J. Math. Soc. Japan
Vol. 62, No. 2 (2010) pp. 623–647
doi: 10.2969/jmsj/06220623

Nontrivial $\mathscr{P}(G)$-matched $\mathfrak{S}$-related pairs for finite gap Oliver groups

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(Received Sep. 11, 2008)
(Revised Mar. 2, 2009)

Abstract. In this paper we construct nontrivial pairs of $\mathfrak{S}$-related (i.e. Smith equivalent) real G-modules for the group $G = P\Sigma L(2,27)$ and the small groups of order 864 and types 2666, 4666. This and a theorem of K. Pawałowski-R. Solomon together show that Laitinen's conjecture is affirmative for any finite nonsolvable gap group. That is, for a finite nonsolvable gap group G, there exists a nontrivial $\mathscr{P}(G)$-matched pair consisting of $\mathfrak{S}$-related real G-modules if and only if the number of all real conjugacy classes of elements in G not of prime power order is greater than or equal to 2.

1. Introduction.

Let G be a finite group. We denote by $\mathscr{S}(G)$ the set of all subgroups of G and by $\mathscr{P}(G)$ the set of all subgroups of G of prime power order. In this paper, a real G-representation space of finite dimension is referred to, briefly, as a real G-module, a smooth manifold as a manifold, and a smooth G-action on a manifold as a G-action on a manifold, unless otherwise stated. Real G-modules V and W are called $\mathfrak{D}$*-related* (resp. $\mathfrak{S}$*-related*) and written as $V \sim_{\mathfrak{D}} W$ (resp. $V \sim_{\mathfrak{S}} W$) if there exists a G-action on a manifold X diffeomorphic to a disk (resp. homotopy sphere) such that $X^G = \{a, b\}$ and the tangential G-representations $T_a(X)$ and $T_b(X)$ are isomorphic to V and W, respectively. If V and W are both $\mathfrak{D}$-related and $\mathfrak{S}$-related then they are called $\mathfrak{D}\mathfrak{S}$-related and written as $V \sim_{\mathfrak{D}\mathfrak{S}} W$. A homotopy sphere Σ with (smooth) G-action is called a 2*-fixed-point sphere*, or 2*fp sphere*, if $|\Sigma^G| = 2$. If V and W are real G-modules and Σ is a 2fp sphere such that $\Sigma^G = \{a, b\}$, $T_a(\Sigma) \cong V$ and $T_b(\Sigma) \cong W$ then we call Σ an $\mathfrak{S}$*-realization* of V and W.

M. Atiyah-R. Bott [**1**] and J. Milnor [**19**] showed that $\mathfrak{S}$-related real G-modules V and W are isomorphic if the G-action of an $\mathfrak{S}$-realization of V and

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2000 *Mathematics Subject Classification.* Primary 57S25; Secondary 55M35, 57S17, 20C15.
Key Words and Phrases. Smith equivalence, Laitinen's conjecture, tangent space, representation, gap condition.
Partially supported by Grant-in-Aid for Scientific Research (KAKENHI) (No. 18540086), Japan Society for the Promotion of Science.

W is semifree. In addition, C. Sanchez [**44**] showed that $\mathfrak{S}$-related real G-modules V and W are isomorphic if the order $|G|$ of G is an odd prime power. On the other hand, many researchers, e.g. T. Petrie, S. Cappell-J. Shaneson, K. H. Dovermann, D.Y. Suh, E. Laitinen-K. Pawałowski, K. Pawałowski-R. Solomon and etc. have found nontrivial pairs (V, W), i.e. $V \not\cong W$, consisting of $\mathfrak{S}$-related G-representations for various groups G. We note that their nontrivial pairs (V, W) satisfy $\dim V^N = \dim W^N$ whenever N is a normal subgroup of G with prime power index. In the present paper, we show the next theorem.

Theorem 1.1. *If $G = P\Sigma L(2, 27)$, $SG(864, 2666)$, or $SG(864, 4666)$, then there exist $\mathfrak{D}\mathfrak{S}$-related pairs (V, W) satisfying the following conditions*:

(1) $\dim V^N > 0$, $\dim W^N = 0$ *for a normal subgroup N of G with index 3, and*
(2) $\dim V^P = \dim W^P \geq 6$ *for every Sylow subgroup P of G.*

In the above, $SG(m, n)$ denote the small group of order m and type n which is obtained as `SmallGroup(m,n)` in GAP [**13**]. We showed in [**22**] that if V and W are $\mathfrak{S}$-related and N is a normal subgroup of G with index 2 then $V^N \cong W^N$ as real G/N-modules. The theorem above shows that $V^N \cong W^N$ does not always hold if $|G/N| = 2$ is replaced by $|G/N| = p$ with an odd prime p.

We recall (see [**28**]) that if there exists a G-action on a disk with exactly two G-fixed points then G is an *Oliver group*, that is G can acts on a disk without G-fixed points, which is also equivalent to that G is not a mod-$\mathscr{P}$ hyperelementary group, namely G never admits a normal series $P \trianglelefteq H \trianglelefteq G$ such that P and G/H have both prime power order and H/P is cyclic. Let a_G denote the number of real conjugacy classes $(g)^{\pm} = (g) \cup (g^{-1})$ of G such that the order of g is not a prime power. In the paper [**17**], we read the following conjecture.

Conjecture (Laitinen's Conjecture). Let G be an Oliver group. Then there exists an $\mathfrak{S}$-realization Σ of G-modules V and W such that Σ^g is connected for every element $g \in G$ having order 2^m with $m \geq 3$, if and only if $a_G \geq 2$.

We have, however, seen in [**22**] and [**14**] that this conjecture fails for the groups $G = \mathrm{Aut}(A_6)$, $SG(1176, 220)$, and $SG(1176, 221)$. In addition, K. Pawałowski-T. Sumi [**36**] showed that the conjecture also fails for the groups $G = SG(72, 44)$, $SG(288, 1025)$, $SG(432, 734)$, and $SG(567, 8654)$.

Let $\mathscr{F}$ be a set of subgroups of G. A real G-module V is called *$\mathscr{F}$-free* if $V^H = 0$ for all $H \in \mathscr{F}$. Real G-modules V and W are called *$\mathscr{F}$-matched* if $\mathrm{res}^G_H V \cong \mathrm{res}^G_H W$ for all $H \in \mathscr{F}$. An $\mathscr{F}$-matched pair (V, W) is said to be of *type 1* if $\dim V^G = 1$ and $\dim W^G = 0$. Let $\mathscr{L}(G)$ be the smallest upper closed subset of $\mathscr{S}(G)$ containing all normal subgroups $N \trianglelefteq G$ such that G/N is of prime power order. We say that V satisfies the *gap condition* if $\dim V^P > 2 \dim V^H$ for all subgroups $P \lneq H$ of G such that P is of prime power order. A real G-module V is called a *gap module* if V is $\mathscr{L}(G)$-free and satisfies the gap condition. A finite group G is called a *gap group* if there exists a gap real G-module. K. Pawałowski-R. Solomon showed [**35**, Theorem B3] that if G is a nonsolvable gap group and G is not isomorphic to $P\Sigma L(2,27)$ then Laitinen's conjecture is affirmative. Thus, our result for the group $G = P\Sigma L(2,27)$ stated above implies the next theorem.

Theorem 1.2. *If G is a nonsolvable gap group then Laitinen's conjecture is affirmative for G.*

Let $\mathrm{RO}(G)$ denote the real representation ring of G. Define

$$\begin{aligned}\mathrm{RO}(G,\mathfrak{D}) &= \{[V]-[W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{D}} W\},\\ \mathrm{RO}(G,\mathfrak{S}) &= \{[V]-[W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{S}} W\},\\ \mathrm{RO}(G,\mathfrak{D}\mathfrak{S}) &= \{[V]-[W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{D}\mathfrak{S}} W\}.\end{aligned}$$

In this paper we will study $\mathrm{RO}(G,\mathfrak{D}\mathfrak{S})$.

For sets $\mathscr{F}$ and $\mathscr{G}$ of subgroups of G and $M \subseteq \mathrm{RO}(G)$, we define

$$\begin{aligned}M_{\mathscr{F}} &= \{[V]-[W] \in M \mid V \text{ and } W \text{ are } \mathscr{F}\text{-matched}\},\\ M^{\mathscr{G}} &= \{[V]-[W] \in M \mid V \text{ and } W \text{ are } \mathscr{G}\text{-free}\},\\ M_{\mathscr{F}}^{\mathscr{G}} &= M_{\mathscr{F}} \cap M^{\mathscr{G}}.\end{aligned}$$

B. Oliver [**27**] showed $\mathrm{RO}(G,\mathfrak{D}) = \mathrm{RO}(G)_{\mathscr{P}(G)}^{\{G\}}$ for an arbitrary Oliver group G. In addition, E. Laitinen-K. Pawałowski [**17**] showed that $\mathrm{rank}_{\boldsymbol{Z}}\ \mathrm{RO}(G)_{\mathscr{P}(G)}^{\{G\}} = \max(a_G - 1, 0)$, which also follows from B. Oliver [**27**]. We will show the equality $\mathrm{RO}(G,\mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G)_{\mathscr{P}(G)}^{\{G\}}$ in the cases $G = P\Sigma L(2,27)$, $SG(864,2666)$, $SG(864,4666)$. This is stated in a slightly general form as the next theorem. In order to state it, we define, for a prime p, the *Dress subgroup* $G^{\{p\}} \le G$ of type p, to be the smallest normal subgroup $N \trianglelefteq G$ such that $|G/N|$ is a power of p (possibly $G = G^{\{p\}}$). Let G^{nil} denote the smallest normal subgroup N of G such that G/N is nilpotent. Then the equality

$$G^{\mathrm{nil}} = \bigcap_{p:\ \mathrm{prime}} G^{\{p\}}$$

holds, cf. [**15**]. Let D_{2n} denote the dihedral group of order $2n$:

$$\langle a,\ b \mid a^n = e,\ b^2 = e,\ bab = a^{-1}\rangle.$$

For a subset S of G, let $\overline{\mathscr{P}}(S)$ denote the set of all elements g of S such that the order of g is not a power of a prime. Here we regard $e \notin \overline{\mathscr{P}}(S)$ for the sake of convenience.

Theorem 1.3. *Let G be an Oliver group satisfying Conditions* (1)–(4) *below. Here N stands for G^{nil}.*

(1) *N has a subquotient group isomorphic to D_{2qr} for distinct primes q and r.*
(2) *G/N is a nontrivial group of odd order.*
(3) *The set $G \smallsetminus N$ contains an element not of prime power order.*
(4) $|\overline{\mathscr{P}}(gN)| = |\overline{\mathscr{P}}(g'N)|$ *for all g, $g' \in G \smallsetminus N$.*

Then there exists a $\mathscr{P}(G)$-matched pair (U_1, U_2) of type 1 consisting of real G-modules such that $U_1^N = \boldsymbol{R}[G/N]$ and $U_2^N = 0$, and $\mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)}$ contains a direct summand $\langle x\rangle_{\boldsymbol{Z}}$ generated by an element $x = [V_1]-[V_2]$ such that $V_1^N = (\boldsymbol{R}[G/N]-\boldsymbol{R}[G/N]^G)^{\oplus m}$ for some $m \geq 1$ and $V_2^N = 0$. For the element x, the implication $\langle x\rangle_{\boldsymbol{Z}} \subseteq \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ $(\neq 0)$ holds and hence $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)} \neq \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$. Moreover in the case $a_G = 2$, the equality $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)}$ holds.

Remark 1.4. In the theorem above, if $|G/N| = 3$ then Condition (4) is automatically satisfied.

In each case $G = P\Sigma L(2, 27)$, $SG(864, 2666)$, $SG(864, 4666)$, it is easy to see that $a_G = 2$, $|G/G^{\{3\}}| = 3$, $G^{\mathrm{nil}} = G^{\{3\}}$, $G^{\{3\}} \supset D_{2qr}$ (q and r are distinct primes), $\mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)} \cong \boldsymbol{Z}$, $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)} = 0$, and $G \smallsetminus G^{\mathrm{nil}}$ contains an element not of prime power order. Thus Theorem 1.1 follows from Theorem 1.3.

The readers familiar with [**35**] would see the next.

Theorem 1.5. *Let G be a gap Oliver group. Then the implication $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)} \subseteq \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ holds. If G^{nil} contains distinct two real conjugacy classes of elements not of prime power order, then $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)} \neq 0$ and hence $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \neq 0$.*

The rest of this paper is organized as follows. We prepare basic facts concerned with $\mathscr{P}(G)$-matched real G-modules in Section 2. A key to proving Theorem 1.3 is observation of the tangent bundle of the real projective space $P(V)$ associated with a real G-module V. In Section 3, we exhibit basic results related to the tangent space. In Section 4 we claim several lemmas showing an outline of the proof of Theorem 1.3, and in Section 5 we explain known facts which are used to prove the lemmas. These lemmas are proved in Sections 6–9. Finally, Theorems 1.3 and 1.5 are proved in Section 10.

The author wishes to express his gratitude to the referee for his carefully

reading the manuscript, pointing out errors in it, and giving invaluable comments.

NOTATION.

$\mathscr{S}(G) =$ the set of all subgroups of G

$\mathscr{P}(G) = \{P \in \mathscr{S}(G) \mid P \text{ is of prime power order}\}$

$\mathscr{L}(G) = \{H \in \mathscr{S}(G) \mid H \supseteq G^{\{p\}} \text{ for some prime } p\}$

$\mathscr{M}(G) = \mathscr{S}(G) \smallsetminus \mathscr{L}(G)$

$\mathscr{N}_2(G) = \{N \in \mathscr{S}(G) \mid N \trianglelefteq G,\ |G : N| \leq 2\}$

$\mathscr{P}\mathscr{C}(G) = \{H \in \mathscr{S}(G) \mid \exists\ P \in \mathscr{P}(G) \text{ such that } P \trianglelefteq H \text{ and } H/P \text{ is cyclic}\}$

$X^{\times m} = X \times \cdots \times X$ (the m-fold cartesian product of X)

$V^{\oplus m} = V \oplus \cdots \oplus V$ (the m-fold direct (Whitney) sum of V)

2. Preliminary on real G-modules.

Let G be a finite group and V a real G-module. If H is a subgroup of G then the H-fixed point set V^H is a real $N_G(H)$-module. Let V_H denote the orthogonal complement of V^H in V with respect to a G-invariant inner product. V_H is uniquely determined up to $N_G(H)$-isomorphisms independently of the choice of a G-invariant inner product on V. Thus we have the direct sum decomposition

$$V = V^H \oplus V_H \quad \text{as real } N_G(H)\text{-modules.}$$

If $x \in \mathrm{RO}(G)$ has the form $x = [V] - [W]$ with real G-modules V and W, then x^H stands for the element $[V^H] - [W^H]$ in $\mathrm{RO}(N_G(H)/H)$ as well as $\mathrm{RO}(N_G(H))$. In the same situation, $\dim x^H$ stands for the integer $\dim V^H - \dim W^H$. Let $V^{\mathscr{L}}$ denote the G-subspace of V spanned by all elements in V^L, where L ranges over $\mathscr{L}(G)$. Namely

$$V^{\mathscr{L}} = \sum_{q:\ \text{prime}} V^{G^{\{q\}}} = V^G \oplus \bigoplus_{q:\ \text{prime}} \left(V^{G^{\{q\}}} - V^G\right).$$

It induces a direct sum decomposition

$$V = V^{\mathscr{L}} \oplus V_{\mathscr{L}} \quad \text{as real } G\text{-modules.}$$

If $G = G^{\{2\}}$ then $V(G) = \boldsymbol{R}[G]_{\mathscr{L}}$ is a gap G-module, cf. Lemma 5.2, and hence G is a gap group.

Each element $x = [V] - [W] \in \mathrm{RO}(G)$ determines the character (function) $\chi_x = \chi_V - \chi_W$. We can regard $\mathrm{RO}(G)$ as a set of functions $G \to \boldsymbol{R}$ taking a same value on a real conjugacy class. Note that for $g \in N_G(H)$,

$$\chi_{x^H}(g) = \frac{1}{|H|} \sum_{h \in H} \chi_x(gh).$$

Thus, for a real conjugacy class function $f : G \to \boldsymbol{R}$, we define $f^H : N_G(H) \to \boldsymbol{R}$ by

$$f^H(g) = \frac{1}{|H|} \sum_{h \in H} f(gh).$$

If $g \in G$ then let $f_{(g)^\pm} : G \to \boldsymbol{Z}$ denote the class function defined by

$$f_{(g)^\pm}(h) = \begin{cases} \frac{|G|}{|(g)^\pm|} & \text{if } h \in (g)^\pm \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let g_1, g_2 be elements not of prime power order of G. Then the class function φ defined by*

$$\varphi = f_{(g_1)^\pm} - f_{(g_2)^\pm}$$

belongs to $\mathrm{RO}(G)_{\mathscr{P}(G)} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Clearly, if $(g_1)^\pm \neq (g_2)^\pm$ then $\varphi \neq 0$. If N is a normal subgroup of G and g_1, $g_2 \in N$ then $\varphi^N = 0$.

Proof. By the character theory, the class function φ above belongs to $\mathrm{RO}(G) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Since $\varphi(a) = 0$ holds for all $a \in G$ of prime power order, $\varphi \in \mathrm{RO}(G)_{\mathscr{P}(G)} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Suppose $N \trianglelefteq G$ and g_1, $g_2 \in N$. Then for $g \in G$,

$$\begin{aligned} \varphi^N(g) &= \frac{1}{|N|} \sum_{a \in N} \varphi(ga) \\ &= \begin{cases} \frac{1}{|N|} \sum_{h \in N} \varphi(h) & \text{if } g \in N \\ 0 & \text{if } g \notin N \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{1}{|N|}\left(|(g_1)^{\pm}|\frac{|G|}{|(g_1)^{\pm}|} - |(g_2)^{\pm}|\frac{|G|}{|(g_2)^{\pm}|}\right) & \text{if } g \in N \\ 0 & \text{if } g \notin N \end{cases}$$
$$= 0.$$

We have checked $\varphi^N = 0$. □

The lemma above immediately implies the next.

COROLLARY 2.2. *Let g_1 and g_2 be elements not of prime power order in G. Suppose $(g_1)^{\pm} \neq (g_2)^{\pm}$ and g_1, $g_2 \in G^{\mathrm{nil}}$. Then $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)}$ is a nontrivial direct summand of* $\mathrm{RO}(G)$. *In particular,* rank $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)} \geq 1$.

On the other hand, we are also interested in the case where $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)} = 0$ and $\mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)} \neq 0$, e.g. $G = P\Sigma L(2,27)$, $SG(864,2666)$, $SG(864,4666)$. The next follows from straightforward computation using the character table.

PROPOSITION 2.3. *Let G be one of $P\Sigma L(2,27)$, $SG(864,2666)$, or $SG(864,4666)$ and $N = G^{\{3\}}$ $(= G^{\mathrm{nil}})$. Then there exist $\mathscr{P}(G)$-matched pairs (U_1, U_2) and (V_1, V_2) such that $U_1^N = \boldsymbol{R}[G/N]$, $U_2^N = 0$, $V_1 = (U_1 - U_1^G)^{\oplus 3} \oplus W$ for some real G-module W with $W^N = 0$, and $V_2^N = 0$, and moreover $\mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)}$ coincides with $\langle [V_1] - [V_2] \rangle_{\boldsymbol{Z}}$, the submodule generated by the element $[V_1] - [V_2]$ in* $\mathrm{RO}(G)$.

Let N be a normal subgroup of G. Suppose

$$|\overline{\mathscr{P}}(gN)| = |\overline{\mathscr{P}}(g'N)| > 0 \quad \text{for all } g,\ g' \in G \smallsetminus N.$$

Set $C = |\overline{\mathscr{P}}(g_0N)|$ for an element $g_0 \in G \smallsetminus N$ and define a function $\phi : G \to \boldsymbol{Q}$ by

$$\phi = \frac{|N|}{C} \sum_{(g)^{\pm}:g\in\overline{\mathscr{P}}(G\smallsetminus N)} \delta_{(g)^{\pm}}$$

where

$$\delta_{(g)^{\pm}}(a) = \begin{cases} 1 & (a \in (g)^{\pm}) \\ 0 & (a \notin (g)^{\pm}) \end{cases}$$

for $a \in G$. Then for $a \in G \smallsetminus N$, we have

$$\begin{aligned}
\phi^N(a) &= \frac{1}{|N|}\sum_{h\in N}\frac{|N|}{C}\sum_{(g)^{\pm}:g\in\overline{\mathscr{P}}(G\smallsetminus N)}\delta_{(g)^{\pm}}(ah)\\
&= \frac{1}{|N|}\frac{|N|}{C}C\\
&= 1.
\end{aligned}$$

If $a \in N$ then $\phi^N(a) = 0$. Thus $|G/N|\phi^N = |G/N|\chi_{\boldsymbol{Q}[G/G]} - \chi_{\boldsymbol{Q}[G/N]}$ as $\boldsymbol{Q}$-valued functions on G/N. The function $|G/N|\phi : G \to \boldsymbol{Q}$ takes a same value on each rationally conjugate class of G. The $\boldsymbol{Q}$-module consisting of all rationally conjugate class functions $G \to \boldsymbol{Q}$ is canonically isomorphic to $\mathrm{R}(G,\boldsymbol{Q}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, where $\mathrm{R}(G,\boldsymbol{Q})$ is the rational representation ring. Thus, for some positive integer m, $m|G/N|\phi$ lies in $\mathrm{RO}(G)$, namely $m|G/N|\phi = \chi_V - \chi_W$ for some real G-modules V and W, and $[V^N] - [W^N] = m|G/N|[\boldsymbol{R}] - m[\boldsymbol{R}[G/N]]$.

Immediately, we get the next lemma.

Lemma 2.4. *Let G be a finite group with a normal subgroup N satisfying $|\overline{\mathscr{P}}(gN)| = |\overline{\mathscr{P}}(g'N)| > 0$ for all g, $g' \in G\smallsetminus N$. Then there exists $x \in \mathrm{RO}(G)_{\mathscr{P}(G)}$ such that $x^N = m|G/N|[\boldsymbol{R}] - m[\boldsymbol{R}[G/N]]$ for some positive integer m.*

3. Real projective spaces and their tangent bundles.

Let V be a real G-module and let M denote the real projective space $P(V)$ and γ_M the canonical line bundle over M. In particular, the total space of γ_M is

$$\bigl\{(\{\pm x\}, v) \mid x \in S(V),\ v \in V \text{ with } v \in L_{\pm x}\bigr\}$$

where $L_{\pm x}$ is the straight line in V containing the points x and $-x$. We often abuse the notation γ_M to denote the total space. The total space has the induced G-action and γ_M is a real G-vector bundle over M. Let $\gamma_M^{\perp}$ denote the complementary G-vector bundle of γ_M in the product bundle $\varepsilon_M(V)$ with fiber V. Thus $\varepsilon_M(V) = \gamma_M \oplus \gamma_M^{\perp}$. Let $T(X)$ denote the tangent bundle of X. Then we have the next basic lemma.

Lemma 3.1. *The following equalities hold up to G-vector bundle isomorphisms.*

(1) $\mathrm{Hom}(\gamma_M, \gamma_M) = \varepsilon_M(\boldsymbol{R})$.
(2) $\mathrm{Hom}(\gamma_M, \varepsilon_M(\boldsymbol{R})) = \gamma_M$.
(3) $T(M) = \mathrm{Hom}(\gamma_M, \gamma_M^{\perp})$.
(4) $T(M) \oplus \varepsilon_M(\boldsymbol{R}) = \mathrm{Hom}(\gamma_M, \varepsilon_M(V))$.

(5) $\mathrm{Hom}(\gamma_M, \varepsilon_M(V)) = \gamma_M \otimes V$.

Proof. The equalities (1)–(4) above follow from the proof of [**20**, Lemma 4.4]. The equality (5) follows from

$$\mathrm{Hom}(\gamma_M, \varepsilon_M(V)) = \mathrm{Hom}(\gamma_M, \varepsilon_M(\boldsymbol{R})) \otimes V = \gamma_M \otimes V. \qquad \square$$

The lemma says that the tangent bundle $T(M)$ is stably isomorphic to $\gamma_M \otimes V - \varepsilon_M(\boldsymbol{R})$. By this, we immediately get the next lemma which is a key to constructing an $\mathfrak{S}$-realization of nonisomorphic real G-modules.

Lemma 3.2. *Let G be a finite group and set $K = G^{\mathrm{nil}}$. Let (U_1, U_2) be a $\mathscr{P}(G)$-matched pair of real G-modules such that $U_1^N = \boldsymbol{R}$ for all $N \in \mathscr{N}_2(G)$, and $U_2^K = 0$. Then the real projective space $M = P(U_1^K)$ and the real G-vector bundle $\xi_M = (\gamma_M \otimes U_1) \oplus (\gamma_M^\perp \otimes U_2)$, where $\gamma_M \oplus \gamma_M^\perp = \varepsilon_M(U_1^K)$, have the following properties,*

(1) *$M^G = \{x_0\}$ and $M^{=N} = M^N \smallsetminus \{x_0\}$ is a closed manifold (possibly the empty set) for any $N \in \mathscr{N}_2(G)$.*
(2) *$T(M) \oplus \varepsilon_M(\boldsymbol{R}) \cong \gamma_M \otimes U_1^K$.*
(3) *$T_{x_0}(M) \cong {U_1^K}_G \; (= U_1^K - U_1^G)$.*
(4) *$\xi_M|_{x_0} \cong T_{x_0}(M) \oplus \boldsymbol{R} \oplus U_{1K} \oplus ({U_1^K}_G \otimes U_2)$.*
(5) *$T(M)^K \oplus \varepsilon_M(\boldsymbol{R})^K \cong {\xi_M}^K$ as real G-vector bundles.*
(6) *ξ_M is $\mathscr{P}(G)$-matched to $\varepsilon_M(U_1^K \otimes U_2)$, i.e. $\mathrm{res}^G_P \xi_M \cong \mathrm{res}^G_P \varepsilon_M(U_1^K \otimes U_2)$ for all $P \in \mathscr{P}(G)$.*

4. Steps to construct $\mathfrak{S}$-realizations.

In this section, we give the outline of our construction of $\mathfrak{S}$-realizations of two real G-modules by describing lemmas in a step by step way.

Let $\mathscr{S}(G)/\mathrm{conj}$ denote the set of all conjugacy classes of subgroups of G. Let $\mathscr{K} = \{K_1, \ldots, K_c\}$ be a complete set of representatives of the conjugacy classes of proper subgroups of G, i.e. $K_i \neq G$. Thus, $\mathscr{S}(G)/\mathrm{conj} = \{(G), (K_1), \ldots, (K_c)\}$ with $c+1 = |\mathscr{S}(G)/\mathrm{conj}|$. As usual, we arrange $\mathscr{K}$ so that if $(K_i) \geq (K_j)$, namely K_j is subconjugate to K_i, then $i \leq j$. By this convention, we have $K_c = \{e\}$. Define a finite G-CW complex R by

$$R = \coprod_{i=1}^{c} G/K_i$$

and refer to R as the *set of reference points*.

If $|G| = p_1^{a_1} \cdots p_n^{a_n}$, where $p_1, \ldots, p_n$ are distinct primes and $a_1, \ldots, a_n \geq 1$, then we denote by $\mathrm{pow}(G)$ the maximum in the set $\{a_1, \dots, a_n\}$.

The first step is constructing a finite contractible G-CW complex Y including a given G-manifold M.

LEMMA 4.1. *Let G be an Oliver group and M a compact G-manifold with $x_0 \in M^G$. Then there exist a finite contractible G-CW complex Y and G-subcomplexes N_Y and Q_Y having the following properties.*

(1) $Y^G = M^G$.
(2) $\chi(Y^H) = 1$ *for all* $H \in \mathscr{M}(G)$.
(3) $N_Y \cap Q_Y = \emptyset$ *and* $Q_Y \supset R$.
(4) $\chi(N_Y^H \amalg Q_Y^H) = 1$ *for all* $H \in \mathscr{M}(G)$.
(5) *Each G-connected component of $Q_Y \setminus R$ is G-diffeomorphic to $G/K \times T$ for some $K \in \mathscr{M}(G)$ and a connected closed orientable 2-dimensional manifold T with the trivial G-action, or to G/K_j for some $K_j \in \mathscr{K}$.*
(6) $N_Y = M \amalg N_1 \amalg \cdots \amalg N_s$ *such that each N_i is G-diffeomorphic to $G/K_{j(i)} \times M$ for some $K_{j(i)} \in \mathscr{K}$.*
(7) $\mathrm{Iso}(G, Y \setminus (N_Y \cup Q_Y)) = \mathscr{P}(G)$.
(8) *For each $P \in \mathscr{P}(G)$, Y^P is simply connected.*
(9) $\dim Y^P > \dim Y^{P'}$ *for all P, $P' \in \mathscr{P}(G)$ with $P \subsetneq P'$.*
(10) $\dim Y = \max(\dim M, 2) + \mathrm{pow}(G) + 1$.

The second step is constructing a finite contractible G-CW complex Z with prescribed G-fixed point set and a real G-vector bundle η_Z over Z which will play like a stable tangent bundle of Z.

LEMMA 4.2. *Let G be an Oliver group, M a compact G-manifold with $x_0 \in M^G$, $\xi_M = \tau_M \oplus \nu_M$ a real G-vector bundle over M, and U a real G-module satisfying the following conditions.*

(i) $T(M) \oplus \varepsilon_M(\boldsymbol{R}^k) \cong \tau_M$.
(ii) $\nu_M^L \cong \varepsilon_{M^L}(0)$ *for all Dress subgroups $L = G^{\{q\}}$.*
(iii) ξ_M *is $\mathscr{P}(G)$-matched to $\varepsilon_M(\xi_M|_{x_0})$.*
(iv) U *is $\mathscr{P}(G)$-matched to $T_{x_0}(M)$.*

Then there exist a finite contractible G-CW complex Z, G-subcomplexes N_Z, Q_Z, and a real G-vector bundle η_Z over Z having the following properties.

(1) $Z^G = M^G$.
(2) $\chi(Z^K) = 1$ *for all* $K \in \mathscr{M}(G)$.
(3) $N_Z \cap Q_Z = \emptyset$ *and* $Q_Z \supset R$.
(4) $\chi(N_Z^H \amalg Q_Z^H) = 1$ *for all* $H \in \mathscr{M}(G)$.
(5) *Each G-connected component of $Q_Z \setminus R$ is G-diffeomorphic to $G/K \times T$ for*

some $K \in \mathscr{M}(G)$ and a connected closed orientable 2-dimensional manifold T with the trivial G-action, or to G/K_j for some $K_j \in \mathscr{K}$.

(6) $N_Z = M \amalg N_1 \amalg \cdots \amalg N_s$ *with G-diffeomorphisms $f_i : N_i \to G/K_{j(i)} \times M$ for some $K_{j(i)} \in \mathscr{K}$, $i = 1, \ldots, s$.*

(7) $\mathrm{Iso}(G, Z \smallsetminus (N_Z \cup Q_Z)) = \mathscr{P}(G)$.

(8) $T(Z^L \smallsetminus Q_Z) \cong \eta_Z^L|_{Z^L \smallsetminus Q_Z}$ *for all Dress subgroups $L = G^{\{q\}}$.*

(9) $\eta_Z|_M \cong T(M) \oplus \nu_M \oplus \varepsilon_M(\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus \dim Z})$.

(10) *For each N_i above, $\eta_Z|_{N_i} \cong f_i^*(G/K_{j(i)} \times (T(M) \oplus \nu_M)) \oplus \varepsilon_{N_i}(\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus \dim Z})$.*

(11) $\eta_Z|_{Q_Z} \cong \varepsilon_{Q_Z}(U \oplus \nu_M|_{x_0} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus \dim Z})$.

(12) *For each $P \in \mathscr{P}(G)$, $\pi_1(Z^P)$ is a finite abelian group of order prime to $|P|$.*

(13) $\dim Z = \max(\dim M, 2) + \mathrm{pow}(G) + 2$.

The third step is constructing a G-manifold D diffeomophic to a disk by equivariantly thickening Z with respect to η_Z.

LEMMA 4.3. *Let G be an Oliver group, M a compact G-manifold with $x_0 \in M^G$, $\xi_M = \tau_M \oplus \nu_M$ a real G-vector bundle over M, and U a real G-module satisfying Conditions* (i)–(iv) *in Lemma* 4.2. *Let Z, $N_Z = M \amalg N_1 \amalg \cdots \amalg N_s$, and $Q_Z \supset R$ be the G-CW complexes described in Lemma* 4.2. *Then there exists a disk D with a smooth G-action having the following properties.*

(1) $D^G = M^G$.

(2) $D \supset N_Z \cup (Q_Z^{(0)} \times D(U))$, *where $Q_Z^{(0)}$ is the union of 0-dimensional connected components of Q_Z.*

(3) $D^L = N_Z^L \cup (Q_Z^{(0)} \times D(U))^L$ *for all Dress subgroups $L = G^{\{q\}}$.*

(4) $T(D)|_M = T(M) \oplus \nu_M \oplus \varepsilon_M(\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(\dim Z+1)})$.

(5) *For each $P \in \mathscr{P}(G)$, $\pi_1(D^P)$ is a finite abelian group of order prime to $|P|$ and the inclusion induced map $j_\# : \pi_1(\partial D^P) \to \pi_1(D^P)$ is an isomorphism.*

Let (V_1, V_2) be a $\mathscr{P}(G)$-matched pair of real G-modules, $y_1 = 0 \in V_1$, and $y_2 = 0 \in V_2$. Applying the lemma above to the case $M = D(V_1) \amalg D(V_2)$, $\xi_M = \tau_M = T(M)$, $\nu_M = \varepsilon_M(0)$, and $U = V_1$, we immediately obtain the next corollary.

COROLLARY 4.4. *Let G be an Oliver group and (V_1, V_2) be a $\mathscr{P}(G)$-matched pair of real G-modules such that $V_1^G = 0$ and $V_2^G = 0$. Then there exists a disk $D(V_1, V_2)$ with a smooth G-action such that*

(1) $D(V_1, V_2) \supset D(V_1) \amalg D(V_2)$,

(2) $D(V_1, V_2)^G = \{y_1, y_2\}$, *and*

(3) $T(D(V_1, V_2))|_{D(V_1) \amalg D(V_2)} \cong (\varepsilon_{D(V_1)}(V_1) \amalg \varepsilon_{D(V_2)}(V_2)) \oplus \varepsilon_{D(V_1) \amalg D(V_2)} (\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(d+1)})$, *where $d = \max(\dim V_1, 2) + \mathrm{pow}(G) + 2$.*

(4) *For each* $P \in \mathscr{P}(G)$, $\pi_1(D(V_1, V_2)^P)$ *is a finite abelian group of order prime to* $|P|$ *and the inclusion induced map* $j_\# : \pi_1(\partial D(V_1, V_2)^P) \to \pi_1(D(V_1, V_2)^P)$ *is an isomorphism.*

Let M be a closed G-manifold and D a G-manifold diffeomorphic to a disk such that $D^G = M^G$. Let D' denote the m-fold cartesian product $D^{\times m}$ of D, where m is a positive integer. The fourth step is constructing a G-manifold D'' diffeomorphic to a disk such that $D''^G = \emptyset$ and $\partial(D'') = \partial(D')$ by a deleting theorem of G-fixed point sets. The union of D' and D'' glued along the boundary is a homotopy sphere Σ having the property $\Sigma^G = M^{\times m G}$.

Lemma 4.5. *Let G be a gap Oliver group, V a gap G-module, and m a positive integer. Let M be a closed G-manifold* (*hence,* $\partial M = \emptyset$) *with* $x_0 \in M^G$, $\xi_M = \tau_M \oplus \nu_M$, *and U a real G-module satisfying Conditions* (i)–(iv) *in Lemma* 4.2. *Let D be a disk with a smooth G-action satisfying the following conditions.*

(v) $D \supset M$ *as a G-submanifold and* $D^G = M^G$.
(vi) *For each* $L = G^{\{p\}}$, $D^L \smallsetminus M^L$ *is a closed subset of* D.
(vii) $T(D)|_M = T(M) \oplus \nu_M \oplus \varepsilon_M(E)$, *for an* $\mathscr{L}(G)$*-free real G-module* E.

Let W be an $\mathscr{L}(G)$-free real G-module. Then for any integers $a \geq m \dim D + \dim W + 3$ *and* $b \geq 3$, *there exists a homotopy sphere Σ with a smooth G-action having the following properties.*

(1) $\Sigma \supset M^{\times m}$ *as a G-submanifold.*
(2) $\Sigma^G = M^{\times m G}$.
(3) $T(\Sigma)|_{M^{\times m}} = (T(M) \oplus \nu_M \oplus \varepsilon_M(E))^{\times m} \oplus \varepsilon_{M^{\times m}}(W \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b})$.

Using the lemma above, we construct an $\mathfrak{S}$-realization of an appropriately given $\mathscr{P}(G)$-matched pair (V_1, V_3).

Lemma 4.6. *Let G be an Oliver group and V a gap real G-module. Set* $K = G^{\mathrm{nil}}$. *Let* (U_1, U_2), (U_3, U_4) *and* (V_1, V_3) *be* $\mathscr{P}(G)$*-matched pairs of real G-modules such that* $U_1^N = \boldsymbol{R}$ *and* $U_3^N = \boldsymbol{R}$ *for all* $N \in \mathscr{N}_2(G)$, $U_2^K = 0 = U_4^K$, $V_1 = (U_1 - \boldsymbol{R})^{\oplus m_1} \oplus W_1$, *and* $V_3 = (U_3 - \boldsymbol{R})^{\oplus m_3} \oplus W_3$, *where m_1 and m_3 are nonnegative integers and W_1 and W_3 are $\mathscr{L}(G)$-free real G-modules. Then there exist positive integers N_1 and N_2 such that for any integers* $a \geq N_1$ *and* $b \geq N_2$, *one has a smooth G-action on a standard sphere S having the following properties.*

(1) $S^G = \{y_1, y_3\}$.
(2) $T_{y_1}(S) \cong V_1 \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b}$.
(3) $T_{y_3}(S) \cong V_3 \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b}$.
(4) $\dim S^H \geq 6$ *for all* $H \in \mathscr{M}(G)$.

In the special case where $(U_3, U_4) = (U_1, U_2)$ and $m_3 = 0$, we have the next corollary.

Corollary 4.7. *Let G be a gap Oliver group and V a gap real G-module. Set $K = G^{\mathrm{nil}}$. Let (U_1, U_2) and (V_1, V_3) be $\mathscr{P}(G)$-matched pairs of real G-modules such that $U_1^N = \boldsymbol{R}$ for all $N \in \mathscr{N}_2(G)$, $V_1 = (U_1 - \boldsymbol{R})^{\oplus m_1} \oplus W_1$, $U_2^K = 0$, and V_3 and W_1 are $\mathscr{L}(G)$-free, where m_1 is a nonnegative integer. Then there exist positive integers N_1 and N_2 such that for arbitrary integers $a \geq N_1$ and $b \geq N_2$, one has a smooth G-action on a standard sphere S satisfying the following conditions.*

(1) $S^G = \{y_1, y_3\}$.
(2) $T_{y_1}(S) \cong V_1 \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b}$.
(3) $T_{y_3}(S) \cong V_3 \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b}$.
(4) $\dim S^H \geq 6$ *for all* $H \in \mathscr{M}(G)$.

The corollary above implies the next result.

Theorem 4.8. *Let G be a gap Oliver group and set $K = G^{\mathrm{nil}}$. Let (U_1, U_2) be a $\mathscr{P}(G)$-matched pair of real G-modules such that $U_1^N = \boldsymbol{R}$ for any $N \in \mathscr{N}_2(G)$ and $U_2^K = 0$. Then for $x = [U_1] - [U_2]$, the implication*

$$\left(\langle x - x^G \rangle_{\boldsymbol{Z}} + \mathrm{RO}(G)^{\mathscr{L}(G)}\right)_{\mathscr{P}(G)} \subseteq \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$$

holds.

We remark that the last implication formula also holds for $x = 0$ if G is a gap Oliver group.

5. Known basic facts.

As was seen in the previous section, our proof of Theorem 4.8 is based on certain knowledge of transformation group theory. For reader's convenience, we recall basic results on the real G-module $V(G) = \boldsymbol{R}[G]_{\mathscr{L}}$, a bundle subtraction lemma, an equivariant thickening theorem, and a deleting theorem of G-fixed point sets.

Lemma 5.1. *A real G-module V is $\mathscr{L}(G)$-free if and only if V is isomorphic to a submodule of $\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus m}$ for some integer m.*

Proof. This immediately follows from the fact that an arbitrary irreducible real G-module is isomorphic to a submodule of $\boldsymbol{R}[G]$. $\square$

Lemma 5.2 ([**15**, Theorem 2.3]). *Let G be a finite group not of prime power order. Then the following properties hold.*

(1) $\boldsymbol{R}[G]_{\mathscr{L}}{}^H \neq 0$ *if and only if* $H \in \mathscr{M}(G)$.
(2) $\dim \boldsymbol{R}[G]_{\mathscr{L}}{}^H \geq |K : H| \dim \boldsymbol{R}[G]_{\mathscr{L}}{}^K$ *if* $H \leq K \in \mathscr{S}(G)$.
(3) *Let* H, $K \in \mathscr{M}(G)$ *with* $H \leq K$. *Then* $\dim \boldsymbol{R}[G]_{\mathscr{L}}{}^H = 2 \dim \boldsymbol{R}[G]_{\mathscr{L}}{}^K$ *if and only if* $|K : H| = 2$, $|KG^{\{2\}} : HG^{\{2\}}| = 2$, *and* $HG^{\{p\}} = G$ *for all odd primes* p.

Lemma 5.3 ([**25**, Proposition 1.9]). *Let G be a finite group not of prime power order and $H \in \mathscr{M}(G)$.*

(1) *If* $|G : H| = p_1^{a_1} \cdots p_n^{a_n}$, $n \geq 2$, *for distinct primes* $p_1, \ldots, p_n$, *and* $a_1, \ldots, a_n \geq 1$, *then*

$$\dim \boldsymbol{R}[G]_{\mathscr{L}}{}^H \geq (p_1^{a_1} - 1) \cdots (p_n^{a_n} - 1).$$

(2) *If* $|G : H|$ *is a power of a prime* p *then* $\dim \boldsymbol{R}[G]_{\mathscr{L}}{}^H \geq p - 1$, *and furthermore, in the case* $p = 2$, $\dim \boldsymbol{R}[G]_{\mathscr{L}}{}^H > 2$.

Lemma 5.4 ([**25**, Proposition 2.3]). *Let G be a finite group not of prime power order. Then for each $H \in \mathscr{M}(G)$, any irreducible real H-module is isomorphic to a submodule of $\mathrm{res}_H^G \boldsymbol{R}[G]_{\mathscr{L}}$.*

Lemma 5.5 (Bundle Subtraction Lemma). *Let G be a finite group, V a real G-module, and W a real G-module such that for any $H \in \mathscr{M}(G)$, each irreducible component of $\mathrm{res}_H^G V$ is isomorphic to a submodule of $\mathrm{res}_H^G W$. Let (Z, X) be a finite G-CW pair ($Z \supseteq X$) such that $\mathrm{Iso}(G, Z \smallsetminus X) \subseteq \mathscr{M}(G)$ and let ℓ be an integer such that $\ell \geq \dim Z$. Let η_Z and ξ_X be real G-vector bundles over Z and X, respectively, such that*

(i) $\eta_Z|_X = \xi_X \oplus \varepsilon_X(V \oplus W^{\oplus \ell})$, *and*
(ii) $\eta_Z|_x \supset \mathrm{res}_{G_x}^G V$ (*as real G_x-modules*) *for all* $x \in Z$.

Then there exist a G-subbundle θ_Z of η_Z and a complementary G-subbundle ν_Z to θ_Z in η_Z, i.e. $\eta_Z = \theta_Z \oplus \nu_Z$, satisfying the following properties.

(1) $\theta_Z \cong \varepsilon_Z(V)$.
(2) $\theta_Z|_X = \varepsilon_X(V)$.
(3) $\nu_Z|_X = \xi_X \oplus \varepsilon_X(W^{\oplus \ell})$.

Proof. This follows from Proof of Theorem 2.2 in [**25**]. $\square$

THEOREM 5.6 (Equivariant Thickening Theorem). *Let G be a finite group. Let X be a compact G-manifold, and ν_X a real G-vector bundle over X such that $\nu_X^L = \varepsilon_{X^L}(0)$ for all Dress subgroups $L = G^{\{p\}}$. Let Z be a finite G-CW complex such that $X \subset Z$ and $\mathrm{Iso}(G, Z \smallsetminus X) \subseteq \mathscr{M}(G)$, and η_Z a real G-vector bundle over Z such that $\eta_Z|_X = T(X) \oplus \nu_X \oplus \varepsilon_X(W)$ for an $\mathscr{L}(G)$-free real G-module W. If the dimension conditions*

(a) $\dim \eta_Z|_x^H > 2 \dim Z^H$ *for all* $H \in \mathscr{M}(G)$ *and* $x \in Z^H$,
(b) $\dim \eta_Z|_x^H - \dim \eta_Z|_x^{>H} > \dim Z^H$ *for all* $H \in \mathscr{M}(G)$ *and* $x \in Z^H$, *and*
(c) $\dim \eta_Z|_x^P > \dim Z^P + 2$ *for all* $P \in \mathscr{P}(G)$ *and* $x \in Z^P$

are satisfied, then there exist a compact G-manifold $N \supset X$ and a strong G-deformation retraction $f : N \to Z$ having the following properties.

(1) *N contains Z as a G-subcomplex.*
(2) *N contains X as a G-submanifold.*
(3) $\mathrm{Iso}(G, N \smallsetminus X) = \mathscr{M}(G)$.
(4) $T(N) \cong f^*\eta_Z$ *(hence,* $T(N)|_Z \cong \eta_Z$ *and* $T(N)|_X \cong T(X) \oplus \nu_X \oplus \varepsilon_X(W)$*).*
(5) $\pi_0(\partial N^P) = \pi_0(N^P)$ *and* $\pi_1(\partial N^P, x) = \pi_1(N^P, x)$ *for all* $P \in \mathscr{P}(G)$ *and* $x \in \partial N^P$.

PROOF. See Proof of Theorem 3.1 in [**25**]. □

THEOREM 5.7 (Deleting Theorem). *Let G be an Oliver group and Y a smooth G-manifold diffeomorphic to a disk with exactly s G-fixed points $y_1, \ldots, y_s$, where $s \geq 1$. Suppose the following conditions.*

(1) $\dim Y^P > 2(\dim Y^H + 1)$ *for any* $P \in \mathscr{P}(G)$, $H \in \mathscr{S}(G)$ *with* $P \subsetneq H$.
(2) $\dim Y^{=H} \geq 3$ *for any* $H \in \mathscr{PC}(G)$, *where* $Y^{=H}$ *denotes the set of all points* y *in* Y *with* $G_y = H$.
(3) $\dim Y^P \geq 5$ *for any* $P \in \mathscr{P}(G)$.
(4) *$\pi_1(Y^P)$ is a finite group of order prime to $|P|$ for each $P \in \mathscr{P}(G)$.*
(5) *The inclusion induced map $\pi_1(\partial Y^P) \to \pi_1(Y^P)$ is an isomorphism for each $P \in \mathscr{P}(G)$.*
(6) *The connected component Y_i^L of Y^L containing y_i is a closed manifold for each $L \in \mathscr{L}(G)$ and each i with $1 \leq i \leq s$.*

Then there exists a smooth G-manifold X diffeomorphic to the disk such that $X^G = \emptyset$ and ∂X is G-diffeomorphic to ∂Y.

PROOF. This follows from Theorem 1.3 of [**23**]. □

6. Proof of Lemma 4.1.

For a finite G-CW complex X, define $\overline{\chi}(X)$ to be the number $\chi(X)-1$, where $\chi(X)$ is the Euler characteristic of X. If H is a subgroup of G then $\chi_H(X)$ and $\overline{\chi}_H(X)$ denote the numbers $\chi(X^H)$ and $\chi(X^H)-1$, respectively. Let $\Omega(G)$ denote the Burnside ring, cf. [**8**], [**21**]. Each element $x \in \Omega(G)$ has the form $[X_1]-[X_2]$ with finite G-CW complexes (or finite G-sets) X_1 and X_2. For each subgroup H of G, we define the homomorphism $\chi_H : \Omega(G) \to \boldsymbol{Z}$ using the Euler characteristic: $\chi_H(x) = \chi(X_1^H) - \chi(X_2^H)$. By definition, $[X_1]-[X_2] = [Y_1]-[Y_2]$ holds if and only if $\chi_H(X_1)-\chi_H(X_2) = \chi_H(Y_1)-\chi_H(Y_2)$ for all subgroups H of G. By Theorem 1.3 of [**15**], we have the next lemma.

Lemma 6.1. *If G is a finite group, then there exists an element $\beta \in \Omega(G)$ such that $\chi_G(\beta) = 0$ and $\chi_H(\beta) = 1$ whenever $H \in \mathscr{M}(G)$.*

Let G be an Oliver group and let $\beta = \sum_{i=1}^{c} b_i[G/K_i]$ be an element given in Lemma 6.1. Then take an element $(-\beta)^{\%} = \sum_{i=1}^{c} b_i'[G/K_i]$ in $\Omega(G)$ such that $b_i' \geq 0$ and

$$b_i' \equiv -b_i \mod 2|G||\widetilde{K}_0(\boldsymbol{Z}[G])|.$$

For finite G-CW complexes X and Y with reference points x_0 and y_0, respectively, having a same isotropy subgroup H, let $X \vee_{G/H} Y$ denote the equivariant wedge sum, namely the union of X and Y identified gx_0 with gy_0 for each $g \in G$. If X has the reference point x_0 of isotropy subgroup H then we regard (eH, x_0) as the reference point of the G-space $G/H \times X$ with the diagonal G-action. Then the isotropy subgroup of (eH, x_0) is H. Take the equivariant wedge sum $X \vee_{G/H} (G/H \times X)$ and denote this space by $([G/G]+[G/H]) \circ X$ for the sake of convenience. It holds that

$$\overline{\chi}_H(X \vee_{G/K} ((G/K) \times X)) = \overline{\chi}_H(X) + |(G/K)^H|\overline{\chi}_H(X).$$

If $X \supset R \ (= \coprod_{i=1}^{c} G/K_i)$, the set of reference points, then we denote by $([G/G]+(-\beta)^{\%}) \circ X$ the space obtained by iterating wedge sum operation on X associated with $(-\beta)^{\%}$. Then we have

$$\begin{aligned} \overline{\chi}_H\big(([G/G]+(-\beta)^{\%}) \circ X\big) &= \big(1+\chi_H((-\beta)^{\%})\big)\overline{\chi}_H(X) \\ &\equiv \big(1-\chi_H(\beta)\big)\overline{\chi}_H(X) \mod 2|G|. \end{aligned}$$

If $H \in \mathscr{S}(G)$ then $(G/K \times X)^H = (G/K)^H \times X^H$, and hence if $(H) > (K)$ then

$(G/K \times X)^H = \emptyset$.

Let M be a compact G-manifold. Set

$$Y_0 = \big([G/G] + (-\beta)^{\%}\big) \circ (M \amalg R).$$

Let Q_{Y_0} denote the subset of Y_0 obtained as $([G/G] + (-\beta)^{\%}) \circ R$. Let $N_{Y_0} = Y_0 \smallsetminus Q_{Y_0}$. If i is the smallest integer such that $K_i \in \mathscr{M}(G)$ and $\overline{\chi}_{K_i}(Y_0) \neq 0$ then $\overline{\chi}_{K_i}(Y_0)$ is divisible by $2|G|$, and hence by $2|N_G(K_i)/K_i|$. Thus there exists a finite G-CW complex Y_1 such that $Y_1 = Y_0 \amalg (G/K_i \times T) \amalg \cdots \amalg (G/K_i \times T)$ for some connected closed orientable 2-dimensional manifold T (with the trivial G-action) and $\overline{\chi}_{K_i}(Y_1) = 0$. We set $Q_{Y_1} = Q_{Y_0} \amalg (Y_1 \smallsetminus Y_0)$ and $N_{Y_1} = N_{Y_0}$. Performing subsequently this procedure, we obtain a finite G-CW complex $Y_2 = N_{Y_2} \amalg Q_{Y_2}$ satisfying the following conditions.

(1) $Y_2^G = M^G$.
(2) $\overline{\chi}_H(Y_2) = 0$ for all $H \in \mathscr{M}(G)$.
(3) $Q_{Y_2} \supset R$.
(4) Each G-connected component of Q_{Y_2} is G-diffeomorphic to $G/K \times T$ for some $K \in \mathscr{M}(G)$ and a connected closed orientable 2-dimensional manifold T with the trivial G-action, or to G/K_j for some $K_j \in \mathscr{K}$.
(5) $N_{Y_2} = M \amalg N_1 \amalg \cdots \amalg N_\ell$ such that for each i, $N_i \cong_G G/K_{j(i)} \times M$ for some $j(i)$.

By the same argument as in [**28**] (alternatively [**30**]), we can obtain a finite G-CW complex Y_3 containing Y_2 such that $\mathrm{Iso}(G, Y_3 \smallsetminus Y_2) \subseteq \mathscr{P}(G)$, and Y_3^P is simply connected as well as $\boldsymbol{Z}_p$-acyclic for every $P \in \mathscr{P}(G)$ with $P \neq \{e\}$, where p is the prime dividing $|P|$. We can also obtain a finite G-CW complex Y_4 containing Y_3 such that $Y_4 \smallsetminus Y_3$ consists of free cells, namely the isotropy type is $\{e\}$, Y_4 is 1-connected, $\dim Y_4 \geq 2$, and $H_i(Y_4, \{x_0\}; \boldsymbol{Z}) = 0$ for all $i < \dim Y_4$. Set $n = \dim Y_4$. Then by Nakayama's theorem, $H_n(Y_4; \boldsymbol{Z})$ is a projective module over $\boldsymbol{Z}[G]$. For $Y_5 = ([G/G] + (-\beta)^{\%}) \circ Y_4$, $H_n(Y_5; \boldsymbol{Z})$ is a stably free module over $\boldsymbol{Z}[G]$. Hence by attaching free cells of dimension n and $n+1$ to Y_5, we can obtain a finite contractible G-CW complex Y. Set $Q_Y = ([G/G] + (-\beta)^{\%}) \circ Q_{Y_2}$. Define N_Y to be the G-manifold contained in Y_5 which is generated by N_{Y_2} via the wedge sum operation on Y_4 associated with $[G/G] + (-\beta)^{\%}$. Then these Y, N_Y, Q_Y satisfy the desired conditions.

7. Proof of Lemma 4.2.

Let G, M, $\xi_M = \tau_M \oplus \nu_M$, $x_0 \in M^G$ be as in Lemma 4.2. Clearly, we have $\xi_M|_{x_0} = \tau_M|_{x_0} \oplus \nu_M|_{x_0}$. Let Y, $R = \coprod_{i=1}^c G/K_i$, Q_Y and N_Y be as in

Lemma 4.1. For each N_i, $1 \leq i \leq s$, define a real G-vector bundle ξ_{N_i} by $\xi_{N_i} = G/K \times \xi_M$ using K such that $N_i \cong G/K \times M$. Set $\xi_{N_Y} = \xi_M \cup \bigcup_{i=1}^{s} \xi_{N_i}$, $\xi_{Q_Y} = \varepsilon_{Q_Y}(U \oplus \boldsymbol{R}^k \oplus \nu_M|_{x_0})$, and $X = N_Y \cup Q_Y$. Then the real G-vector bundle $\xi_X = \xi_{N_Y} \cup \xi_{Q_Y}$ over X has the following properties.

(a) ξ_X has the form $\xi'_X \oplus \varepsilon_X(\boldsymbol{R}^k)$.
(b) $\mathrm{res}^G_P \xi_X = 0$ in $\widetilde{KO}_P(X)$ for all $P \in \mathscr{P}(G)$.
(c) $\chi(X^H) = \chi(Y^H)$ for all $H \in \mathscr{S}(G)$.
(d) $\chi(X^H) = \chi(Y^H) = 1$ for all $H \in \mathscr{M}(G)$.

Let $B_G O$ and $B^*_G O$ be the G-spaces and $L_G : B_G O \to B^*_G O$ be the G-map defined in [**27**]. Let $f_X : X \to B_G O$ denote the classifying map of ξ_X. Then $g_X = L_G \circ f_X$ is G-homotopic to a constant map. Thus g_X extends to a G-map $g_Y : Y \to B^*_G O$ which is G-homotopic to a constant map.

We wish to lift g_Y to a G-map $Y \to B_G O$, although it is impossible in general. Hence we need some modification. Observe the G-homotopically commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_X"] \arrow[d, "\varphi_X"'] & B_G O \arrow[d, "L_G"] \\ Y \arrow[r, "g_Y"'] & B^*_G O. \end{tikzcd}$$

Diagram (D1)

By Proposition 2.3 of [**27**], Diagram (D1) extends to a G-homotopically commutative diagram

$$\begin{tikzcd} Z \arrow[r, "f_Z"] \arrow[d, "\varphi_Z"'] & B_B O \arrow[d, "LG"] \\ Y \arrow[r, "g_Y"'] & B^*_G O, \end{tikzcd}$$

Diagram (D2)

where Z is a finite contractible G-CW complex containing X with $\mathrm{Iso}(G, Z \smallsetminus X) \subseteq \mathscr{P}(G)$, and f_Z and φ_Z are extensions of f_X and φ_X, respectively. Furthermore, we can obtain Z so that $\pi_1(Z^P)$ is a finite abelian group of order prime to $|P|$ for each $P \in \mathscr{P}(G)$. This fact follows from that $\pi_1(Y^P)$ is trivial and $\mathrm{Ker}(\pi_1(\beta\alpha_1))$ appearing in Proof, Finite Case of [**27**, Lemma 2.2] is finite abelian of order prime to p (see Proof, Finite Case, Step 1 of [**27**, Proposition 2.3], too). Here we can choose Z so that $\dim Z = \dim Y + 1$. Define N_Z and Q_Z by $N_Z = N_Y$ and $Q_Z = Q_Y$.

Let ω_Z be a real G-vector bundle over Z associated with f_Z. By Lemma 5.1, each $\mathscr{L}(G)$-free irreducible real G-module is isomorphic to a submodule of $\boldsymbol{R}[G]_{\mathscr{L}}$. We can take ω_Z so that

$$\omega_Z|_X = \xi_X \oplus \varepsilon_X\bigl(V \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus \ell}\bigr)$$

for some real G-module V and some integer ℓ. Here we may suppose $\ell \geq \dim Z$. Since $\mathrm{Iso}(G, Z \smallsetminus X) \subseteq \mathscr{P}(G)$ and Z^P is connected for every $P \in \mathscr{P}(G)$, we see

$$\omega_Z|_x \supseteq \mathrm{res}_{G_x}^G\bigl(\boldsymbol{R}^k \oplus V \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(\ell - \dim Z)}\bigr)$$

for all $x \in Z$. By Bundle Subtraction Lemma (Lemma 5.5), there exists an actual G-subbundle θ_Z of ω_Z such that $\theta_Z \cong \varepsilon_Z(\boldsymbol{R}^k \oplus V \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(\ell - \dim Z)})$ and

$$\eta_Z|_X \cong \xi'_X \oplus \varepsilon_X\bigl(\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus \dim Z}\bigr), \tag{7.1}$$

where η_Z is the complementary bundle of θ_Z in ω_Z, i.e. $\omega_Z = \theta_Z \oplus \eta_Z$, These Z, N_Z, Q_Z and η_Z are desired ones in Lemma 4.2.

8. Proofs of Lemmas 4.3 and 4.5.

Let G be an Oliver group, M a compact G-manifold with $x_0 \in M^G$, $\xi_M = \tau_M \oplus \nu_M$ a real G-vector bundle over M, and U a real G-module satisfying (i)–(iv) in Lemma 4.2. Let Z, $N_Z = M \amalg N_1 \amalg \cdots \amalg N_s$, $Q_Z \supset R$ and η_Z be those stated in Lemma 4.2. Set

$$\eta'_Z = \eta_Z \oplus \varepsilon_Z(\boldsymbol{R}[G]_{\mathscr{L}}).$$

Using Lemmas 5.2 and 5.3, we can check that η'_Z satisfies the dimension condition (a)–(c) in Theorem 5.6 for η_Z replaced by η'_Z.

Proof of Lemma 4.3. Set $X = N_Z \amalg (Q_Z^{(0)} \times D(U))$. Note that X equivariantly simply collapses to $N_Z \amalg Q_Z^{(0)}$. In addition, $\eta_Z|^L_{N_Z \cup Q_Z^{(0)}} = T(N_Z^L) \amalg T((Q_Z^{(0)})^L \times D(U)^L)|_{(Q_Z^{(0)})^L}$ for all Dress subgroups $L = G^{\{p\}}$. Now use Equivariant Thickening Theorem (Theorem 5.6) for the initial manifold X and the real G-vector bundle η'_Z over Z, instead of η_Z, and obtain a disk D as stated in Lemma 4.3. □

Proof of Lemma 4.5. Let D be the disk with a G-action, V the gap G-module, E and W the real G-modules, and a and b integers stated in Lemma 4.5. Then the disk $D_1 = D^{\times m} \times D(W \oplus V^{\oplus a})$ satisfies the strong gap condition

$$\dim D_1^P > 2\big(\dim D_1^H + 1\big)$$

for all $P \in \mathscr{P}(G)$, $H \in \mathscr{S}(G)$ with $P \subsetneq H$. Thus $D_2 = D^{\times m} \times D(W \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b})$ satisfies the following conditions:

(1) $\dim D_2^H \geq 6$ for all $H \in \mathscr{M}(G)$.
(2) $\dim D_2^P > 2(\dim D_2^H + 1)$ for all $P \in \mathscr{P}(G)$, $H \in \mathscr{S}(G)$ with $P \subsetneq H$.
(3) $D_2 \supseteq M^{\times m} \supset {M^{\times m}}^G \ni x_1 = (x_0, \ldots, x_0)$.
(4) $T(D_2)|_{M^{\times m}} \cong (T(M) \oplus \nu_M \oplus \varepsilon_M(E))^{\times m} \oplus \varepsilon_{M^{\times m}}(W \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b})$.

Now we are ready to use Deleting Theorem (Theorem 5.7). We can use D_2 as Y of Deleting Theorem to obtain a smooth G-action on a disk D_3 such that $D_3^G = \emptyset$ and $\partial D_3 = \partial D_2$. The union $\Sigma = D_2 \cup_\partial D_3$ glued along the boundary is a homotopy sphere. □

We close this section with the next proposition.

Proposition 8.1. *The homotopy sphere Σ above can be converted to the standard sphere having the desired properties in Lemma* 4.5.

Proof. Let Σ be as above. Note for each Sylow subgroup P of G with $|P| = q^a > 1$, the set $\Sigma^{=P}$ is a connected open dense subset of the $\boldsymbol{Z}_q$-homology sphere Σ^P of dimension ≥ 6. By [**16**, Proposition 2.1], taking an equivariant connected sum of copies of Σ, we obtain a smooth G-action on the sphere S such that $\dim S = \dim \Sigma$, $S^G = \Sigma^G$, and the normal bundle $\nu(S^G, S)$ is G-isomorphic to the normal bundle $\nu(\Sigma^G, \Sigma)$. This S satisfies the properties required in Lemma 4.5 in place of Σ. □

9. Proof of Lemma 4.6.

Let G be an Oliver group with a gap G-module V. Let (U_1, U_2), (U_3, U_4) and (V_1, V_3) be the real $\mathscr{P}(G)$-matched pairs described in Lemma 4.6. We note that the dimension of each of these G-modules is greater than or equal to 3.

For each $i = 1$, 3, let $M_i = P(U_i^K)$, $\tau_{M_i} = \gamma_{M_i} \otimes U_i^K$, $\nu_{M_i} = (\gamma_{M_i} \otimes U_{iK}) \oplus (\gamma_{M_i}^\perp \otimes U_{i+1})$, where $\gamma_{M_i} \oplus \gamma_{M_i}^\perp = \varepsilon_{M_i}(U_i^K)$, and $\xi_{M_i} = \tau_{M_i} \oplus \nu_{M_i}$. Since U_{i+1} is $\mathscr{L}(G)$-free, we have $\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus n_i} = ({U_i^K}_G \otimes U_{i+1}) \oplus A_{i+1}$ for some positive integer n_i and an $\mathscr{L}(G)$-free real G-module A_{i+1}. By Lemma 3.2, $T(M_i) \oplus \varepsilon_{M_i}(\boldsymbol{R}) \cong \tau_{M_i}$. Using these data, we obtain a finite contractible G-CW complex Z_i ($\supset M_i$) such that $\dim Z_i = d_i + \mathrm{pow}(G) + 2$ with $d_i = \dim U_i$, and a real G-vector bundle η_{Z_i} described in Lemma 4.2. Apply Lemma 4.3 for these Z_i, η_{Z_i} and $U = {U_i^K}_G$ to obtain a disk D_i ($\supset M_i$) with a G-action having Properties (1)–(5) in Lemma 4.3. In particular,

$$T(D_i)|_{M_i} = T(M_i) \oplus (\gamma_{M_i} \otimes U_{iK}) \oplus (\gamma_{M_i}^{\perp} \otimes U_{i+1}) \oplus \varepsilon_{M_i}(E_i),$$

where

$$E_i = \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(d_i+\mathrm{pow}(G)+3)}.$$

Apply Lemma 4.5 for the disk D_i and the integer m_i to obtain a homotopy sphere Σ_i ($\supset M_i^{\times m_i}$) with a G-action stated in Lemma 4.5 such that

$$\begin{aligned} T(\Sigma_i)\big|_{M_i^{\times m_i}} = {} & \big(T(M_i) \oplus (\gamma_{M_i} \otimes U_{iK}) \oplus (\gamma_{M_i}^{\perp} \otimes U_{i+1}) \oplus \varepsilon_{M_i}(E_i)\big)^{\times m_i} \\ & \oplus \varepsilon_{M_i^{\times m_i}}\big(A_{i+1}^{\oplus m_i} \oplus W_i \oplus V^{\oplus a_i} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b_i}\big), \end{aligned}$$

where $a_i \geq m_i(d_i - 1 + n_i r) + m_i r(d_i + \mathrm{pow}(G) + 3) + \dim W_i + 3$ with $r = \dim \boldsymbol{R}[G]_{\mathscr{L}}$ and $b_i \geq 3$ can be arbitrarily chosen. Let x_i $(i = 1, 3)$ be the unique G-fixed point of Σ_i. Then we have

$$T_{x_i}(\Sigma_i) \cong V_i \oplus E_i^{\oplus m_i} \oplus V^{\oplus a_i} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(m_i n_i + b_i)}.$$

Thus there exist positive integers N_1 and N_2 such that for arbitrary $a \geq N_1$, $b \geq N_2$, we have one-fixed-point G-actions on spheres Σ_1 and Σ_3 such that

$$T_{x_i}(\Sigma_i) \cong V_i \oplus V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus b}.$$

Let $M_1' = D(V_1)$, $M_3' = D(V_3)$, $M' = M_1' \amalg M_3'$, $\tau_{M_1'} = \varepsilon_{M_1'}(V_1)$, $\tau_{M_3'} = \varepsilon_{M_3'}(V_3)$, $\tau_{M'} = \tau_{M_1'} \amalg \tau_{M_3'}$, $\nu_{M'} = \varepsilon_{M'}(0)$, and $\xi_{M'} = \tau_{M'}$. Then there exists a G-action on a disk $D(V_1, V_3)$ described in Corollary 4.4. Let y_1 and y_3 denote origins in V_1 and V_3, respectively. The G-fixed points of $D(V_1, V_3)$ are y_1 and y_3. It holds that

$$T(D(V_1, V_3))|_{M_1' \amalg M_3'} \cong (\varepsilon_{M_1'}(V_1) \amalg \varepsilon_{M_3'}(V_3)) \oplus \varepsilon_{M_1' \amalg M_3'}\big(\boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(d+1)}\big)$$

with $d = \max(\dim V_1, 2) + \mathrm{pow}(G) + 2$.

We may assume $N_2 \geq d + 1$. Then let

$$\Delta = D(V_1, V_3) \times D\big(V^{\oplus a} \oplus \boldsymbol{R}[G]_{\mathscr{L}}^{\oplus(b-d-1)}\big)$$

and take the double $\Sigma_5 = \Delta \cup_\partial \Delta'$ (a sphere) of Δ, where Δ' is a copy of Δ. Obviously, we have $\Sigma_5^G = \{y_1, y_3, y_1', y_3'\}$, $T_{y_1}(\Sigma_5) \cong T_{y_1'}(\Sigma_5) \cong T_{x_1}(\Sigma_1)$, $T_{y_3}(\Sigma_5) \cong T_{y_3'}(\Sigma_5) \cong T_{x_3}(\Sigma_3)$. Thus we can construct the G-connected sum Σ of Σ_5 with the spheres Σ_1 and Σ_3 at the point data (y_1', x_1) and (y_3', x_3). Then $\Sigma^G = \{y_1, y_3\}$. By [**16**, Proposition 1.3], the homotopy sphere Σ can be modified to a G-manifold diffeomorphic to the standard sphere, without changing the local data around y_1 and y_3. The sphere S has the desired properties.

10. Proofs of Theorems 1.3 and 1.5.

Now we are ready to prove our main theorem.

Proof of Theorem 1.3. Let G be a finite group satisfying the hypotheses in Theorem 1.3. Set $N = G^{\mathrm{nil}}$. Since $|G/N|$ is odd, $\boldsymbol{R}[G]_{\mathscr{L}}$ is a gap G-module. For N has a subquotient group isomorphic to D_{2qr}, there exists a $\mathscr{P}(N)$-matched pair (W_1, W_2) of type 1 consisting of real N-modules. Let $U_1 = \mathrm{ind}_N^G W_1$ and $U_2 = \mathrm{ind}_N^G W_2$. Then (U_1, U_2) is a $\mathscr{P}(G)$-matched pair of type 1, $U_1^N = \boldsymbol{R}[G/N]$, and $U_2^N = 0$. By Lemma 2.4, there exists a $\mathscr{P}(G)$-matched pair (M_1, M_2) such that $[M_1^N] - [M_2^N] = m[\boldsymbol{R}[G/N]] - m|G/N|[\boldsymbol{R}]$ for some positive integer m. Then

$$\begin{aligned}
&\big([M_1^N] - [M_2^N]\big) + (m|G/N| - m)\big([U_1^N] - [U_2^N]\big) \\
&\quad = m[\boldsymbol{R}[G/N]] - m|G/N|[\boldsymbol{R}] + (m|G/N| - m)[\boldsymbol{R}[G/N]] \\
&\quad = m|G/N|([\boldsymbol{R}[G/N]] - [\boldsymbol{R}]) \\
&\quad = m|G/N|[\boldsymbol{R}[G/N] - \boldsymbol{R}[G/N]^G].
\end{aligned}$$

Thus there exists a $\mathscr{P}(G)$-matched pair (V_1, V_2) such that $V_1^N = (\boldsymbol{R}[G/N] - \boldsymbol{R}[G/N]^G)^{\oplus n}$ and $V_2^N = 0$, where n is a positive integer. Set $x = [V_1] - [V_2]$. Replacing (V_1, V_2) if necessary, we may suppose that if $y \in \mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)}$ satisfies $ky = x$ for an integer k then $k = 1$ or -1. Namely the element x is a basis element of $\mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)}$. By Theorem 4.8, we have $\langle x \rangle_{\boldsymbol{Z}} \subseteq \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$. If additionally $a_G = 2$, then $\mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)} = \langle x \rangle_{\boldsymbol{Z}}$. Since $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \subseteq \mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)}$, we conclude $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G)^{\{G\}}_{\mathscr{P}(G)}$. □

Remark 10.1. By a little further work, we can see the following. Let G be a gap Oliver group containing a subgroup K with the following properties. Set $N = K^{\mathrm{nil}}$.

(1) K is an Oliver group
(2) N has a subquotient group isomorphic to a dihedral group D_{2qr} of order $2pq$ with distinct primes q and r.
(3) K/N is a nontrivial group of odd order.
(4) $K \smallsetminus N$ contains an element not of prime power order, i.e. $|\overline{\mathscr{P}}(K \smallsetminus N)| > 0$.
(5) $|\overline{\mathscr{P}}(gN)| = |\overline{\mathscr{P}}(g'N)|$ for all g, $g' \in K \smallsetminus N$.

Then $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ contains an element $x = [V]-[W]$ such that $\dim V^N \neq \dim W^N$, and hence $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \neq 0$.

Proof of Theorem 1.5. Let x be an element in $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)}$. We have a $\mathscr{P}(G)$-matched pair (V_1, V_2) such that $x = [V_1]-[V_2]$ and V_1 and V_2 are $\mathscr{L}(G)$-free. By hypothesis, G has a gap real G-module V. By Lemma 5.4, any irreducible real H-module, where $H \in \mathscr{M}(G)$, is isomorphic to a submodule of $\mathrm{res}^G_H \boldsymbol{R}[G]_{\mathscr{L}}$. By [**35**, Theorem 4.1] or Corollary 4.4, $V_1 \oplus (V \oplus \boldsymbol{R}[G]_{\mathscr{L}})^{\oplus h}$ and $V_2 \oplus (V \oplus \boldsymbol{R}[G]_{\mathscr{L}})^{\oplus h}$ are $\mathfrak{D}$-related whenever h is sufficiently large. Moreover, by [**35**, Theorem 4.3], the real G-modules $V_1 \oplus (V \oplus \boldsymbol{R}[G]_{\mathscr{L}})^{\oplus k}$ and $V_2 \oplus (V \oplus \boldsymbol{R}[G]_{\mathscr{L}})^{\oplus k}$ are $\mathfrak{S}$-related whenever k is sufficiently large. Thus the real G-modules $V_1 \oplus (V \oplus \boldsymbol{R}[G]_{\mathscr{L}})^{\oplus \ell}$ and $V_2 \oplus (V \oplus \boldsymbol{R}[G]_{\mathscr{L}})^{\oplus \ell}$ are $\mathfrak{D}\mathfrak{S}$-related whenever ℓ is sufficiently large.

If G^{nil} contains distinct two real conjugacy classes of elements not of prime power order, then by Lemma 2.1 we have the nontriviality $\mathrm{RO}(G)^{\mathscr{L}(G)}_{\mathscr{P}(G)} \neq 0$. □

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