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Chow rings of nonabelian p-groups of order p^3

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Abstract. Let G be a nonabelian p group of order p^3 (i.e., extraspecial p-group), and BG its classifying space. Then $CH^*(BG) \cong H^{2*}(BG)$ where $CH^*(-)$ is the Chow ring over the field $k = \boldsymbol{C}$. We also compute mod(2) motivic cohomology and motivic cobordism of BQ_8 and BD_8.

1. Introduction.

For a smooth algebraic variety over $k = \boldsymbol{C}$, let $CH^*(X)$ be the Chow ring (over $\boldsymbol{C}$) and $BP^*(X)$ the Brown-Peterson theory. Then Totaro [**To1**] defined the modified cycle map

$$\tilde{cl} : CH^*(X)_{(p)} \to BP^{2*}(X) \otimes_{BP^*} \boldsymbol{Z}_{(p)}$$

such that the composition with the Thom map $\rho : BP^*(X) \to H^*(X)_{(p)}$, is the usual cycle map.

Let G be an algebraic group over $\boldsymbol{C}$ and BG the classifying space. Totaro conjectured that the map $\tilde{cl}$ is an isomorphism for $X = BG$. This conjecture is correct for connected groups $O(n), SO(n), G_2, Spin_7, Spin_8, PGL_p$ ([**To2**], [**Mo-Vi**], [**In-Ya**], [**Gu1**], [**Mo**], [**Ka-Ya**], [**Vi**]), and finite abelian groups [**To1**].

We will show it holds for each nonabelian p-group of order p^3.

Theorem 1.1. *If G is an extraspecial p-group of order p^3 (i.e., p_+^{1+2} or p_-^{1+2} for an odd prime, and Q_8 or D_8 for $p = 2$). Then*

$$CH^*(BG)_{(p)} \cong BP^{2*}(BG) \otimes_{BP^*} \boldsymbol{Z}_{(p)} \cong H^{2*}(BG)_{(p)}.$$

Its proof is given in Section 3 for $G = p_+^{1+2}$ and in Section 4 for other cases.

This is the first example for nonabelian p-group ($p > 2$) which satisfies Totaro's conjecture. Note that the cycle map $cl : CH^*(BG) \to H^{2*}(BG)$ is not

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surjective for $G = (\boldsymbol{Z}/p)^3$, and not injective for the central product $D_8 \cdot D_8 \times \boldsymbol{Z}/2$ (see [**To1**]).

It is known [**Te-Ya**], that for each of the above groups, the Brown-Peterson cohomology is given

$$BP^*(BG) \cong BP^*[[y_1, y_2, c_1, \dots, c_p]]/(\text{relations})$$

where y_1, y_2 are the first Chern classes of linear representations of G, and c_i is the i-th Chern class of some p-dimensional representation of G. Moreover we know

$$BP^{2*}(BG) \otimes_{BP^*} \boldsymbol{Z}_{(p)} \cong H^{2*}(BG)_{(p)}.$$

It is shown in [**Ya1**] that if $CH^*(BG)$ is generated as a ring by $y_1, y_2, c_1, \dots, c_p$, then Totaro's conjecture holds. In this paper, we will prove this fact and hence Totaro's conjecture for the above extraspecial p-groups.

Let $MU^*(X)$ be the complex cobordism theory so that $MU^*(X)_{(p)} \cong MU^*_{(p)} \otimes_{BP^*} BP^*(X)$. Let $MGL^{*,*'}(X)$ and $MGL^{*,*'}(X; \boldsymbol{Z}/p)$ be the motivic cobordism defined by Voevodsky [**Vo1**] and its mod(p) theory [**Ya3**].

From the above theorem and Proposition 9.4 in [**Ya3**], we have,

Corollary 1.2. *For an extraspecial p-group G of order p^3, we have the isomorphism $MGL^{2*,*}(BG)_{(p)} \cong MU^{2*}(BG)_{(p)}$.*

When $p = 2$, we get the rather strong results. Let $H^{*,*'}(X; \boldsymbol{Z}/2)$ be the mod(2) motivic cohomology and $0 \neq \tau \in H^{0,1}(\mathrm{Spec}(\boldsymbol{C}); \boldsymbol{Z}/2)$. Then we prove;

Theorem 1.3. *Let $G = Q_8$ or D_8. Then there is a filtration of $H^*(BG; \boldsymbol{Z}/2)$ such that*

$$H^{*,*'}(BG; \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2[\tau] \otimes \mathrm{gr}^{*'} H^*(BG; \boldsymbol{Z}/2).$$

This theorem comes back as Theorem 6.1, 6.3. Using this theorem, we prove;

Theorem 1.4. *Let $G = Q_8$ or D_8. Then we have the isomorphism*

$$MGL^{*,*'}(BG; \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2[\tau] \otimes MU^{2*}(BG).$$

This theorem comes back as Theorem 7.1 in the last section.

2. Extraspecial p-groups.

Throughout this paper, let G be a non abelian p-group of order p^3. Then the group is called an extraspecial p-group so that there is the central extension

$$0 \to C \to G \xrightarrow{q} V \to 0$$

where $C \cong \boldsymbol{Z}/p$ is the center and $V \cong \boldsymbol{Z}/p \oplus \boldsymbol{Z}/p$. We can take $a, b, c \in G$ such that $[a, b] = c$ here c generates C and the q-images of a, b generate V. (See [**Le**], [**Ly**], [**Gr-Ly**], [**Te-Ya**] for details.)

These groups have two types for each prime p. For an odd prime p, they are written as p_-^{1+2}, p_+^{1+2} where $a^p = c$ for the first type but $a^p = b^p = 1$ for the other type. When $p = 2$, the groups are the quaternion group Q_8 and the dihedral group D_8, where $a^2 = b^2 = c$ for Q_8 but $a^2 = c$, $b^2 = 1$ for D_8.

Define the linear representation a^* by $a^* : G \xrightarrow{q} V \xrightarrow{\bar{a}} \boldsymbol{C}^*$ where $\bar{a}$ is the dual of $q(a)$, i.e., $\bar{a}(q(a)) = \zeta$ and $\bar{a}(q(b)) = 1$ for a primitive p-th root ζ of unity. Similarly we define $b^* : G \to V \to \boldsymbol{C}^*$. Let $c^* : \langle c, a \rangle \to \boldsymbol{C}^*$ (resp. $a' : \langle a \rangle \to \boldsymbol{C}^*$) be the linear representation which is the dual of c (resp. a) for the case $G = p_+^{1+2}$ (resp. other cases). Define the representation $\tilde{c}$ of G by

$$\tilde{c} = \begin{cases} \mathrm{Ind}_{\langle a,c \rangle}^G (c^*) & \text{for } G = p_+^{1+2} \\ \mathrm{Ind}_{\langle a \rangle}^G (a') & \text{otherwise.} \end{cases} \tag{2.1}$$

For example when $G = p_+^{1+2}$, we can take as

$$\tilde{c}(c) = \mathrm{diag}(\zeta, \dots, \zeta), \quad \tilde{c}(a) = \mathrm{diag}(1, \zeta, \dots, \zeta^{p-1}) \tag{2.2}$$

are diagonal matrices, and

$$\tilde{c}(b) = \begin{pmatrix} 0 & 0 & \dots & \cdot & 1 \\ 1 & 0 & \dots & \cdot & 0 \\ 0 & 1 & \dots & \cdot & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{2.3}$$

is the permutation matrix in $GL_p(\boldsymbol{C})$.

Here we recall the definition of classifying space. Let V_n be a G-vector space such that G acts freely on $U_n = V_n - S_n$ for some closed set S_n with $\mathrm{codim}_{V_n} S_n > n$. Then the classifying space is defined as $BG = \mathrm{colim}_{n \to \infty} U_n/G$ and for G-space

X, the Borel cohomology (equivariant Chow ring) is defined

$$CH_G^*(X) = CH^*(U_n \times_G X) \quad \text{for } * < n,$$

which does not depend on the choice of U_n (when $* < n$) [**To1**], [**To2**], [**Vo3**].

For an integer $N \geq 1$, representations $N\tilde{c}$, Na^* and Nb^* give the G-action on

$$U_N = \boldsymbol{C}^{pN*} \times \boldsymbol{C}^{N*} \times \boldsymbol{C}^{N*},$$

where $\boldsymbol{C}^{pN*} = \boldsymbol{C}^{pN} - \{0\}$ and $\boldsymbol{C}^{N*} = \boldsymbol{C}^N - \{0\}$. Namely, given $g \in G$ and $(x, y, z) \in U_N$, we define the G-action by

$$g(x, y, z) = (N\tilde{c}(g)x, Na^*(g)y, Nb^*(g)z).$$

Here G acts freely on $U_N = \boldsymbol{C}^{N(p+2)} - H_N$ with $\mathrm{codim}(H_N) \geq N$. Hence given G-variety X, the Borel cohomology (equivariant Chow ring) can be defined by

$$CH_G^*(X) = CH^*(U_N \times_G X) \quad \text{when } * < N.$$

Of course $CH_G^*(pt.) = CH_G^* \cong CH^*(BG)$ the Chow ring of the classifying space BG.

Let us write by $y_1, y_2 \in CH^*(BG)$ the first Chern classes of a^* and b^* respectively. Let c_i be the i-th Chern class of $\tilde{c}$. We consider $CH_G^*(U_N)$ when $N = 1$. We use the stratified methods by Molina-Vistoli [**Mo-Vi**] which was used to compute the Chow rings of BG for classical groups G.

Lemma 2.1.

$$CH_G^*(\boldsymbol{C}^{p*} \times \boldsymbol{C}^* \times \boldsymbol{C}^*) \cong CH^*(BG)/(y_1, y_2, c_p).$$

Proof. We first consider the localized exact sequence ([**To1**], [**To2**])

$$CH_G^*(\{0\} \times \boldsymbol{C} \times \boldsymbol{C}) \xrightarrow{i_*} CH_G^{*+p}(\boldsymbol{C}^p \times \boldsymbol{C} \times \boldsymbol{C}) \to CH_G^{*+p}(\boldsymbol{C}^{p*} \times \boldsymbol{C} \times \boldsymbol{C}) \to 0.$$

Here i_* is the multiplying c_p. So we have

$$CH_G^*(\boldsymbol{C}^{p*} \times \boldsymbol{C} \times \boldsymbol{C}) \cong CH_G^*/(c_p).$$

Next consider

$$CH_G^*(\boldsymbol{C}^{p*} \times \{0\} \times \boldsymbol{C}) \xrightarrow{i_*} CH_G^{*+p}(\boldsymbol{C}^{p*} \times \boldsymbol{C} \times \boldsymbol{C}) \to CH_G^{*+p}(\boldsymbol{C}^{p*} \times \boldsymbol{C}^* \times \boldsymbol{C}) \to 0.$$

Since $c_1(a^*) = y_1$ and $i_* = y_1$, we see

$$CH_G^*(\boldsymbol{C}^{p*} \times \boldsymbol{C}^* \times \boldsymbol{C}) \cong CH_G^*/(c_p, y_1).$$

Similarly, using $c_1(b^*) = y_2$, we have the lemma. □

Corollary 2.2. *The Chow ring $CH^*(BG)$ is generated as a ring by elements of degree $\leq p+2$.*

Proof. First note that the G-action on $\boldsymbol{C}^{p*} \times \boldsymbol{C}^* \times \boldsymbol{C}^*$ is free. Hence

$$CH_G^*(\boldsymbol{C}^{p*} \times \boldsymbol{C}^* \times \boldsymbol{C}^*) \cong CH^*((\boldsymbol{C}^{p*} \times \boldsymbol{C}^* \times \boldsymbol{C}^*)/G).$$

Since $(\boldsymbol{C}^{p*} \times \boldsymbol{C}^* \times \boldsymbol{C}^*)/G$ is a smooth variety of (complex) dimension $p+2$, we see $CH_G^*/(y_1, y_2, c_p)$ is generated by elements of degree $\leq p+2$. □

Recall that the Brown-Peterson theory also has Chern classes. It is known [**Te-Ya**], that for each of the above groups, the Brown-Peterson cohomology is given

$$BP^*(BG) \cong BP^*[[y_1, y_2, c_1, \dots, c_p]]/(\text{relations}).$$

Moreover we know $BP^{2*}(BG) \otimes_{BP^*} \boldsymbol{Z}_{(p)} \cong H^{2*}(BG)$. Hence $H^{2*}(BG)$ is generated as a ring by Chern classes of degree $\leq 2p$.

Corollary 2.3. *If the cycle map $cl : CH^*(BG) \to H^{2*}(BG)$ is injective for $* \leq 2p-2$ (for $* \leq p+2$ when $p \leq 3$), then $CH^*(BG) \cong H^{2*}(BG)$ for all $* \geq 0$.*

Proof. Since $H^{2*}(BG)$ is generated as a ring by y_1, y_2, c_i, we see from Corollary 2.2 that $CH^*(BG)$ is generated by the same elements y_1, y_2, c_i. It is known that all relations between the above ring generators are in cohomological degree $\leq 4p-4$ (for the explicit relations of the ordinary cohomology, see Theorem 2.4–2.7 below). Hence we get the corollary. □

Of course the usual cohomology of BG is explicitly known as follows.

Theorem 2.4 (Lewis [**Le**], see also [**Ly**], [**Te-Ya**]).

$$\begin{aligned} H^{even}(Bp_+^{1+2}) \cong (&\boldsymbol{Z}[y_1, y_2]/(y_1 y_2^p - y_1^p y_2, p y_i) \\ &\oplus \boldsymbol{Z}/p\{c_2, \dots, c_{p-1}\}) \otimes \boldsymbol{Z}[c_p]/(p^2 c_p), \end{aligned}$$

$$H^{odd}\big(Bp_+^{1+2}\big) \cong H^{even}(Bp_+^{1+2})/(p)\{e\} \quad |e| = 3.$$

Here $c_i y_j = c_i c_k = 0$ *for* $i < p-1$, *but* $y_j c_{p-1} = y_j^p$, $c_{p-1}^2 = y_1^{p-1} y_2^{p-1}$.

In fact, the degree of relations in the above cohomology are given

$$\big|y_1 y_2^p - y_1^p y_2\big| = 2p+2,\ |py_i| = 2,\ \ldots,\ \big|c_{p-1}^2 - y_1^{p-1} y_2^{p-1}\big| = 4p-4.$$

They are all deg $\le 4p-4$. Similar facts happen for cohomology of other types.

Theorem 2.5 (Lewis [**Le**], [**Ly**]).

$$H^{even}\big(Bp_-^{1+2}\big) \cong (\boldsymbol{Z}[y_2]/(py_2) \oplus \boldsymbol{Z}/p\{y_1 = c_1, c_2, \ldots, c_{p-1}\}) \otimes \boldsymbol{Z}[c_p]/(p^2 c_p),$$
$$H^{odd}\big(Bp_-^{1+2}\big) \cong \boldsymbol{Z}/p[y_2, c_p]\{e\} \quad \textit{with } |e| = 2p+1.$$

Here $c_i y_j = c_i c_k = 0$ *for* $i \le p-1$.

Theorem 2.6 (Evens [**Ev**]).

$$H^{even}(BD_8) \cong \boldsymbol{Z}[y_1, y_2, c_2]/(y_1 y_2, 2y_i, 4c_2),$$
$$H^{odd}(BD_8) \cong H^{even}(BD_8)/(2)\{e\} \quad \textit{with } |e| = 3.$$

Theorem 2.7 (Atiyah [**At**]).

$$H^{even}(BQ_8) \cong \boldsymbol{Z}[y_1, y_2, c_2]/(y_i^2, 2y_i, 4c_2 = y_1 y_2),$$
$$H^{odd}(BQ_8) \cong 0.$$

The following lemma is used in the proof of Lemma 3.3 in Section 3.

Lemma 2.8. *If* $H^{2*}(X)_{(p)}$ *is generated as a ring by Chern classes for all* $* \le p$, *then we have the isomorphisms for* $* < p$,

$$CH^*(X)_{(p)} \cong BP^{2*}(X) \otimes_{BP^*} \boldsymbol{Z}_{(p)} \cong H^{2*}(X)_{(p)}.$$

Moreover, if $H^1(X)_{(p)} = 0$ *or* $pH^{2p}(X)_{(p)} = 0$, *then the isomorphisms hold also for* $* = p$.

Proof. Recall that the usual K-theory $K^*(X)_{(p)}$ localized at p can be decomposed to the integral Morava K-theory $\tilde{K}(1)^*(X)$ with the coefficient ring

$\tilde{K}(1) = \boldsymbol{Z}_{(p)}[v_1, v_1^{-1}]$, $|v_1| = -2p+2$. We consider the Atiyah-Hirzebruch spectral sequence ([**Te-Ya**], [**Ya3**])

$$E(K)_2^{*,*'} \cong H^*(X) \otimes \tilde{K}(1)^{*'} \Longrightarrow \tilde{K}(1)^*(X).$$

The first nonzero differential is known

$$d_{2p-1}(x) = v_1 \otimes \beta P^1(x) \quad (= v_1 \otimes Q_1(x) \bmod(p)).$$

Since $H^{2*}(X)_{(p)}$ is generated by Chern classes, each element is a permanent cycle because $|\beta P^1| = 2p-1$. In fact

$$E(K)_\infty^{2*,*'} \cong H^{2*}(X) \otimes \tilde{K}(1)^{*'} \quad \text{for } * < p.$$

This implies from the definition of $\mathrm{gr}_{geo}^i K^0(X)$ ([**Th**], [**To2**])

$$(1) \quad \mathrm{gr}_{geo}^i K^0(X)_{(p)} \cong H^{2i}(X)_{(p)} \quad \text{for } i < p.$$

Next consider the Atiyah-Hirzebruch spectral sequence for $BP^*(X)$

$$E(BP)_2^{*,*'} \cong H^*(X) \otimes BP^{*'} \Longrightarrow BP^*(X).$$

Similarly we have $E(BP)_\infty^{2*,*'} \cong BP^{*'} \otimes H^{2*}(X)$ for $* < p$. (The differential d_{2p-1} is the same as the case $\tilde{K}(1)^*(-)$.) Hence we have

$$(2) \quad (BP^*(X) \otimes_{BP^*} \boldsymbol{Z}_{(p)})^{2i} \cong H^{2i}(X)_{(p)}.$$

On the other hand, there is the natural map

$$CH^i(X) \to \mathrm{gr}_{geo}^i K^0(X) \xrightarrow{c_i} CH^i(X),$$

which is the multiplication by $(-1)^{i-1}(i-1)!$ by Riemann-Roch with denominators. (See the proof of Corollary 3.2 in [**To2**].) Moreover the first map is epic. Hence $CH^i(X)_{(p)} \cong \mathrm{gr}_{geo}^i K^0(X)_{(p)}$ for $i \le p$. Thus we have the desired result from (1) and (2).

Next suppose that $H^1(X)_{(p)} = 0$ or $pH^{2p}(X)_{(p)} = 0$. Then each nonzero element in $H^{2p}(X) \otimes \tilde{K}(1)^*$ is not the target of the differential d_{2p-1} in the spectral sequence $E(K)_r^{*,*'}$. Indeed, $P^1H^1(X) = 0 \bmod(p)$ and

$$E(K)_\infty^{2*,*'} \cong H^{2*}(X) \otimes \tilde{K}(1)^{*'} \quad \text{for } * \leq p.$$

Hence all isomorphism above hold also for $* = p$. □

COROLLARY 2.9 (Lemma 6.1 in [**Ya1**]). *We have the isomorphism*

$$CH^*(BG)_{(p)} \cong H^{2*}(BG)_{(p)} \quad \textit{for } * \leq p.$$

3. The group $E = p_+^{1+2}$.

Throughout this section, we assume $p \geq 3$ and $G = E = p_+^{1+2}$. Recall that E is generated by a, b, c such that $[a, b] = c,\ a^p = b^p = c^p = 1$. Recall also the p-dimensional representation $\tilde{c} = \mathrm{Ind}_{\langle a,c \rangle}^G (c^*)$ so that

$$\tilde{c}(c) = \mathrm{diag}(\zeta, \dots, \zeta), \quad \tilde{c}(a) = \mathrm{diag}(1, \zeta, \dots, \zeta^{p-1}),$$

and $\tilde{c}(b)$ is the permutation matrix (2.3) in Section 2.

The group E does not act freely on $\boldsymbol{C}^{p*}$. We consider fixed points for small subgroups. Let $W = \boldsymbol{C}^{p*}$. Since $\tilde{c}(a) = \mathrm{diag}(1, \zeta, \dots, \zeta^{p-1})$, the fixed points of the subgroup $\langle a \rangle$ is given by

$$W^{\langle a \rangle} = \{(x, 0, \dots, 0) \mid x \in \boldsymbol{C}^*\} = \boldsymbol{C}^*\{e\} \quad e = (1, 0, \dots, 0).$$

Since $b^{-i}ab^i = ac^i$ in E, we see

$$ac^ib^{-i}e = b^{-i}ab^ib^{-i}e = b^{-i}ae = b^{-i}e.$$

This means $W^{\langle ac^i \rangle} = \boldsymbol{C}^*\{b^{-i}e\}$. Let us write

$$H_0 = \boldsymbol{C}^*\{e, be, \dots, b^{p-1}e\} = \boldsymbol{C}^*\{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}.$$

(It is the disjoint union of p-th (complex) lines in $\boldsymbol{C}^{p*}$ generated by $(0, \dots, 1, \dots, 0)$.) Then the group E acts on H_0, namely, H_0 is a smooth E-variety.

In $GL_p(\boldsymbol{C})$, the elements $\tilde{c}(ab^i)$, $\tilde{c}(b)$ have the trace zero and are p-th roots of the identity. Hence there is a $g_j \in GL_p(\boldsymbol{C})$ for $0 \leq j \leq p$ such that $g_j^{-1}ag_j = ab^j$ for $j < p$ and $g_p^{-1}ag_p = b$. Then we see $ab^jg_j^{-1}e = g_j^{-1}e$ as above arguments, and so $\boldsymbol{C}^*\{g_j^{-1}e\} = W^{\langle ab^j \rangle}$. Hence we can define E-equivariant set $H_j = g_j^{-1}H_0$. Here note $H_j \cap H_{j'} = \emptyset$ for $j \neq j'$, in fact the stabilizer group of each point in H_j is $\langle ab^jc^i \rangle$ and they are not equal for $j \neq j'$. Let us write the disjoint union

$$H = H_0 \coprod H_1 \coprod \cdots \coprod H_p.$$

(It is a disjoint union of $p(p+1)$ (complex) lines in $\boldsymbol{C}^{p*}$.)

LEMMA 3.1. *The group E acts freely on $(\boldsymbol{C}^{p*} - H)$.*

PROOF. The stabilizer of any points, if it were nontrivial, would contain a subgroup of E isomorphic to $\boldsymbol{Z}/p$. All subgroups of E isomorphic to $\boldsymbol{Z}/p$ are written as $\langle ab^j c^i \rangle$, $\langle bc^i \rangle$ or $\langle c \rangle$. But c is not a stabilizer of any element in $\boldsymbol{C}^*$. Hence all points which have nontrivial stabilizer groups are contained in H. Thus we have the lemma. □

Let $i : H \subset \boldsymbol{C}^{p*}$. Let us write $i^*(y_i) \in H_E^*(H)$ by the same letter y_i.

LEMMA 3.2. *We have the isomorphism $H_E^*(H_i) \cong H_E^*(H_0)$ and*

$$H_E^*(H_0; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p[y_1] \otimes \Lambda(x_1, z), \quad \textit{with } |x_1| = |z| = 1,$$
$$H_E^*(H_0) \cong \boldsymbol{Z}[y_1]/(py_1)\{1, z\}.$$

PROOF. We consider the group extension

$$0 \to \langle b, c \rangle \to E \to \langle a \rangle \to 0$$

and the induced Hochschild-Serre spectral sequence

$$E_2^{*,*} \cong H^*\big(B\langle a \rangle; H^*_{\langle b,c \rangle}(H_0; \boldsymbol{Z}/p)\big) \Longrightarrow H_E^*(H_0; \boldsymbol{Z}/p).$$

Here we have

$$H^*_{\langle b,c \rangle}(H_0; \boldsymbol{Z}/p) \cong H^*_{\langle b,c \rangle}(\langle b \rangle \times \boldsymbol{C}^*; \boldsymbol{Z}/p) \cong H^*_{\langle c \rangle}(\boldsymbol{C}^*; \boldsymbol{Z}/p) \cong \Lambda(z).$$

Of course $\langle a \rangle \cong \boldsymbol{Z}/p$ acts on $\Lambda(z)$ trivially. Hence the $E_2^{*,*}$ is isomorphic to

$$H^*(B\langle a \rangle; \Lambda(z)) \cong \boldsymbol{Z}/p[y_1] \otimes \Lambda(x_1) \otimes \Lambda(z) \cong \boldsymbol{Z}/p[y_1]\{1, x_1, z, x_1 z\}.$$

In particular, we note

$$(1) \quad \dim(H^*(B\langle a \rangle; \Lambda(z))) = 2 \quad \text{for each } * > 0.$$

We will see that $d_2(z) = 0$ and this spectral sequence collapses from the dimensional reason.

Consider the localized exact sequence for the cohomology

$$H_E^{*+2p-1}(\boldsymbol{C}^{p*}-H)\to H_E^{*+2}(H)\to H_E^{*+2p}(\boldsymbol{C}^{p*})\to H_E^{*+2p}(\boldsymbol{C}^{p*}-H)\to\cdots$$

Since E acts on $\boldsymbol{C}^{p*}-H$ freely, we see

$$H_E^{*+2p}(\boldsymbol{C}^{p*}-H)\cong H^{*+2p}((\boldsymbol{C}^{p*}-H)/E),$$

which is zero if $*>0$ since $(C^{p*}-H)/E$ is a $2p$-dimensional (p-dimensional complex) manifold. Thus for $*>0$, we have the isomorphism

$$(2)\quad H_E^{*+2}(H)\cong H_E^{*+2p}(\boldsymbol{C}^{p*}).$$

On the other hand, we recall from Theorem 2.4

$$H^{even}(BE)\cong\big(\boldsymbol{Z}[y_1,y_2]/(y_1y_2^p-y_1^py_2,py_i)\oplus\boldsymbol{Z}/p\{c_2,\dots,c_{p-1}\}\big)\otimes\boldsymbol{Z}[c_p]/(p^2c_p),$$
$$H^{odd}(BE)\cong H^{even}(BG)/(p)\{e\}\quad |e|=3.$$

We consider the long exact sequence

$$\to H_E^*(\{0\})\xrightarrow{i_{H*}=\times c_p}H_E^{*+2p}(\boldsymbol{C}^p)\to H_E^{*+2p}(\boldsymbol{C}^{p*})\to\cdots$$

However, this sequence becomes a short exact sequence because $\times c_p|H^*(BE)$ is an injection for $*>0$ from the above isomorphisms. Hence

$$(3)\quad H_E^*(\boldsymbol{C}^{p*})\cong H^*(BE)/(c_p)\quad\text{for } *>0.$$

In particular, we have for $*>0$

$$H_E^{2*+2p}(\boldsymbol{C}^{p*})\cong H^{2*+2p}(BE)/(c_p)\cong\big(\boldsymbol{Z}/p[y_1,y_2]/(y_1^py_2-y_1y_2^p)\big)^{2*+2p}$$
$$\cong\boldsymbol{Z}/p\{y_1^{*+p},y_1^{*+p-1}y_2,\dots,y_1^{*+1}y_2^{p-1},y_2^{*+p}\}$$

and $H_E^{2*+2p+3}(\boldsymbol{C}^{p*})\cong H_E^{2*+2p}(\boldsymbol{C}^{p*})\{e\}$. Hence from (2), we have for $*'\le p$

$$\dim H_E^{2*'+2}(H)=\dim H_E^{2*'+3}(H)=p+1.$$

Here we recall the universal coefficient theorem such as

$$\dim H^*(X;\boldsymbol{Z}/p)=\dim(H^*(X)/p)+\dim(p\text{-torsion}(H^{*+1}(X)).$$

Since all elements in $H^{*+2p}(BE)/(c_p)$ are p-torsion for $*\geq 0$, we see

$$\dim H_E^{2*'+2}(H;\boldsymbol{Z}/p)=2\dim H_E^{2*'+2}(H)=2(p+1).$$

For each $0\leq j\leq p$, since $H_0\cong H_j$ as E-varieties, we see $H_E^*(H_j;\boldsymbol{Z}/p)\cong H_E^*(H_0;\boldsymbol{Z}/p)$. Hence $\dim H_E^*(H_0;\boldsymbol{Z}/p)=2$.

From (1), the above fact means $E_2^{*,*}\cong E_\infty^{*,*}$ (in fact if $d_2(z)\neq 0$, then $\dim H_E^*(H_0;\boldsymbol{Z}/p)<2$). Hence we get the result for $\boldsymbol{Z}/p$ coefficient.

The integral coefficient case follows from the universal coefficient theorem (as stated above), e.g., $\dim(H^*(H_0)/p)=1$ for $*>0$. Indeed, $\beta(x_1)=y_1$, and we see that y_1 is p-torsion element in $H^*(H_0)$ but $x_1\notin H^1(H_0)$, and so $z\in H^1(H_0)$. □

Lemma 3.3. *The cycle map $cl: CH_E^*(\boldsymbol{C}^{p*})\to H_E^{2*}(\boldsymbol{C}^{p*})$ is an isomorphism for $*\leq 2p-1$.*

Proof. Since $H_E^*(\boldsymbol{C}^{p*})\cong H_E^*/(c_p)$ is generated by Chern classes (and $H_E^1(\boldsymbol{C}^{p*})=0$), we see the above cycle map cl is an isomorphism for $*\leq p$ from Lemma 2.8.

Let $*>0$. Consider the diagram

$$\begin{array}{ccccc}
CH_E^{*+1}(H) & \xrightarrow{i_{CH*}} & CH_E^{*+p}(\boldsymbol{C}^{p*}) & \longrightarrow & CH_E^{*+p}(\boldsymbol{C}^{p*}-H)=0\\
\downarrow{\scriptstyle cl_1} & & \downarrow{\scriptstyle cl_2} & & \downarrow{\scriptstyle cl_3}\\
0\to H_E^{2*+2}(H) & \xrightarrow{i_{H*}} & H_E^{2*+2p}(\boldsymbol{C}^{p*}) & \longrightarrow & H_E^{2*+2p}(\boldsymbol{C}^{p*}-H)=0.
\end{array}$$

Here note that

$$H_E^*(\boldsymbol{C}^{p*}-H)=H^*((\boldsymbol{C}^{p*}-H)/E)=0\quad\text{for } *>2p$$

since $(\boldsymbol{C}^{p*}-H)/E$ is a $2p$-dimensional manifold. So $H_E^{2*+2p-1}(\boldsymbol{C}^{p*}-H)=0$ and we see i_{H*} is an isomorphism. From the preceding lemma, $H_E^{2*}(H_j)$ generated by Chern classes (e.g., y_1^* for H_0). Hence the cycle map cl_1 is isomorphic for $*\leq p-1$ from Lemma 2.8. Therefore

$$cl_2\cdot i_{CH*}=i_{H*}\cdot cl_1$$

is an isomorphism and so is cl_2 for $* \leq p-1$. □

LEMMA 3.4. *The cycle map* $cl : CH^*(BE) \to H^{2*}(BE)$ *is an isomorphism for* $* \leq 2p-1$.

PROOF. Let $0 < * < p-1$. Consider the diagram

$$\begin{array}{ccccccc} CH_E^*(\{0\}) & \xrightarrow{i_{CH^*}=\times c_p} & CH_E^{*+p}(\boldsymbol{C}^p) & \longrightarrow & CH_E^{*+p}(\boldsymbol{C}^{p*}) & \to 0 \\ \downarrow{\scriptstyle cl_1} & & \downarrow{\scriptstyle cl_2} & & \downarrow{\scriptstyle cl_3} & \\ 0 \to H_E^{2*}(\{0\}) & \xrightarrow{i_{H^*}=\times c_p} & H_E^{2*+2p}(\boldsymbol{C}^p) & \longrightarrow & H_E^{2*+2p}(\boldsymbol{C}^{p*}) & \to 0. \end{array}$$

Here the lower short exactness follows from the fact that $\times c_p|H^{2*}(BE)$ is an injection for $0 < *$ (see (3) in the proof of Lemma 3.2). The map cl_3 is an isomorphism for all $* \leq p-1$, from the preceding lemma. We still know that the map cl_1 is an isomorphism for $* \leq p$ from Lemma 2.8. Hence we see cl_2 is also an isomorphism for $* \leq p-1$. □

From Corollary 2.3, we have the isomorphism $CH^*(BE) \cong H^{2*}(BE)$ for all $* \geq 0$. Thus we prove Theorem 1.1 in the introduction when $G = p_+^{1+2}$.

4. Other groups $M = p_-^{1+2}$, D_8 and Q_8.

We consider the other groups cases in this section. Let $M = p_-^{1+2}$ for an odd prime. In this case $a^p = c$ and the representation $\tilde{c}$ is given as

$$\tilde{c}(a) = \mathrm{diag}\left(\xi, \xi^{1+p}, \xi^{1+2p}, \ldots, \xi^{1+(p-1)p}\right)$$

and $\tilde{c}(b)$ is the permutation matrix (2.3) as in the case E, where ξ is a p^2-th primitive root of the unity, i.e., $\xi^p = \zeta$. So M acts freely on $\boldsymbol{C}^{p*} \times \boldsymbol{C}^*$.

The fixed points set on $W = \boldsymbol{C}^{p*}$ of the subgroup $\langle b\rangle$ is given by

$$W^{\langle b\rangle} = \{(x, \ldots, x) \mid x \in \boldsymbol{C}^*\} = \boldsymbol{C}^*\{e'\} \quad e' = (1, \ldots, 1).$$

Since $a^{-i}ba^i = bc^i$, we see $W^{\langle bc^i\rangle} = \boldsymbol{C}^*\{a^{-i}e'\}$. So M acts on

$$H = \boldsymbol{C}^*\{e', ae', \ldots, a^{p-1}e'\}.$$

Note $(a^ibc^j)^p = c^i$ for $1 \leq i \leq p-1$ (but $(ab)^2 = 1$ for $G = D_8$). Hence for all

$x \in \boldsymbol{C}^{p*}$, $a^i bc^j(x) \neq x$. Thus we can see that M acts freely on $U - H$, i.e., Lemma 3.1 holds for $G = M$.

Next we will see Lemma 3.2 by $H = H_0$ for $G = M$. We consider the group extension

$$0 \to \langle a \rangle \to M \to \langle b \rangle \to 0$$

and induced spectral sequence

$$E_2^{*,*'} = H^*\big(\langle b \rangle; H^{*'}_{\langle a \rangle}(H; \boldsymbol{Z}/p)\big) \Longrightarrow H^*_M(H; \boldsymbol{Z}/p).$$

Since $\langle a \rangle$ acts freely on H, we see

$$H/\langle a \rangle \cong \boldsymbol{C}^*\{e', \ldots, a^{p-1}e'\}/\langle a \rangle \cong \boldsymbol{C}^*/\langle a^p \rangle.$$

Therefore we have $H_{\langle a \rangle}(H; \boldsymbol{Z}/p) \cong H^*(\boldsymbol{C}^*/\langle a^p \rangle; \boldsymbol{Z}/p) \cong \Lambda(z)$ as in the case $G = E$. From Theorem 2.5, we know

$$H_M^{2*+2p}(\boldsymbol{C}^{p*}) \cong \boldsymbol{Z}/p\big\{y_2^{*+p}\big\}.$$

This implies $\dim H_M^{2*+2p}(H) = 1$. Therefore the spectral sequence collapses. Lemma 3.3 holds for $G = M$ and we see $CH^*(BM) \cong H^{2*}(BM)$.

Next, we consider the case $G = D_8$ and $p = 2$. Then the representation can be taken as in the case $G = M$. Take

$$H_0 = \boldsymbol{C}^*\{e', ae'\}, \quad H_1 = \boldsymbol{C}^*\{g^{-1}e', g^{-1}ae'\}$$

where $g \in GL_2(\boldsymbol{C})$ with $g^{-1}bg = ab$ (note $(ab)^2 = 1$). Let $H = H_0 \coprod H_1$. Then D_8 acts freely on $\boldsymbol{C}^{2*} - H$. In fact from Theorem 2.6, we know

$$H_{D_8}^{2*+4}(\boldsymbol{C}^{2*}) \cong \boldsymbol{Z}/2\big\{y_1^{*+2}, y_2^{*+2}\big\}.$$

Hence arguments work as in the case E or M.

At last we consider the case $G = Q_8$. The representation $\tilde{c}$ is given

$$\tilde{c}(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{c}(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can easily see that Q_8 acts freely on $\boldsymbol{C}^{2*}$. Therefore

$$CH_{Q_8}(\boldsymbol{C}^{2*}) \cong CH^*(\boldsymbol{C}^{2*}/Q_8)$$

which is generated by degree ≤ 2. In fact from Theorem 2.7

$$H^*(BD_8)/(c_2) \cong \boldsymbol{Z}[y_1, y_2]/\big(y_i^2, 2y_i, y_1y_2\big),$$

which shows $H^*(BD_8)/(c_2) = 0$ for $* \geq 3$.

5. Motivic cohomology.

We recall the motivic cohomology, in this section. Let X be a smooth (quasi projective) variety over a field $k \subset \boldsymbol{C}$. Let $H^{*,*'}(X; \boldsymbol{Z}/p)$ be the mod(p) motivic cohomology defined by Voevodsky and Suslin ([**Vo1**], [**Vo2**], [**Vo3**], [**Vo4**]). Recall that the Beilinson-Lichtenbaum conjecture holds if

$$H^{m,n}(X; \boldsymbol{Z}/p) \cong H^m_{et}\big(X; \mu_p^{\otimes n}\big) \qquad \text{for all } m \leq n.$$

Recently M. Rost and V. Voevodsky ([**Vo5**], [**Su-Jo**]) proved the Bloch-Kato conjecture. The Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture.

We assume that k contains a p-th root ζ of unity. Then there is the isomorphism $H^m_{et}(X; \mu_p^{\otimes n}) \cong H^m_{et}(X; \boldsymbol{Z}/p)$. Let τ be a generator of $H^{0,1}(\mathrm{Spec}(k); \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p \cong \mu_p$, so that ([**Vo2**], [**Vo3**], Lemma 2.4 in [**Or-Vi-Vo**])

$$\mathrm{colim}_i\, \tau^i H^{*,*'}(X; \boldsymbol{Z}/p) \cong H^*_{et}(X; \boldsymbol{Z}/p).$$

We define the weight degree $w(x) = 2n - m$ if $0 \neq x \in H^{m,n}(X; \boldsymbol{Z}/p)$. Then it is known $w(x) \geq 0$ for smooth X.

Let $H^*(X; H^{*'}_{\boldsymbol{Z}/p})$ be the cohomology of the Zariski sheaf induced from the presheaf $H^*_{et}(V; \boldsymbol{Z}/p)$ for open subsets V of X. This sheaf cohomology is isomorphic to the E_2-term

$$E_2^{*,*'} \cong H^*\big(X; H^{*'}_{\boldsymbol{Z}/p}\big) \Longrightarrow H^*_{et}(X; \boldsymbol{Z}/p)$$

of the coniveau spectral sequence by Bloch-Ogus [**Bl-Og**]. We also note

$$H^0\big(X; H^{*'}_{\boldsymbol{Z}/p}\big) \subset H^{*'}(k(X); \boldsymbol{Z}/p)$$

for the function field of X.

The relation between this cohomology and the motivic cohomology is given as follows.

Theorem 5.1 ([**Or-Vi-Vo**], [**Vo5**]). *We have the long exact sequence*

$$\to H^{m,n-1}(X;\boldsymbol{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X;\boldsymbol{Z}/p) \\ \to H^{m-n}\big(X;H^n_{\boldsymbol{Z}/p}\big) \xrightarrow{\partial} H^{m+1,n-1}(X;\boldsymbol{Z}/p) \xrightarrow{\times\tau} \cdots$$

In particular, we have

Corollary 5.2. *The cohomology $H^{m-n}(X;H^n_{\boldsymbol{Z}/p})$ is (additively) isomorphic to*

$$H^{m,n}(X;\boldsymbol{Z}/p)/(\tau) \oplus \mathrm{Ker}(\tau)|H^{m+1,n-1}(X;\boldsymbol{Z}/p)$$

where $H^{m,n}(X;\boldsymbol{Z}/p)/(\tau) = H^{m,n}(X;\boldsymbol{Z}/p)/(\tau H^{m,n-1}(X;\boldsymbol{Z}/p))$.

Corollary 5.3. *The map $\times\tau : H^{m,m-1}(X;\boldsymbol{Z}/p) \to H^{m,m}(X;\boldsymbol{Z}/p)$ is injective.*

By using above theorems, we can do some computations for concrete cases. Suppose $k=\boldsymbol{C}$. Then the realization (cycle map)

$$t_{\boldsymbol{C}} = cl : H^{*,*'}(X;\boldsymbol{Z}/p) \to H^*_{et}(X;\boldsymbol{Z}/p) \cong H^*(X;\boldsymbol{Z}/p)$$

can be identified with

$$\times\tau^{*-*'} : H^{*,*'}(X;\boldsymbol{Z}/p) \to H^{*,*}(X;\boldsymbol{Z}/p) \cong H^*_{et}(X;\boldsymbol{Z}/p),$$

from the Beilinson-Lichtenbaum conjecture.

We define the motivic filtration of $H^*(X;\boldsymbol{Z}/p)$ by

$$F_i^* = \mathrm{Im}\big(t_{\boldsymbol{C}}^{*,*(i)}\big) = t_{\boldsymbol{C}}\big(H^{*,*(i)}(X;\boldsymbol{Z}/p)\big),$$

where $*(i) = [(*+i)/2]$ so that $x \in F_i^*$ if $x = t_{\boldsymbol{C}}(x')$ for some $x' \in H^{*,*'}(X;\boldsymbol{Z}/p)$ with $w(x') \le i$. Let us write the associated graded ring $F_i^*/F_{i-1}^* = \mathrm{gr}^i H^*(X;\boldsymbol{Z}/p)$. In [**Ya2**], we define

$$h^{*,*'}(X;\boldsymbol{Z}/p) = H^{*,*'}(X;\boldsymbol{Z}/p)/\big(\mathrm{Ker}(t_{\boldsymbol{C}}^{*,*'})\big),$$

and compute them for some cases of $X = BG$. It is immediate that

$$h^{m,n}(X;\boldsymbol{Z}/p) \cong \bigoplus_{i=0} \mathrm{gr}^{2(n-i)-m} H^m(X;\boldsymbol{Z}/p)\{\tau^i\}.$$

We will simply write (for ease of notations) the above isomorphism by

$$h^{*,*'}(X;\boldsymbol{Z}/p) \cong \mathrm{gr}^{*'} H^*(X;\boldsymbol{Z}/p) \otimes \boldsymbol{Z}/p[\tau].$$

LEMMA 5.4. *Let X be a smooth variety (over $k = \boldsymbol{C}$) of $\dim(X) = 2$. Then we have the isomorphism $H^{*,*'}(X;\boldsymbol{Z}/p) \cong h^{*,*'}(X;\boldsymbol{Z}/p)$.*

PROOF. By the definition of $h^{*,*'}(X;\boldsymbol{Z}/p)$, we see

$$H^{*,*'}(X;\boldsymbol{Z}/p) \cong h^{*,*'}(X;\boldsymbol{Z}/p) \oplus \mathrm{Ker}\left(t_{\boldsymbol{C}}^{*,*'}\right).$$

We still know $\mathrm{Ker}(t_{\boldsymbol{C}}^{*,*'}) = \mathrm{Ker}(\times\tau^{*-*'})$ and we will show this is zero.

It is known ([**Vo1**], [**Vo2**]) that

$$H^{*,*'}(X;\boldsymbol{Z}/p) \cong 0 \quad \text{if } * - *' > \dim(X).$$

Hence we only need to consider $H^{*,*'}(X;\boldsymbol{Z}/p)$ for $* - *' \le 2$. If $* - *' \le 1$, then from the Beilinson-Lichtenbaum conjecture and Corollary 5.3, $H^{*,*'}(X;\boldsymbol{Z}/p)$ has no τ-torsion elements.

Hence we consider the case $*' = * - 2$. From the exact sequence in Theorem 5.1,

$$\to H^0\big(X;H^{*-1}_{\boldsymbol{Z}/p}\big) \xrightarrow{\partial} H^{*,*-2}(X;\boldsymbol{Z}/p) \xrightarrow{\times\tau} \cdots$$

we see $\mathrm{Ker}(\tau|H^{*,*-2}(X;\boldsymbol{Z}/p)) = \mathrm{Im}(\partial|H^0(X;H^{*-1}_{\boldsymbol{Z}/p}))$.

Moreover we know $H^0(X;H^{*-1}_{\boldsymbol{Z}/p}) \subset H^{*-1}(k(X);\boldsymbol{Z}/p)$ where $k(X)$ is the function field of X. It is well known from Serre (Chapter II 4.2 Proposition 11, Corollary in [**Se**]) that the Galois group G_F for a function field F in two variables over an algebraically closed field k has the cohomological dimension $\mathrm{cd}(G_F) = 2$. (By a function field in r variables over k, we mean a finitely generated extension of k of transcendence degree r.)

Since $\dim(X) = 2$, the function field $k(X)$ satisfies $\mathrm{cd}(G_{k(X)}) = 2$ for $k = \boldsymbol{C}$, that is, $H^*(k(X);\boldsymbol{Z}/p) = 0$ for $* \ge 3$. This implies

$$H^0\big(X;H^{*-1}_{\boldsymbol{Z}/p}\big) \subset H^{*-1}(\boldsymbol{C}(X);\boldsymbol{Z}/p) = 0 \quad \text{for } * \ge 4.$$

Hence $\mathrm{Ker}(\tau|H^{*,*-2}(X;\boldsymbol{Z}/p)) = 0$ for $* \geq 0$. (The cases $* < 4$ follow from $* > 2(*-2)$.) □

Here we give an example of a function field. We consider the function field $\boldsymbol{C}(X)$ of $X = (\boldsymbol{C}^{2*} - H)/D_8$ for the action given in Section 4.

Let $\boldsymbol{C}^2//G = \mathrm{Spec}(\boldsymbol{C}[t,s]^G)$ be the geometric quotient by G. Then $X = (\boldsymbol{C}^2 - H)/G$ is an open set in $\boldsymbol{C}^2//G$. So $\boldsymbol{C}(X) \cong \boldsymbol{C}(t,s)^G$; the quotient field of the invariant ring $\boldsymbol{C}[t,s]^G$. The group $G = D_8$ satisfies Noether's problem so that $\boldsymbol{C}(X)$ is purely transcendental over $\boldsymbol{C}$, i.e. $\boldsymbol{C}(X) \cong \boldsymbol{C}(t',s')$. This fact is easily seen since

$$\boldsymbol{C}[t,s]^{D_8} = \boldsymbol{C}[ts, t^4 + s^4] \subset \boldsymbol{C}[t,s],$$

where the action is given by $a : \begin{cases} t \mapsto it \\ s \mapsto -is \end{cases}$, $b : \begin{cases} t \mapsto s \\ s \mapsto t \end{cases}$.

6. Motivic cohomology of BD_8 and BQ_8.

In this section, we compute the mod(2) motivic cohomology of BD_8 and BQ_8.

At first, we consider the case Q_8. The mod 2 (usual) cohomology is well known (see Theorem 2.7)

$$H^*(BQ_8;\boldsymbol{Z}/2) \cong \boldsymbol{Z}/2\{1, x_1, y_1, x_2, y_2, w\} \otimes \boldsymbol{Z}/2[c_2]$$

where $x_i^2 = \beta x_i = y_i$ and $|w| = 3$. The graded algebra $\mathrm{gr}^{*'} H^*(BQ_8;\boldsymbol{Z}/2)$ is given by letting the weight degree by

$$w(y_i) = w(c_2) = 0, \qquad w(x_i) = w(w) = 1.$$

The facts $w(y_i) = w(c_2) = 0$ follows from that they are Chern classes. The fact $w(w) = 1$ (in fact, we can take $w \in H^{3,2}(BQ_8;\boldsymbol{Z}/2)$) follows from the proof the following theorem.

THEOREM 6.1. *We have the bidegree isomorphism*

$$H^{*,*'}(BQ_8;\boldsymbol{Z}/2) \cong h^{*,*'}(BQ_8;\boldsymbol{Z}/2) \cong \boldsymbol{Z}/2[\tau] \otimes \mathrm{gr}^{*'} H^*(BQ_8;\boldsymbol{Z}/2).$$

PROOF. Let $G = Q_8$. In the usual mod(2) cohomology

$$H^*_G(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) \cong H^*(BG;\boldsymbol{Z}/2)/(c_2) \cong \boldsymbol{Z}/2\{1, x_1, y_1, x_2, y_2, w\},$$

which is isomorphic to $H^*(\boldsymbol{C}^{2*}/Q_8;\boldsymbol{Z}/2)$. Hence we can use Lemma 5.4

$$H_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2)\cong\boldsymbol{Z}/2[\tau]\otimes\boldsymbol{Z}/2\{1,x_1,y_1,x_2,y_2,w\}.$$

Here $\deg(w)=(3,2)$ by the following reason. The Bockstein exact sequence also exists in the motivic cohomology

$$\to H^{*-1,*'}(BG;\boldsymbol{Z}/2)\xrightarrow{\bar\beta}H^{*,*'}(BG;\boldsymbol{Z})\xrightarrow{\times 2}H^{*,*'}(BG;\boldsymbol{Z})\to\cdots$$

Since $c_2\in H^{4,2}(BG)$ and $4c_2=0$, we can take $w\in H^{3,2}(BG;\boldsymbol{Z}/2)$ with $\bar\beta(w)=2c_2$.

Using above facts (indeed, $\operatorname{gr}H^*(BG;\boldsymbol{Z}/2)$ and $\operatorname{gr}H_G^*(\boldsymbol{C}^{2*};\boldsymbol{Z}/2)$ are computed), we can show the lower sequence in the following diagram is exact

$$\begin{array}{ccccc}
\to H^{*-4,*'-2}(BG;\boldsymbol{Z}/2) & \xrightarrow{c_2} & H^{*,*'}(BG;\boldsymbol{Z}/2) & \longrightarrow & H_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2)\to\\
\downarrow j_1 & & \downarrow j_2 & & \downarrow j_3\cong\\
\to h^{*-4,*'-2}(BG;\boldsymbol{Z}/2) & \xrightarrow{c_2} & h^{*,*'}(BG;\boldsymbol{Z}/2) & \longrightarrow & h_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2)\to
\end{array}$$

where $h_G^{*,*'}(X;\boldsymbol{Z}/2)=\boldsymbol{Z}/2[\tau]\otimes\operatorname{gr}^{*'}H_G^*(X;\boldsymbol{Z}/2)$.

Since $H_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2)\cong H^{*,*'}(\boldsymbol{C}^{2*}/G;\boldsymbol{Z}/2)$, the map j_3 is always an isomorphism, from Lemma 5.4. When $*<0$, we know $H^{*,*'}(X;\boldsymbol{Z}/p)=0$ from $H^{*,<0}(X;\boldsymbol{Z}/p)=0$ and the Beilinson-Lichtenbaum conjecture. Of course, for $*=4$, the map j_1 is an isomorphism, namely both are isomorphic to $\boldsymbol{Z}/2[\tau]$. Hence we have the isomorphism of j_2 for $*\le 4$. By induction on $*\ge 0$ and the five lemma, we easily see that the vertical maps are isomorphisms. □

Now we consider the case $G=D_8$. We recall the mod(2) cohomology.

$$\begin{aligned}H^*(BD_8;\boldsymbol{Z}/2)&\cong(\boldsymbol{Z}/2[x_1,x_2]/(x_1x_2))\otimes\boldsymbol{Z}/2[u]\\ &\cong\left(\bigoplus_{i=1}^{2}\boldsymbol{Z}/2[y_i]\{y_i,x_i,y_iu,x_iu\}\oplus\boldsymbol{Z}/2\{1,u\}\right)\otimes\boldsymbol{Z}/2[c_2].\end{aligned}$$

Here we identify, $y_i=x_i^2$ and $c_2=u^2$. The cohomology operations on $H^*(BD_8;\boldsymbol{Z}/2)$ is well known, e.g., (see [**Te-Ya**])

$$Q_0(u)=(x_1+x_2)u=e,\quad Q_1Q_0(u)=(y_1+y_2)c_2.$$

Lemma 6.2. *There exist $u_1', u_2' \in H^{3,2}(BD_8; \boldsymbol{Z}/2)$ with $\tau u_i' = x_i u \in H^{3,3}(BD_8; \boldsymbol{Z}/2)$ (so $u_i' = \tau^{-1} x_i u$).*

Proof. First note that we can take $u \in H^{2,2}(BG; \boldsymbol{Z}/2)$ (since it is not in Chow ring and $Q_0(u) \neq 0$). Of course y_i and c_2 are represented by Chern classes. Hence

$$H^{3,2}(BG; \boldsymbol{Z}) \supset \boldsymbol{Z}/2\{Q_0(u)\}, \quad H^{4,2}(BG; \boldsymbol{Z}) \cong \boldsymbol{Z}/2\{y_1^2, y_2^2\} \oplus \boldsymbol{Z}/4\{c_2\}.$$

By using the universal coefficient theorem such that

$$\dim H^{*,*'}(X; \boldsymbol{Z}/p) = \dim \left(H^{*,*'}(X)/p\right) + \dim \left(p\text{-torsion}(H^{*+1,*'}(X))\right),$$

(since there is the Bockstein exact sequence also in the motivic theory), we see

$$\dim H^{3,2}(BG; \boldsymbol{Z}/2) \geq 1 + 3 = 4.$$

From the Beilinson-Lichtenbaum conjecture and Corollary 5.3, we see that $H^{*,*'}(X; \boldsymbol{Z}/p) \to H^*(X; \boldsymbol{Z}/p)$ is injective for $* \leq 3$. On the other hand

$$H^3(BG; \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2\{x_1 u, x_2 u, x_1 y_1, x_2 y_2\}.$$

Hence each element in $H^3(BG; \boldsymbol{Z}/2)$ must be in $H^{3,2}(BG; \boldsymbol{Z}/2)$. (Indeed, $Q_0(x_i y_i) = y_i^2$, $Q_0(u) = u_1' + u_2'$ and $\bar{\beta}(u_i') = 2c_2$.) □

Therefore we get $\mathrm{gr}^{*'} H^*(BD_8; \boldsymbol{Z}/2)$ which is isomorphic to

$$\left(\bigoplus_{i=1}^{2} \boldsymbol{Z}/2[y_i]\{y_i, x_i, x_i u_i', u_i'\} \oplus \boldsymbol{Z}/2\{1, u\}\right) \otimes \boldsymbol{Z}/2[c_2]$$

with $w(y_i) = w(c_2) = 0$, $w(x_i) = w(u_i') = 1$ and $w(u) = w(x_i u_i') = 2$. (Note $u, x_i u_i' \notin CH^*(BG)/2$, and $x_i u_i' = y_i u$).

Theorem 6.3. *We have the bidegree module isomorphism*

$$H^{*,*'}(BD_8; \boldsymbol{Z}/2) \cong h^{*,*'}(BD_8; \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2[\tau] \otimes \mathrm{gr}^{*'} H^*(BD_8; \boldsymbol{Z}/2).$$

Before the proof of this theorem, we give a lemma.

Lemma 6.4.

$$H^{*,*'}_{D_8}(H_0,\boldsymbol{Z}/2) \cong h^{*,*'}_{D_8}(H_0,\boldsymbol{Z}/2) \cong \boldsymbol{Z}/2[\tau]\otimes \boldsymbol{Z}/2[y_1]\otimes \Lambda(x_1,z)$$

with $\deg(z)=(1,1)$.

Proof. Let $G=D_8$. We consider the exact sequence

$$\to H^{*-2,*'-1}_G(\{0\}\times H_0;\boldsymbol{Z}/2) \xrightarrow{y_1} H^{*,*'}_G(\boldsymbol{C}\times H_0;\boldsymbol{Z}/2) \to H^{*,*'}_G(\boldsymbol{C}^*\times H_0;\boldsymbol{Z}/2) \to \cdots$$

where G acts on $\boldsymbol{C}\times H_0$ by

$$g(x,y)=(b^*(g)(x),g(y)) \quad \text{for } x\in \boldsymbol{C},\ y\in H_0.$$

Note that G acts freely on $\boldsymbol{C}^*\times H_0$ (but H_0 itself has the stabilizer group $\langle b\rangle$) and

$$\begin{aligned} H^{*,*'}_G(\boldsymbol{C}^*\times H_0;\boldsymbol{Z}/2) &\cong H^{*,*'}\big((\boldsymbol{C}^*\times H_0)/G;\boldsymbol{Z}/2\big)\\ &\cong H^{*,*'}(\boldsymbol{C}^*/\langle b\rangle\times \boldsymbol{C}^*/\langle a^2\rangle;\boldsymbol{Z}/2)\\ &\cong H^{*,*'}(\boldsymbol{C}^*/\langle b\rangle;\boldsymbol{Z}/2)\otimes_{\boldsymbol{Z}/2[\tau]} H^{*,*'}(\boldsymbol{C}^*/\langle a^2\rangle;\boldsymbol{Z}/2)\\ &\cong \boldsymbol{Z}/2[\tau]\otimes\Lambda(x_1,z)\end{aligned}$$

since $H^{*,*}(\boldsymbol{C}^{n*}/(\boldsymbol{Z}/2);\boldsymbol{Z}/2)$ holds the Kunneth formula. (See Proposition 6.6 and Lemma 6.7 in [**Vo3**], and the arguments work, if we take $\boldsymbol{C}^{n*}/2$ instead of $B\boldsymbol{Z}/2=\mathrm{colim}_n\,\boldsymbol{C}^{n*}/\boldsymbol{Z}/2$.)

The natural map $H^{*,*'}_G(H_0;\boldsymbol{Z}/2)\to \boldsymbol{Z}/2[\tau]\otimes H^*_G(H_0;\boldsymbol{Z}/2)$ induces the diagram for two exact sequences similar to the above exact sequence. We can prove the lemma by induction on $*\ge 0$ and the five lemma. □

Proof of Theorem 6.3. Let $G=D_8$. First we consider the exact sequence

$$\to H^{*-2}_G(H;\boldsymbol{Z}/2) \xrightarrow{i_*} H^*_G(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) \to H^*_G(\boldsymbol{C}^{2*}-H;\boldsymbol{Z}/2)\to\cdots$$

We write the map i_* explicitly

$$\begin{array}{ccc} H_G^{*-2}(H;\boldsymbol{Z}/2) & \xrightarrow{\quad i_* \quad} & H_G^*(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) \\ \cong\downarrow & & \cong\downarrow \\ \bigoplus_{j=1}^{2} \boldsymbol{Z}/2[y_j]\{1_j, x_j, z_j, x_j z_j\} & \xrightarrow{i_*} & \left(\bigoplus_{j=1}^{2} \boldsymbol{Z}/2[y_j]\{y_j, x_j, u_j', x_j u_j'\}\right) \oplus \boldsymbol{Z}/2\{1, u\} \end{array}$$

where $1_j, z_j$ are the generators in $H_G^*(H_{j-1};\boldsymbol{Z}/2)$. Using the fact that i_* is isomorphic for $* > 4$, the map i_* is given explicitly

$$i_*(1_j) = y_j, \quad i_*(x_j) = y_j x_j, \quad i_*(z_j) = u_j', \quad i_*(z_j x_j) = u_j' x_j.$$

(In particular, i_* is injective.) Therefore

$$H^*((\boldsymbol{C}^{2*} - H)/G;\boldsymbol{Z}/2) \cong \boldsymbol{Z}/2\{1, x_1, x_2, u\}.$$

We still get the weight degree $w(x)$, and we have the exact sequence

$$0 \to \mathrm{gr}^{*'} H_G^{*-2}(H;\boldsymbol{Z}/2) \xrightarrow{i_*} \mathrm{gr}^{*'} H_G^*(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) \to \mathrm{gr}^{*'} H_G^*(\boldsymbol{C}^{2*} - H;\boldsymbol{Z}/2) \to 0.$$

Next we consider the following diagram

$$\begin{array}{ccccccc} \to H_G^{*-2,*'-1}(H;\boldsymbol{Z}/2) & \xrightarrow{i_*} & H_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) & \longrightarrow & H_G^{*,*'}(\boldsymbol{C}^{2*} - H;\boldsymbol{Z}/2) & \to \cdots \\ d_1\downarrow & & d_2\downarrow & & d_3\downarrow & \\ \to h_G^{*-2,*'-1}(H;\boldsymbol{Z}/2) & \xrightarrow{i_*} & h_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) & \longrightarrow & h_G^{*,*'}(\boldsymbol{C}^{2*} - H;\boldsymbol{Z}/2) & \to \cdots \end{array}$$

Here the lower sequence is also (split) exact from the above sequence for $\mathrm{gr}^{*'} H_G^*(-;\boldsymbol{Z}/2)$. The map d_3 is an isomorphism from Lemma 5.4 since $H_G^*(\boldsymbol{C}^{2*} - H;\boldsymbol{Z}/2) \cong H^*((\boldsymbol{C}^{2*} - H)/G;\boldsymbol{Z}/2)$. The map d_1 is also an isomorphism from the preceding lemma. By using the five lemma, we get $H_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) \cong h_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2)$.

Using the exact sequence

$$\to H^{*-4,*'-2}(BG;\boldsymbol{Z}/2) \xrightarrow{c_2} H^{*,*'}(BG;\boldsymbol{Z}/2) \longrightarrow H_G^{*,*'}(\boldsymbol{C}^{2*};\boldsymbol{Z}/2) \to,$$

as in the case of $G = Q_8$, we can see $H^{*,*'}(BG;\boldsymbol{Z}/2) \cong h^{*,*'}(BG;\boldsymbol{Z}/2)$. □

7. Motivic cobordism of BQ_8 and BD_8.

Let $MU^*(X)$ and $MU^*(X;\boldsymbol{Z}/p)$ be the usual complex cobordism theory and its mod p theory. Let $MGL^{*,*'}(X)$ be th motivic cobordism theory defined by Voevodsky [**Vo1**]. Since $t_{\boldsymbol{C}}|CH^*(BG)$ is injective, from Proposition 9.4 in [**Ya3**], we have the isomorphism

$$MGL^{2*,*}(BG) \cong MU^{2*}(BG)$$

for each group of order p^3.

In this section, we give rather strong results for only Q_8 and D_8. Let $MGL^{*,*'}(X;\boldsymbol{Z}/p)$ be the mod p theory defined by the exact sequence

$$\to MGL^{*,*'}(X) \xrightarrow{\times p} MGL^{*,*'}(X) \xrightarrow{\rho} MGL^{*,*'}(X;\boldsymbol{Z}/p) \xrightarrow{\delta} \cdots$$

Then we have the following theorem (which holds also for $(\boldsymbol{Z}/2)^n, O_n, SO_n$).

Theorem 7.1. *Let $G = Q_8$ or D_8. Then there are isomorphisms*

$$MGL^{*,*'}(BG;\boldsymbol{Z}/2) \cong MGL^{2*,*}(BG;\boldsymbol{Z}/2) \otimes \boldsymbol{Z}/2[\tau],$$
$$MGL^{2*,*}(BG;\boldsymbol{Z}/2) \cong MU^{2*}(BG;\boldsymbol{Z}/2) \cong MU^{2*}(BG)/2.$$

Proof. Let $G = Q_8$ or D_8. Let $E(MGL)_r^{*,*',*''}$ (resp. $E(MU)_r^{*,*''}$) be the Atiyah-Hirzebruch spectral sequence converging to $MGL^{*,*'}(BG;\boldsymbol{Z}/2)$ (resp. $MU^*(BG;\boldsymbol{Z}/2)$) (see [**Ya3**]), namely,

$$E(MGL)_2^{*,*',*''} \cong H^{*,*'}(BG;\boldsymbol{Z}/2) \otimes MU^{*''} \Longrightarrow MGL^{*,*'}(BG;\boldsymbol{Z}/2),$$
$$E(MU)_2^{*,*''} \cong H^{*}(BG;\boldsymbol{Z}/2) \otimes MU^{*''} \Longrightarrow MU^{*}(BG;\boldsymbol{Z}/2).$$

The realization map $t_{\boldsymbol{C}}$ induces the map $t_{\boldsymbol{C}}^{*,*',*''} : E(MGL)_r^{*,*',*''} \to E(MU)_r^{*,*''}$ of spectral sequences.

From Theorem 6.1 and 6.3, we know

$$H^{*,*'}(BG;\boldsymbol{Z}/2) \cong \boldsymbol{Z}/2[\tau] \otimes \mathrm{gr}^{*'} H^*(BG;\boldsymbol{Z}/2).$$

Let us write $\mathrm{gr}^{*'} E(MU)_2^{*,*''} = \mathrm{gr}^{*'} H^*(BG;\boldsymbol{Z}/2) \otimes MU^{*''}$ so that we have the bidegree module isomorphism

$$E(MGL)_2^{*,*',*''} \cong \boldsymbol{Z}/2[\tau] \otimes \mathrm{gr}^{*'} E(MU)_2^{*,*''}.$$

Suppose that for all $x \in \mathrm{gr}^{*'} E(MU)_2^{*,*''} \subset E(MGL)_2^{*,*',*''}$,

$$(1) \quad d_2(x) \in \mathrm{gr}^{*'} E(MU)_2^{*,*''} \quad (\text{i.e., } d_2(x) \neq \tau y \text{ for some } \tau y \neq 0).$$

Then from the naturality of the map $t_{\boldsymbol{C}}^{*,*''}$ of spectral sequences, we have

$$E(MGL)_3^{*,*',*''} \cong \boldsymbol{Z}/2[\tau] \otimes \mathrm{gr}^{*'} E(MU)_3^{*,*''}$$

where $\mathrm{gr}^{*'} E(MU)_3^{*,*''}$ is the bidegree module made from $\mathrm{gr}\, E(MU)_3^{*,*''}$ giving the same second degree. Moreover, if for all $x \in \mathrm{gr}^{*'} E(MU)_r^{*,*''}$, $r \geq 2$

$$(2) \quad d_r(x) \in \mathrm{gr}^{*'} E(MU)_r^{*,*''},$$

then we have the bidegree isomorphism

$$E(MGL)_\infty^{*,*',*''} \cong \boldsymbol{Z}/2[\tau] \otimes \mathrm{gr}^{*'} E(MU)_\infty^{*,*''},$$

and we can prove this theorem.

To see (1), (2), we note that $\mathrm{gr}^{*'} H^*(BG; \boldsymbol{Z}/2)$ is generated by elements x of degree $w(x) \leq 1$ (resp. $w(x) \leq 2$ e.g., $w(u) = 2$) for $G = Q_8$ (resp. $G = D_8$). Hence $w(d_r(x)) = w(x) - 1 \leq 1$. Since $w(\tau) = 2$, all elements x' of $w(x') \leq 1$ are contained in

$$H^{2*,*}(BG; \boldsymbol{Z}/2) \oplus H^{2*+1,*}(BG; \boldsymbol{Z}/2) \subset \mathrm{gr}^{*'} H^*(BG; \boldsymbol{Z}/2).$$

Thus we get (1), (2). □

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