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Commutators of C^∞-diffeomorphisms preserving a submanifold

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Abstract. We consider the group of C^∞-diffeomorphisms of M which is isotopic to the identity through C^∞-diffeomorphisms preserving N for a compact manifold pair (M, N) and prove that the group is perfect. Also we prove that it is uniformly perfect for a certain compact manifold with boundary.

1. Introduction and statement of results.

Let M be a connected C^∞-manifold without boundary and let $D_c^\infty(M)$ denote the group of all C^∞-diffeomorphisms of M which are isotopic to the identity through C^∞-diffeomorphisms with compact support. It is well known by the results of M. Herman [**8**] and W. Thurston [**16**] that $D_c^\infty(M)$ is perfect, that is, every element of $D_c^\infty(M)$ is represented by a product of commutators. There are many analogous results on the group of diffeomorphisms preserving a geometric structure of M.

In this paper we consider the relative case. Let M be an m-dimensional connected C^∞-manifold and N a proper n-dimensional C^∞-submanifold and let $D_c^\infty(M, N)$ denote the group of all C^∞-diffeomorphisms of M which are isotopic to the identity through C^∞-diffeomorphisms preserving N with compact support.

The first purpose of this paper is to prove the perfectness of $D_c^\infty(M, N)$. We have the following.

Theorem 1.1. *$D_c^\infty(M, N)$ is perfect for $n \geq 1$.*

Corollary 1.2. *Let M be an m-dimensional C^∞-manifold with boundary. Then $D_c^\infty(M, \partial M)$ is perfect for $m \geq 2$.*

By the fragmentation argument, the proof of Theorem 1.1 is reduced to prove

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the following.

Theorem 1.3. *$D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is perfect for $n \geq 1$.*

In the case $n = 0$, the second author [**6**] proved that $H_1(D_c^\infty(M, N)) \cong \boldsymbol{R} \times \cdots \times \boldsymbol{R}$ (k times) for $N = \{p_1, \cdots, p_k\}$. Here $H_1(D_c^\infty(M, N))$ is defined by the quotient group of $D_c^\infty(M, N)$ by its commutator subgroup. In [**1**], [**2**] we treated the case when M has a smooth action of a compact Lie group, and calculated the first homology group of the equivariant diffeomorphism group of M.

The second purpose of this paper is to study the uniform perfectness of $D_c^\infty(M, N)$. Since $D_c^\infty(M)$ is perfect by Thurston [**16**], each element f of $D_c^\infty(M)$ can be represented as a product $\prod_{i=1}^k [g_{2i-1}, g_{2i}]$, where $g_i \in D_c^\infty(M)$. If every element of $D_c^\infty(M)$ can be represented as a product of a bounded number k of commutators of its elements, then the group is said to be *uniformly perfect*. For example, it is known that $D^\infty(S^1)$ is uniformly perfect. In fact $D^\infty(S^1)$ is represented as a product of at most two commutators (M. Herman [**9**]).

Recently T. Tsuboi [**18**] has studied the uniform perfectness of $\mathrm{Diff}_c^r(M)$ and proved that it is uniformly perfect if $1 \leq r \leq \infty$ and $r \neq \dim M + 1$ and M belongs to a certain wide class of manifolds. In [**3**] Burago, Ivanov and Polterovich, they obtained the remarkable results on the uniform perfectness of $\mathrm{Diff}_c^r(M)$ achieved with excellent methods. On the other hand, if we consider the diffeomorphisms of M preserving a geometric structure of M, then there exist certain cases that the group is not uniformly perfect (c.f. J. Gambaudo-É. Ghys [**7**], M. Entov [**4**]).

From Corollary 1.2, each element f of $D_c^\infty(M, \partial M)$ can be represented as a product of commutators. Then we can prove the following uniform perfectness for $D_c^\infty(M, \partial M)$.

Theorem 1.4. *Let M be an m-dimensional compact manifold with boundary such that both groups $D_c^\infty(\mathrm{int}M)$ and $D^\infty(\partial M)$ are uniformly perfect. Then $D^\infty(M, \partial M)$ is a uniformly perfect group for $m \geq 2$.*

2. Basic lemmas and the group of leaf preserving diffeomorphisms.

In this section, we prepare basic lemmas and a result which are necessary to prove Theorem 1.3.

Let $G : \boldsymbol{R}^n \to GL(m-n, \boldsymbol{R})$ be a C^∞-mapping satisfying that G is C^1-close to the constant mapping $e : \boldsymbol{R}^n \to GL(m-n, \boldsymbol{R})$, $x \mapsto I_{m-n}$ and the support of G is contained in the open ball $B_\delta^n = \{x \in \boldsymbol{R}^n \mid \|x\| < \delta\}$. Then we have the following by Lemma 4 of [**1**].

Lemma 2.1. *There exist C^∞-mappings $G_i : \boldsymbol{R}^n \to GL(m-n, \boldsymbol{R})$ and*

$\varphi_i \in D_c^\infty(\boldsymbol{R}^n)$ $(i = 1, 2, \cdots, q = (m-n)^2)$ *satisfying that*

(1) *each* φ_i *is* C^1*-close to the identity and is supported in* B_δ*,*

(2) *each* G_i *is supported in* $B^n_{2\sqrt{3}\delta}$ *and is* C^1*-close to* e*, and*

(3) $G = (G_1^{-1} \cdot (G_1 \circ \varphi_1)) \cdot \cdots \cdot (G_q^{-1} \cdot (G_q \circ \varphi_q))$.

Let $\mathscr{F}_0$ be the product foliation of $\boldsymbol{R}^m$ with leaves of form $\{\boldsymbol{R}^n \times \{y\}\}$ where (x, y) is a coordinate of $\boldsymbol{R}^m = \boldsymbol{R}^n \times \boldsymbol{R}^{m-n}$. By $D^\infty_{L,c}(\boldsymbol{R}^m, \mathscr{F}_0)$ we denote the group of leaf preserving C^∞-diffeomorphisms of $(\boldsymbol{R}^m, \mathscr{F}_0)$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms with compact support. Then we have the following splitting lemma.

LEMMA 2.2 (Splitting lemma). *Suppose that* $f \in D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *is* C^2*-close to* $1_{\boldsymbol{R}^m}$ *and the support of* f *is contained in* B^m_δ*. Then there are* $g_1, g_2 \in D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *such that*

(1) $f = g_2 \circ g_1$,

(2) g_1 *and* g_2 *are* C^2*-close to* $1_{\boldsymbol{R}^m}$ *and their supports are contained in* B^m_δ*, and*

(3) $g_1 \in D^\infty_{L,c}(\boldsymbol{R}^m, \mathscr{F}_0)$, $\hat{g}_2(x) \in D_c^\infty(\boldsymbol{R}^{m-n}, 0)$ *for any* $x \in \boldsymbol{R}^n$*, where* $\hat{g}_2(x) : \boldsymbol{R}^{m-n} \to \boldsymbol{R}^{m-n}$ *is defined by* $\hat{g}_2(x)(y) = g_2(x, y)$ *for* $y \in \boldsymbol{R}^{m-n}$*.*

PROOF. Take $f \in D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. We have $\|f - 1_{\boldsymbol{R}^m}\|_2 < \varepsilon$ for a sufficiently small $\varepsilon > 0$, where $\|\cdot\|_2$ denotes the C^2 norm. We put $f(x, y) = (f_1(x, y), f_2(x, y)) \in \boldsymbol{R}^m = \boldsymbol{R}^n \times \boldsymbol{R}^{m-n}$. Note that $f_2(x, 0) = 0$. Put $g_1(x, y) = (f_1(x, y), y)$. Then we have $g_1 \in D^\infty_{L,c}(\boldsymbol{R}^m, \mathscr{F}_0)$ and g_1 is C^2-close to $1_{\boldsymbol{R}^m}$ since $\|g_1 - 1_{\boldsymbol{R}^m}\|_2 \leq \|f - 1_{\boldsymbol{R}^m}\|_2 < \varepsilon$. We define a map $\hat{f}_1 : \boldsymbol{R}^{m-n} \to D_c^\infty(\boldsymbol{R}^n)$ by $\hat{f}_1(y)(x) = f_1(x, y)$ for $y \in \boldsymbol{R}^{m-n}$ and $x \in \boldsymbol{R}^n$. Then $\hat{f}_1$ satisfies $\hat{f}_1(y) = 1_{\boldsymbol{R}^n}$ for $\|y\| \geq \delta$. Then we have by easy calculations that $g_1^{-1}(x, y) = (\hat{f}_1(y)^{-1}(x), y)$ for $x \in \boldsymbol{R}^n$ and $y \in \boldsymbol{R}^{m-n}$. We put $g_2 = f \circ g_1^{-1}$. Then we have $g_2(x, y) = f \circ g_1^{-1}(x, y) = (x, f_2(\hat{f}_1(y)^{-1}(x), y))$ for $(x, y) \in \boldsymbol{R}^m = \boldsymbol{R}^n \times \boldsymbol{R}^{m-n}$. Note that $g_2 \in D^\infty_{L,c}(\boldsymbol{R}^m, \mathscr{F}_1)$, where $\mathscr{F}_1$ is the product foliation of $\boldsymbol{R}^m$ with leaves of form $\{\{x\} \times \boldsymbol{R}^{m-n}\}$. By putting $\hat{g}_2(x)(y) = f_2(\hat{f}_1(y)^{-1}(x), y)$, we have a map $\hat{g}_2 : \boldsymbol{R}^n \to D_c^\infty(\boldsymbol{R}^{m-n}, 0)$ satisfying that $\hat{g}_2(x) = 1_{\boldsymbol{R}^{m-n}}$ for $\|x\| \geq \delta$. For, take any $x \in \boldsymbol{R}^n$. Then the Jacobian of $\hat{g}_2(x)$ at $y = 0$ satisfies the following:

$$\frac{\partial \hat{g}_2(x)_i}{\partial y_j} = \sum_{k=1}^{n} \frac{\partial f_{2,i}}{\partial x_k} (\hat{f}_1(y)^{-1}(x), y) \cdot \frac{\partial (\hat{f}_1(y)_k^{-1})(x)}{\partial y_j} + \frac{\partial f_{2,i}}{\partial y_j} (\hat{f}_1(y)^{-1}(x), y)$$

thus,

$$\begin{aligned}
&\left|\left(\frac{\partial \hat{g}_2(x)_i}{\partial y_j}\right) - I_{m-n}\right| \\
&\quad \le \sum_{i,j=1}^{m-n} \sum_{k=1}^{n} \left|\frac{\partial f_{2,i}}{\partial x_k}(\hat{f}_1(y)^{-1}(x), y) \cdot \frac{\partial (\hat{f}_1(y)_k^{-1})(x)}{\partial y_j}\right| \\
&\qquad + \sum_{i \ne j} \left|\frac{\partial f_{2,i}}{\partial y_j}(\hat{f}_1(y)^{-1}(x), y)\right| + \sum_{i=1}^{m-n} \left|\frac{\partial f_{2,i}}{\partial y_i}(\hat{f}_1(y)^{-1}(x), y) - 1\right| \\
&\quad \le (m-n)^2 (n\varepsilon + 1)\varepsilon.
\end{aligned}$$

Hence we have $\hat{g}_2(x) \in D_c^\infty(\boldsymbol{R}^{m-n}, 0)$ for $x \in \boldsymbol{R}^n$. This completes the proof of Lemma 2.2. □

Let $\mathscr{F}$ be a C^∞-foliation of a C^∞-manifold M. By $D_{L,c}^\infty(M, \mathscr{F})$ we denote the group of leaf preserving C^∞-diffeomorphisms of $(M, \mathscr{F})$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms with compact support.

Then T. Tsuboi (and T. Rybicki [**11**] independently) proved the following by looking at the proofs in [**8**] and [**16**].

Theorem 2.3 (Theorem 1.1 of [**17**]). *$D_{L,c}^\infty(M, \mathscr{F})$ is perfect.*

3. Proof of Theorem 1.3.

Take $f \in D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. We may assume that f is C^2-close to $1_{\boldsymbol{R}^m}$ and is supported in B_δ^m. From Lemma 2.2, there are g_1, $g_2 \in D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ such that (1) $f = g_2 \circ g_1$, (2) g_1 and g_2 are C^2-close to $1_{\boldsymbol{R}^m}$ and are supported in B_δ^m, and (3) $g_1 \in D_{L,c}^\infty(\boldsymbol{R}^m, \mathscr{F}_0)$, $\hat{g}_2(x) \in D_c^\infty(\boldsymbol{R}^{m-n}, 0)$ for any $x \in \boldsymbol{R}^n$, where $\hat{g}_2(x)(y) = f_2(\hat{f}_1(y)^{-1}(x), y)$.

By Theorem 2.3, g_1 is represented as a product of commutators of elements in $D_{L,c}^\infty(\boldsymbol{R}^m, \mathscr{F}_0)$.

Thus we shall prove that g_2 can be represented as a product of commutators of elements in $D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. Let $\mathrm{d}\hat{g}_2(x)(0)$ be the differential of $\hat{g}_2(x)$ at $y = 0$ for each $x \in \boldsymbol{R}^n$. Since g_2 is C^2-close to $1_{\boldsymbol{R}^m}$, $\mathrm{d}\hat{g}_2(\cdot)(0)$ is C^1-close to the constant mapping e ($e(x) = I_{m-n}$) and is supported in B_δ^n. Then we have by Lemma 2.1 that there exist C^∞-mappings $G_i : \boldsymbol{R}^n \to GL(m-n, \boldsymbol{R})$ and $\varphi_i \in D_c^\infty(\boldsymbol{R}^n)$ ($i = 1, 2, \cdots, q = (m-n)^2$) satisfying that (1) each φ_i is C^1-close to the identity and is supported in B_δ^n, (2) each G_i is supported in $B_{2\sqrt{3}\delta}^n$ and is C^1-close to e, and (3) $\mathrm{d}\hat{g}_2(\cdot)(0) = (G_1^{-1} \cdot (G_1 \circ \varphi_1)) \cdot \cdots \cdot (G_q^{-1} \cdot (G_q \circ \varphi_q))$.

Let $\lambda : [0, \infty) \to [0, 1]$ be a C^∞-monotone decreasing function satisfying that $\lambda(t) = 1$ for $t \le (1/2)\delta$ and $\lambda(t) = 0$ for $t \ge \delta$ and put a C^∞ function $\mu : \boldsymbol{R}^{m-n} \to [0, 1]$ by $\mu(y) = \lambda(\|y\|)$. For G_i, we define a C^∞-mapping $h_{G_i} : \boldsymbol{R}^m \to \boldsymbol{R}^m$ by

$$h_{G_i}(x,y) = (x, (\mu(y)\cdot G_i(x) + (1-\mu(y))\cdot I_{m-n})y).$$

We see that h_{G_i} is a C^∞-diffeomorphism of $\boldsymbol{R}^m$ since G_i is C^1-close to e. Note that $h_{G_i}(x,y) = (x, G_i(x)y)$ for $\|y\| \le (1/2)\delta$.
For $\varphi_i \in D_c^\infty(\boldsymbol{R}^n)$, we put

$$F_{\varphi_i}(x,y) = (\mu(y)\varphi_i(x) + (1-\mu(y))x, y).$$

We see that F_{φ_i} is a C^∞-diffeomorphism of $\boldsymbol{R}^m$ since each φ_i is sufficiently C^1-close to the identity. Then, since $F_{\varphi_i}^{-1}(x,y) = (\varphi_i^{-1}(x), y)$ for $\|y\| \le (1/2)\delta$, we have that

$$h_{G_i}^{-1} \circ F_{\varphi_i}^{-1} \circ h_{G_i} \circ F_{\varphi_i}(x,y) = (x, G_i(x)^{-1} \cdot G_i(\varphi_i(x)) \cdot y)$$

for $\|y\| \le (1/2)\delta$.

LEMMA 3.1. $h_{\mathrm{d}\hat{g}_2}(\cdot)(0) = \prod_{i=1}^q [h_{G_i}^{-1}, F_{\varphi_i}^{-1}]$ *on a small neighborhood of* $\boldsymbol{R}^n \times \{0\}$, *furthermore* $h_{\mathrm{d}\hat{g}_2}(\cdot)(0) \in [D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n), D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)]$.

PROOF. We note that $h_{\mathrm{d}\hat{g}_2(\cdot)(0)}(x,y) = (x, \mathrm{d}\hat{g}_2(x)(0)\cdot y)$ for $x \in \boldsymbol{R}^n$ and $y \in \boldsymbol{R}^{m-n}(\|y\| \le (1/2)\delta)$. Then the equality $h_{\mathrm{d}\hat{g}_2(\cdot)(0)} = \prod_{i=1}^q [h_{G_i}^{-1}, F_{\varphi_i}^{-1}]$ on the small neighborhood $\{(x,y) \in \boldsymbol{R}^n \times \boldsymbol{R}^{m-n} \mid \|y\| \le (1/2)\delta\}$ of $\boldsymbol{R}^n \times \{0\}$ follows from a simple calculation. Since $h_{\mathrm{d}\hat{g}_2(\cdot)(0)} \circ (\prod_{i=1}^q [h_{G_i}^{-1}, F_{\varphi_i}^{-1}])^{-1}$ is supported in a compact subset of $\boldsymbol{R}^m - \boldsymbol{R}^n$, the proof of the rest follows from the perfectness of $D_c^\infty(\boldsymbol{R}^m - \boldsymbol{R}^n)$ (Thurston [**16**]). $\square$

PROOF OF THEOREM 1.3 continued. We put $g_3 = h_{\mathrm{d}\hat{g}_2(\cdot)(0)}^{-1} \circ g_2$. Then we have $g_3(x,y) = (x, \mathrm{d}\hat{g}_2(x)(0)^{-1} \cdot f_2(\hat{f}_1(y)^{-1}(x), y))$ for $\|y\| \le (1/2)\delta$. We remark that the Jacobian of $\hat{g}_3(x)$ at $y=0$, $\left(\partial\hat{g}_3(x)_i/\partial y_j\right) = I_{m-n}$ for any $x \in \boldsymbol{R}^n$.

The rest is proved by considering a parameter version of Sternberg [**15**] (cf. [**2**]) in the following. We may assume that the support of g_3 is contained in B_δ^m for a small $\delta > 0$. We put $\psi(x,y) = (x, cy)$ for $0 < c < 1$. Then the eigenvalues of the Jacobian $J(g_3 \circ \psi(\cdot, y))$ at $y = 0$ are all c. This satisfies the condition of Theorem 1 of [**15**]. By the parameter version of Theorem 1 of [**15**], there exists a C^∞-diffeomorphism R of $\boldsymbol{R}^n \times \boldsymbol{R}^{m-n}$ of the form $R(x,y) = (x, R_1(x,y))$ with $R_1(x,0) = 0$ satisfying that $R^{-1} \circ (g_3 \circ \psi) \circ R = \psi$ on B_δ^m.

Put a C^∞-function $\nu : \boldsymbol{R}^m \to [0,1]$ by $\nu(x,y) = \lambda(\|(x,y)\|)$ for $(x,y) \in \boldsymbol{R}^m$. Note that the support of ν is contained in B_δ^m. We define a C^∞-diffeomorphism $\tilde{\psi}$ of $\boldsymbol{R}^n \times \boldsymbol{R}^{m-n}$ by $\tilde{\psi}(x,y) = (x, (\nu(x,y)c + 1 - \nu(x,y))\cdot y)$. Then we have $\tilde{\psi} = \psi$ on $B_{(1/2)\delta}^m$. By replacing ψ to $\tilde{\psi}$ and considering R to be the identity outside of B_δ^m, g_3 is written as a commutator of elements in $D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ on $B_{(1/2)\delta}^m$. Outside of

$B^m_{(1/2)\delta}$, $g_3 \circ ([R, \tilde{\psi}])^{-1}$ is also represented by a product of commutators of elements in $D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$, by the perfectness of $D^\infty_c(\boldsymbol{R}^m - \boldsymbol{R}^n)$ (Thurston [**16**]). Thus we have that g_3 is contained in $[D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n), D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)]$, hence g_2 is so. This completes the proof. □

Remark 3.2. Theorem 1.1 was proved by T. Rybicki in [**12**], [**13**]. The proof was given by following the Mather'rolling up method [**10**] and using some estimate in Epstein [**5**]. But it is not easy to check whether the proof is complete. We have prove it by the another method.

4. The uniform perfectness.

Let M be an m-dimensional compact connected C^∞-manifold with boundary ∂M. In this section we study the uniform perfectness of $D^\infty(M, \partial M)$.

Definition 4.1. The group $D^\infty(M, \partial M)$ is *uniformly perfect* if any element of the group can be represented as a product of a bounded number of commutators of its elements.

First we investigate the uniform perfectness of $D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$ $(n \geq 1)$.

Theorem 4.2. *$D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is uniformly perfect for $n \geq 1$. In fact, any $f \in D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$ can be represented as a product of two commutators of elements in $D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$.*

Proof. We prove Theorem 4.2 by the parallel way to Tsuboi [**18**]. Take $f \in D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$. By Theorem 1.3, f can be represented by a product of commutators as

$$f = \prod_{i=1}^k [a_i, b_i], \quad \text{where} \quad a_i, b_i \in D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n).$$

Let U be an bounded open set of $\boldsymbol{R}^m$ containing the supports of a_i's and b_i's. Take $\phi \in D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$ such that $\{\phi^i(U)\}_{i=1}^k$ are disjoint. This is possible because $n \geq 1$. We put $F = \prod_{j=1}^k \phi^j(\prod_{i=j}^k [a_i, b_i])\phi^{-j}$ which defines an element in $D^\infty_c(\boldsymbol{R}^m, \boldsymbol{R}^n)$. Then we have

$$\phi^{-1} \circ F \circ \phi \circ F^{-1} = f \circ (\prod_{j=1}^k \phi^j [a_j, b_j]^{-1} \phi^{-j}) = f \circ [\prod_{j=1}^k \phi^j b_j \phi^{-j}, \prod_{j=1}^k \phi^j a_j \phi^{-j}].$$

Thus $f = [\phi^{-1}, F] \circ [\prod_{j=1}^k \phi^j a_j \phi^{-j}, \prod_{j=1}^k \phi^j b_j \phi^{-j}]$. Therefore, f can be represented as a product of two commutators of elements in $D_c^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. This completes the proof. $\square$

REMARK 4.3. Put $\boldsymbol{H}^m = \{(x_1, x_2, \ldots, x_m) \in \boldsymbol{R}^m \mid x_m \geq 0\}$. By the same way as the proof of Theorem 4.2, we can prove that each element of $D_c^\infty(\boldsymbol{H}^m, \boldsymbol{R}^{m-1})$ can be represented as a product of two commutators of its elements.

From Theorem 4.2 we can prove Theorem 1.4.

PROOF OF THEOREM 1.4. Let $\pi : D^\infty(M, \partial M) \to D^\infty(\partial M)$ be the map which is defined by the restriction. By the isotopy extension theorem, π is epimorphic. Take an element $f \in D^\infty(M, \partial M)$ and put $\bar{f} = \pi(f)$. Since $D^\infty(\partial M)$ is a uniformly perfect group, there exists a bounded number k such that each element of the group is represented as a product of k commutators of its elements. Then $\bar{f}$ is written as $\bar{f} = \prod_{j=1}^k [\bar{g}_j, \bar{h}_j]$ for $\bar{g}_j, \bar{h}_j \in D^\infty(\partial M)$. We take g_j and h_j in $D^\infty(M, \partial M)$ satisfying $\pi(g_j) = \bar{g}_j$ and $\pi(h_j) = \bar{h}_j$. Let $\hat{f} = (\prod_{j=1}^k [g_j, h_j])^{-1} \circ f$. Then $\hat{f} \in \ker \pi$.

Let W be a collar neighborhood of ∂M. Then W can be identified with $[0,1) \times \partial M$. For a positive number ε put $W_\varepsilon = [0, \varepsilon) \times \partial M$. If we take ε sufficiently small, then $\hat{f}(W_\varepsilon) \subset W$. Thus $\hat{f}$ is represented on W_ε as $\hat{f}(t,x) = (\hat{f}_1(t,x), \hat{f}_2(t,x))$ for $(t,x) \in W_\varepsilon$ with $\hat{f}_1(t,x) \in [0,1)$, $\hat{f}_2(t,x) \in \partial M$. By the Taylor expansion formula, for $(t,x) \in W_\varepsilon$

$$\hat{f}_1(t,x) = t\, \frac{\partial \hat{f}_1}{\partial t}(0,x) + t^2 \int_0^1 (1-r) \frac{\partial^2 \hat{f}_1}{\partial t^2}(tr, x) dr.$$

Let $\mu : [0,1) \to [0,1]$ be a C^∞-function such that $\mu(t) = 1$ for $t \in [0, 2/3]$ and $\mu(t) = 0$ for $t \in [3/4, 1)$. Take a positive number $\delta \leq \varepsilon$ and put $\lambda_\delta(t) = \mu(t/\delta) \quad (0 \leq t < \delta)$. For $(t,x) \in W_\delta$, let

$$g_1(t,x) = t\, \frac{\partial \hat{f}_1}{\partial t}(0,x) + (1 - \lambda_\delta(t)) t^2 \int_0^1 (1-r) \frac{\partial^2 \hat{f}_1}{\partial t^2}(tr, x) dr.$$

Define $g : M \to M$ such that $g = (g_1, \hat{f}_2)$ on W_δ and $g = \hat{f}$ outside of W_δ.

There exist positive numbers K, L such that

$$|\mu'(t)| \leq K, \quad \left| \frac{\partial^i \hat{f}_1}{\partial t^i}(t,x) \right| \leq L \quad (i = 2, 3) \quad \text{for } (t,x) \in W_\delta.$$

Then

$$\left| g_1(t,x) - \frac{\partial \hat{f}_1}{\partial t}(0,x)t \right| \le \frac{\delta^2}{2} L,$$
$$\left| \frac{\partial g_1}{\partial t}(t,x) - \frac{\partial \hat{f}_1}{\partial t}(0,x) \right| \le \frac{\delta}{2}(K+2)L + \frac{\delta^2}{6} L.$$

Note that $\hat{f}_2(0,x) = (0,x)$ for $x \in \partial M$. Then, if we take δ sufficiently small, we see that g is a diffeomorphism of M.

If we define $\tilde{g}_s$ $(0 \le s \le 1)$ by

$$\tilde{g}_s(t,x) = \left(t \frac{\partial \hat{f}_1}{\partial t}(0,x) + (1 - s\lambda_\delta(t))t^2 \int_0^1 (1-r) \frac{\partial^2 \hat{f}_1}{\partial t^2}(tr,x)dr,\ \hat{f}_2(t,x) \right)$$

for $(t,x) \in W_\delta$ and $\tilde{g}_s = \hat{f}$ outside of W_δ, then we see that $\{\tilde{g}_s\}$ is an isotopy of M such that $\tilde{g}_0 = \hat{f}$, $\tilde{g}_1 = g$. By the definition $g(t,x) = ((\partial \hat{f}_1/\partial t)(0,x) \cdot t,\ \hat{f}_2(t,x))$ on $W_{(2/3)\delta}$. Then it is easy to see that g is isotopic to a diffeomorphism $\hat{g}$ which is equal to the identity on $W_{(1/2)\delta}$ and is equal to $\hat{f}$ on $[(3/4)\delta, \delta) \times \partial M$. Thus we have an isotopy $\tilde{f}_s$ $(0 \le s \le 1)$ of M such that $\hat{f}_s = \hat{f}$ on $[(4/5)\delta, \delta) \times \partial M$ for any s and $\tilde{f}_0 = \hat{f}$, $\tilde{f}_1 = \hat{g}$.

Let ℓ be the category number of ∂M and $\mathscr{V} = \{V_i\}_{i=1}^{\ell+1}$ be a covering of ∂M such that each V_i is diffeomorphic to a disjoint union of open balls in ∂M. Let $\{\mu_i\}_{i=1}^{\ell+1}$ be a partition of unity subordinate to the covering $\mathscr{V}$. Define $h_i \in D_c^\infty(M)$ $(i = 1, 2, \cdots, \ell+1)$ supported in W_δ such that for $(t,x) \in W_\delta$

$$h_1(t,x) = \left(\hat{f} \circ \tilde{f}^{-1}_{\mu_1(x)} \right)(t,x),$$
$$h_i(t,x) = \left((h_1 \circ \cdots \circ h_{i-1})^{-1} \circ \hat{f} \circ \tilde{f}^{-1}_{\mu_1(x)+\cdots+\mu_i(x)} \right)(t,x) \quad (i = 2, \cdots, \ell+1).$$

Then $\hat{f} \circ \hat{g}^{-1} = h_1 \circ h_2 \circ \cdots \circ h_{\ell+1}$ and the support of h_i is contained in $[0,\delta) \times V_i$. We can assume that each h_i is contained in $D_c^\infty(\boldsymbol{H}^m, \boldsymbol{R}^{m-1})$. Hence from Remark 4.3, each h_i is represented as a product of two commutators of elements in $D_c^\infty(\boldsymbol{H}^m, \boldsymbol{R}^{m-1})$.

Note that $\hat{g}$ is isotopic to the identity being supported in $\mathrm{int}M$. By the assumption of Theorem 1.4, there exists a bounded number s such that $\hat{g}$ is represented as a product of s commutators of elements in $D_c^\infty(\mathrm{int}M)$. Hence f is represented as a product of $k + 2(\ell+1) + s$ commutators of elements in $D^\infty(M, \partial M)$. Since k, ℓ, s are bounded numbers, this completes the proof of Theorem 1.4. $\square$

Remark 4.4. T. Tsuboi [**18**] studied the uniform perfectness of $\mathrm{Diff}_c^r(M)$. He has proved that it is uniformly perfect if $1 \leq r \leq \infty$ $(r \neq \dim M + 1)$ and M is one of the following cases

(1) an odd dimensional compact manifold without boundary,

(2) an even dimensional compact manifold without boundary which has a handle decomposition without handles of the middle index,

(3) the interior of an odd dimensional compact manifold W and

(4) the interior of an even dimensional compact manifold W which has a handle decomposition only with handles of indices not greater than $(\dim W - 1)/2$.

The following is an immediate consequence of Theorem 1.4 and Remark 4.4.

Corollary 4.5. *Let M be a compact manifold with boundary ∂M such that* int$M$ *and ∂M are manifolds in the cases of* Remark 4.4. *Then $D^\infty(M, \partial M)$ is a uniformly perfect group.*

Example 4.6. Since intD^m and $S^{m-1} (m \geq 2)$ satisfy the condition in Corollary 4.5, $D^\infty(D^m, S^{m-1})$ is a uniformly perfect group.

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