3.2, X has an open-valued discrete sieve $\mathscr{L} : \mathbb{H}_\tau[T] \to 2^X$ such that $\mathscr{L}$ is a multi-selection for $\mathscr{P}$. Consider the inverse polar mapping $\mho_{\mathscr{L}} : X \to 2^{\mathscr{B}(\mathbb{H}_\tau[T])}$ defined by $\mho_{\mathscr{L}}(x) = \Omega_{\mathscr{L}}^{-1}[\{x\}]$, $x \in X$. Since each family $\{\mathscr{L}(\sigma) : \sigma \in \mathbb{H}_\tau[T](n)\}$, $n < \omega$, is pairwise disjoint, the mapping $\mho_{\mathscr{L}}$ is singleton-valued. It is also continuous because $\mho_{\mathscr{L}}^{-1}[\mathscr{O}(\sigma)] = \mathscr{L}(\sigma)$ for every $\sigma \in \mathbb{H}_\tau[T]$, see (2.1). We are going to show that $Z = \mathscr{B}(\mathbb{H}_\tau[T])$, $g = \mho_{\mathscr{L}}$ and $\psi = \Omega_{\overline{\mathscr{R}}}$ are as required. By Theorem 2.2, $\psi : Z \to \mathscr{F}_\tau(Y)$ and is ρ-continuous. Take a point $x \in X$, and let $\boldsymbol{\beta} \in \mathscr{B}(\mathbb{H}_\tau[T])$ be a branch such that $x \in \mathscr{L}(\sigma)$ for every $\sigma \in \boldsymbol{\beta}$. Write $\boldsymbol{\beta} = \{\sigma_n : n < \omega\}$, where $\sigma_n \in \mathbb{H}_\tau[T](n)$ for each $n < \omega$. Since $\mathscr{L}(\sigma) \subset \mathscr{P}(\sigma)$ for every $\sigma \in \boldsymbol{\beta}$, by (3.1), for every $n < \omega$ there exists $t_n \in \sigma_n$ such that $\Phi(x) \cap \mathscr{M}(t_n) \neq \emptyset$. Take a branch $\gamma_n \in \mathscr{B}\big(\bigcup \boldsymbol{\beta}\big)$, with $t_n \in \gamma_n$. Then,

$$\emptyset \neq \Omega_{\overline{\mathscr{M}}}(\gamma_n) \subset \Omega_{\overline{\mathscr{R}}}(\boldsymbol{\beta}) = \psi(g(x)) \quad \text{and} \quad \Omega_{\overline{\mathscr{M}}}(\gamma_n) \subset \overline{\mathscr{M}}(t_n).$$

Since $\mathbf{diam}_\rho(\mathscr{M}(t_n)) < 2^{-n}$ for every $n < \omega$, it follows that $\boldsymbol{D}_\rho(\psi(g(x)), \Phi(x)) = 0$, and the proof of this implication is completed.

Since (b) $\Rightarrow$ (c) is obvious by taking $\varphi = \psi \circ g$, we complete the proof by showing that (c) $\Rightarrow$ (a). Take an open cover $\mathscr{U}$ of X, and endow it with the discrete metric $\rho(V, U) = 1$ if $V, U \in \mathscr{U}$ are distinct elements. Next, define a mapping $\Phi : X \to \mathscr{F}(\mathscr{U})$ by $\Phi(x) = \{U \in \mathscr{U} : x \in U\}$, $x \in X$. Since Φ is l.s.c., by (c), there is a ρ-continuous mapping $\varphi : X \to \mathscr{F}_\tau(\mathscr{U})$ such that $\boldsymbol{D}_\rho(\varphi(x), \Phi(x)) = 0$ for every $x \in X$. According to the definition of ρ, this implies that φ is a section for Φ. Consider the family $\Gamma = \{\varphi(x) : x \in X\} \subset \mathscr{F}_\tau(\mathscr{U}) = [\mathscr{U}]^{<\tau}$. Next, for every $\mathscr{G} \in \Gamma$, let $V_{\mathscr{G}} = \{x \in X : \varphi(x) = \mathscr{G}\}$. By the definition of ρ, we have that $B_1^\rho(\varphi(x)) = \varphi(x)$ for every $x \in X$. Since φ is ρ-continuous, each $V_{\mathscr{G}}$, $\mathscr{G} \in \Gamma$, is a clopen subset of X. Finally, observe that $W_{\mathscr{G}} = \Phi^{-1}[\mathscr{G}] = \bigcup \mathscr{G} \in \mathscr{U}^\tau$ for every $\mathscr{G} \in \Gamma$ because $\Gamma \subset [\mathscr{U}]^{<\tau}$. If $x \in V_{\mathscr{G}}$ for some $\mathscr{G} \in \Gamma$, then $\varphi(x) = \mathscr{G}$ and $\varphi(x) \cap \Phi(x) \neq \emptyset$. Hence, it follows that $\Phi(x) \cap \mathscr{G} \neq \emptyset$ and, therefore, $x \in W_{\mathscr{G}}$. Thus, $\{V_{\mathscr{G}} : \mathscr{G} \in \Gamma\}$ is a refinement of $\mathscr{U}^\tau$. That is, $\mathbf{sp}(X) \leq \tau$. □

The following is an immediate consequence of Theorem 3.3; it follows by Proposition 3.1 and the fact that $F \cap K \neq \emptyset$ provided that F is a nonempty closed subset of (Y, ρ), K is a nonempty compact subset of Y and $\boldsymbol{D}_\rho(F, K) = 0$.

Corollary 3.4. *For a space X, the following are equivalent*:

(a) *X is super-paracompact.*
(b) *Whenever Y is a completely metrizable space and $\Phi : X \to \mathscr{F}(Y)$ is an l.s.c. mapping, there exists a non-Archimedean completely metrizable space Z, a continuous map $g : X \to Z$ and a continuous mapping $\psi : Z \to \mathscr{C}(Y)$ such that $\psi \circ g$ is a section for Φ.*
(c) *Whenever Y is a completely metrizable space, every l.s.c. $\Phi : X \to \mathscr{F}(Y)$ has a continuous section $\varphi : X \to \mathscr{C}(Y)$.*

Corollary 3.4 provides an alternative proof of Theorem 1.2; while this proof is similar to that one given in [**2**], the theorem is now obtained as a special case of a more general result. Indeed, by Theorem 3.3 we get also the following slight generalisation of Theorem 1.3.

COROLLARY 3.5. *For a space X, the following are equivalent*:

(a) *X is strongly paracompact.*
(b) *Whenever (Y,ρ) is a complete metric space and $\Phi : X \to \mathscr{F}(Y)$ is an l.s.c. mapping, there exists a non-Archimedean completely metrizable space Z, a continuous map $g : X \to Z$ and a ρ-continuous mapping $\psi : Z \to \mathscr{L}(Y)$ such that the composition $\psi \circ g : X \to \mathscr{L}(Y)$ is a section for Φ.*
(c) *Whenever (Y,ρ) is a complete metric space and $\Phi : X \to \mathscr{F}(Y)$ is an l.s.c. mapping, there exists a ρ-continuous mapping $\varphi : X \to \mathscr{L}(Y)$ such that $\boldsymbol{D}_\rho(\varphi(x), \Phi(x)) = 0$ for every $x \in X$.*

PROOF. (a) $\Rightarrow$ (b). Let X be a strongly paracompact space, (Y,ρ) be a complete metric space, and let $\Phi : X \to \mathscr{F}(Y)$ be an l.s.c. mapping. Since X is paracompact (being strongly paracompact), by [**12**, Theorem 1.1], Φ has an l.s.c. multi-selection $\Psi : X \to \mathscr{C}(Y)$. By Proposition 3.1, $\mathbf{sp}(X) \le \omega_1$. Hence, by Theorem 3.3, there exists a non-Archimedean completely metrizable space Z, a continuous map $g : X \to Z$ and a ρ-continuous mapping $\psi : Z \to \mathscr{F}_{\omega_1}(Y) = \mathscr{L}(Y)$ such that $\boldsymbol{D}_\rho(\psi(g(x)), \Psi(x)) = 0$ for every $x \in X$. Since Ψ is compact-valued, it follows that $\psi \circ g$ is a section for Ψ, hence for Φ as well.

The implication (b) $\Rightarrow$ (c) is obvious, while (c) $\Rightarrow$ (a) follows by Proposition 3.1 and Theorem 3.3. The proof is completed. □

REMARK 3.6. Let Y be a completely metrizable space, and let ρ be a metric on Y compatible with the topology of Y. Then, there exists another compatible metric d on Y such that d is complete and $\rho \le d$. If $\varphi : X \to \mathscr{F}(Y)$ is a d-continuous mapping, then it will be also ρ-continuous because $B^d_\varepsilon(A) \subset B^\rho_\varepsilon(A)$ for every $A \subset Y$. In particular, Theorems 1.3 and 3.3, also Corollary 3.5, will remain valid provided that Y is only assumed to be completely metrizable and ρ is a compatible metric on it.

4. Some possible applications.

Given a metric space (Y,ρ), let us recall some natural ways of introducing a topology on $\mathscr{F}(Y)$. Using the metric ρ on Y, $\mathscr{F}(Y)$ can be equipped with the topology $\tau_{H(\rho)}$ generated by the *Hausdorff distance*

$$H(\rho)(S,T) = \sup\{\rho(S,y) + \rho(y,T) : y \in S \cup T\}, \quad S,T \in \mathscr{F}(Y).$$

Using only the topology of Y, $\mathscr{F}(Y)$ can be equipped with the *Vietoris topology* τ_V generated by all collections of the form

$$\langle \mathscr{V} \rangle = \Big\{S \in \mathscr{F}(Y) : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathscr{V}\Big\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of Y. It is well known that these two topologies coincide on the subfamily $\mathscr{C}(Y)$ of all nonempty compact subsets of Y, but, in general, $\tau_{H(\rho)}$ and τ_V are not comparable.

Suppose that X is a space, and $\varphi : X \to \mathscr{F}(Y)$ is ρ-continuous. Then, φ is continuous as a single-valued map from X to the space $(\mathscr{F}(Y), \tau_{H(\rho)})$. If X is Lindelöf and $\varphi : X \to \mathscr{L}(Y)$, then $\varphi(X) = \{\varphi(x) : x \in X\} \subset \mathscr{L}(Y)$ is also Lindelöf, hence it is separable because $(\mathscr{L}(Y), \tau_{H(\rho)})$ is metrizable. So, there exists a countable subset $A \subset X$ such that $\{\varphi(a) : a \in A\}$ is dense in $\varphi(X)$. Since each $\varphi(a)$, $a \in A$, is also separable (as a subset of Y), it implies that $\varphi[A]$ is itself separable. Since it is dense in $\varphi[X] = \bigcup\{\varphi(x) : x \in X\}$, so is $\varphi[X]$. Thus, we have the following simple proposition.

Proposition 4.1. *Let X be a Lindelöf space, (Y, ρ) be a metric space, and let $\varphi : X \to \mathscr{L}(Y)$ be ρ-continuous. Then, $\overline{\varphi[X]} \in \mathscr{L}(Y)$.*

This implies the following characterisation of Lindelöf spaces.

Corollary 4.2. *A space X is Lindelöf if and only if for every completely metrizable space Y and every l.s.c. $\Phi : X \to \mathscr{F}(Y)$, there exists a separable subset $B \subset Y$ such that $B \cap \Phi(x) \neq \emptyset$ for every $x \in X$, i.e. $X = \Phi^{-1}[B]$.*

Proof. Suppose that X is Lindelöf, (Y, ρ) is complete metric and $\Phi : X \to \mathscr{F}(Y)$ is l.s.c. Then, X is strongly paracompact [**18**, Corollary 1] (see, also, [**7**, Corollary 5.3.11]) and, by Theorem 1.3, the mapping Φ has a ρ-continuous section $\varphi : X \to \mathscr{L}(Y)$. By Proposition 4.1, $B = \overline{\varphi[X]}$ is a separable subset of Y such that $B \cap \Phi(x) \supset \varphi(x) \cap \Phi(x) \neq \emptyset$ for every $x \in X$. Conversely, take an open cover $\mathscr{U}$ of X, endow it with the discrete topology, and define an l.s.c. $\Phi : X \to \mathscr{F}(\mathscr{U})$ by $\Phi(x) = \{U \in \mathscr{U} : x \in U\}$, $x \in X$. If $\mathscr{V} \subset \mathscr{U}$ is a separable subset (hence, countable) such that $X = \Phi^{-1}[\mathscr{V}]$, then $\mathscr{V}$ is a countable subcover of X. The proof is completed. $\square$

Corollary 4.2 can be compared with [**19**, Proposition 3.4] that a space X is Lindelöf if and only if for every completely metrizable space Y and every l.s.c. $\Phi : X \to \mathscr{F}(Y)$, there exists a pair of mappings $\langle \varphi, \psi \rangle : X \to \mathscr{C}(Y)$ such that φ is l.s.c., ψ is u.s.c., $\varphi(x) \subset \psi(x) \subset \Phi(x)$ for all $x \in X$, and $\psi[X]$ is separable. It also implies a part of [**5**, Theorem 10] that a space X is Lindelöf if and only if for every completely metrizable space Y, every l.s.c. $\Phi : X \to \mathscr{F}(Y)$ has a selection $g : X \to Y$ such that $g(X)$ is separable.

Our next application provides a simple proof of a result of Smirnov [**18**, Theorem 5] (see, also, [**14**, Theorem 10.3]).

Corollary 4.3. *For every strongly paracompact metrizable space X there exists a non-Archimedean metrizable space Z and a continuous map $g : X \to Z$ such that each $g^{-1}(z)$, $z \in Z$, is separable.*

Proof. Let ρ be a metric on X compatible with the topology of X, and let (Y, ρ) be the completion of (X, ρ). Consider the l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ defined by $\Phi(x) = \{x\}$, $x \in X$. By Corollary 3.5, there exists a non-Archimedean metrizable space Z, a continuous map $g : X \to Z$ and a ρ-continuous mapping $\psi : Z \to \mathscr{L}(Y)$ such that

$\varphi = \psi \circ g : Z \to \mathscr{L}(Y)$ is a section for Φ. Then, g is as required. To this end, it suffices to show that $g^{-1}(z) \subset \psi(z)$, $z \in Z$. Indeed, take a point $z \in Z$ and $x \in g^{-1}(z)$. Then, $\emptyset \neq \varphi(x) \cap \Phi(x) = \psi(g(x)) \cap \Phi(x) = \psi(z) \cap \Phi(x)$, and we have $x \in \psi(z)$ because $\Phi(x) = \{x\}$. Thus, each $g^{-1}(z)$, $z \in Z$, is separable, and the proof is completed. $\square$

Exactly the same proof but now using Corollary 3.4 instead of Corollary 3.5, gives the following

COROLLARY 4.4. *For every super-paracompact metrizable space X there exists a non-Archimedean metrizable space Z and a continuous map $g : X \to Z$ such that each $g^{-1}(z)$, $z \in Z$, is compact.*

Recall that a mapping $\Phi : Z \to \mathscr{F}(Y)$ from a space Z to the subsets of a metric space (Y, ρ) is *ρ-u.s.c.* (respectively, *ρ-l.s.c.*) if for every $\varepsilon > 0$, every $z_0 \in Z$ has a neighbourhood U such that $\Phi(z) \subset B^{\rho}_{\varepsilon}(\Phi(z_0))$ (respectively, $\Phi(z_0) \subset B^{\rho}_{\varepsilon}(\Phi(z)))$ for every $z \in U$. A mapping $\Phi : Z \to \mathscr{F}(Y)$ is ρ-continuous iff it is both ρ-l.s.c. and ρ-u.s.c. We conclude this paper with another interesting application of Theorem 1.3.

THEOREM 4.5. *For a paracompact space X, the following are equivalent*:

(a) *X is strongly paracompact.*
(b) *For every complete metric space (Y, ρ) and ρ-u.s.c. mapping $\theta : X \to \mathscr{L}(Y)$ there exists a ρ-continuous mapping $\psi : X \to \mathscr{L}(Y)$ such that $\theta(x) \subset \psi(x)$ for all $x \in X$.*
(c) *For every complete metric space (Y, ρ) and usco mapping $\theta : X \to \mathscr{C}(Y)$ there exists a ρ-continuous mapping $\psi : X \to \mathscr{L}(Y)$ such that $\theta(x) \subset \psi(x)$ for all $x \in X$.*

PROOF. (a) $\Rightarrow$ (b). Let X be strongly paracompact, (Y, ρ) be a complete metric space, and $\theta : X \to \mathscr{L}(Y)$ be ρ-u.s.c. If d is a metric on Y obtained by bounding ρ by any constant, for instance $d(y, z) = \min\{\rho(y, z), 1\}$, $y, z \in Y$, then a mapping $\psi : X \to \mathscr{F}(Y)$ is ρ-continuous if and only if it is d-continuous. Thus, to show (b), we may assume that ρ is itself bounded. Consider the metric space $(\mathscr{L}(Y), \tau_{H(\rho)})$, and define a mapping $\Phi : X \to 2^{\mathscr{L}(Y)}$ by

$$\Phi(x) = \{K \in \mathscr{L}(Y) : \theta(x) \subset K\}, \quad x \in X.$$

Then, $\Phi : X \to \mathscr{F}(\mathscr{L}(Y))$ is l.s.c. with respect to $\tau_{H(\rho)}$, see the proof of [**8**, Theorem 3.2]. Indeed, take $x_0 \in X$, $K_0 \in \Phi(x_0)$ and $\varepsilon > 0$. Since $\theta(x_0) \subset K_0$, it follows that $U_0 = \theta^{\#}\left[B^{\rho}_{\varepsilon}(K_0)\right]$ is a neighbourhood of x_0. If $x \in U_0$, then $K = \theta(x) \cup K_0 \in \Phi(x)$ and $H(\rho)(K, K_0) < \varepsilon$, and therefore $\Phi(x) \cap B^{H(\rho)}_{\varepsilon}(K_0) \neq \emptyset$. Thus, Φ is l.s.c.

Since $(\mathscr{L}(Y), H(\rho))$ is a complete metric space because so is (Y, ρ), by Theorem 1.3, Φ has a $H(\rho)$-continuous section $\varphi : X \to \mathscr{L}(\mathscr{L}(Y))$. Define $\psi : X \to 2^Y$ by $\psi(x) = \overline{\bigcup \varphi(x)}$, $x \in X$, and let us show that the mapping ψ is as required in this part of the proof. Take a point $x_0 \in X$. By considering the identity mapping $K \longrightarrow K$, $K \in \varphi(x_0)$, and applying Proposition 4.1, we get that $\psi(x_0) \in \mathscr{L}(Y)$. To show that ψ is ρ-continuous, let $\varepsilon > 0$. Since φ is $H(\rho)$-continuous, x_0 has a neighbourhood U such that

$$\varphi(x_0) \subset B^{H(\rho)}_{\varepsilon/2}(\varphi(x)) \text{ and } \varphi(x) \subset B^{H(\rho)}_{\varepsilon/2}(\varphi(x_0)), \quad x \in U.$$

According to the definition of $H(\rho)$, whenever $x \in U$ we have that

$$\bigcup \varphi(x_0) \subset \bigcup \{B^{\rho}_{\varepsilon/2}(K) : K \in \varphi(x)\} \subset B^{\rho}_{\varepsilon/2}\Big(\bigcup \varphi(x)\Big),$$

and therefore $\psi(x_0) \subset B^{\rho}_{\varepsilon}(\psi(x))$. Similarly, $\psi(x) \subset B^{\rho}_{\varepsilon}(\psi(x_0))$, $x \in U$. Thus, ψ is ρ-continuous. Finally, $\theta(x_0) \subset \psi(x_0)$ because $\theta(x_0) \subset K$ for every $K \in \varphi(x_0)$.

Since (b) $\Rightarrow$ (c) is obvious, we complete the proof by showing that (c) $\Rightarrow$ (a). Suppose that for every complete metric space (Y, ρ) and every usco mapping $\theta : X \to \mathscr{C}(Y)$, there exists a ρ-continuous mapping $\psi : X \to \mathscr{L}(Y)$ such that $\theta(x) \subset \psi(x)$ for all $x \in X$. Let (Y, ρ) be a complete metric space and $\Phi : X \to \mathscr{F}(Y)$ be an l.s.c. mapping. Since X is paracompact, by [**12**, Theorem 1.1], Φ has an usco multi-selection $\theta : X \to \mathscr{C}(Y)$. By hypothesis, there exists a ρ-continuous mapping $\psi : X \to \mathscr{L}(Y)$ such that $\theta(x) \subset \psi(x)$ for all $x \in X$. Since θ is a multi-selection for Φ, we get that ψ is a section for Φ, and, by Theorem 1.3, X must be strongly paracompact. The proof is completed. $\square$

Just like before, following the pattern of the proof of Theorem 4.5, we get also the following characterisation of super-paracompact spaces.

THEOREM 4.6. *A paracompact space X is super-paracompact if and only if for every completely metrizable space Y and usco mapping $\theta : X \to \mathscr{C}(Y)$ there exists a continuous mapping $\psi : X \to \mathscr{C}(Y)$ such that $\theta(x) \subset \psi(x)$ for all $x \in X$.*

ACKNOWLEDGMENTS. The author would like to express his best gratitude to Professor David Buhagiar who made several valuable comments concerning the proof of Theorem 2.2.

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