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Transfinite large inductive dimensions modulo absolute Borel classes

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Abstract. The following inequalities between transfinite large inductive dimensions modulo absolutely additive (resp. multiplicative) Borel classes $A(\alpha)$ (resp. $M(\alpha)$) hold in separable metrizable spaces:

(i) $A(0)\text{-trInd} \geq M(0)\text{-trInd} \geq \max\{A(1)\text{-trInd}, M(1)\text{-trInd}\}$, and
(ii) $\min\{A(\alpha)\text{-trInd}, M(\alpha)\text{-trInd}\} \geq \max\{A(\beta)\text{-trInd}, M(\beta)\text{-trInd}\}$, where $1 \leq \alpha < \beta < \omega_1$.

We show that for any two functions a and m from the set of ordinals $\Omega = \{\alpha : \alpha < \omega_1\}$ to the set $\{-1\} \cup \Omega \cup \{\infty\}$ such that

(i) $a(0) \geq m(0) \geq \max\{a(1), m(1)\}$, and
(ii) $\min\{a(\alpha), m(\alpha)\} \geq \max\{a(\beta), m(\beta)\}$, whenever $1 \leq \alpha < \beta < \omega_1$,

there is a separable metrizable space X such that $A(\alpha)\text{-trInd}\, X = a(\alpha)$ and $M(\alpha)\text{-trInd}\, X = m(\alpha)$ for each $0 \leq \alpha < \omega_1$.

1. Introduction.

All topological spaces in this paper are assumed to be separable and metrizable unless we mention something different. Our terminology mostly follows [**2**] and [**5**].

In 1964 Lelek defined *the small* (*large*) *inductive dimension modulo a class* $\mathscr{P}$ *of topological spaces,* $\mathscr{P}$-ind ($\mathscr{P}$-Ind). Recall that for a space X we have $\mathscr{P}\text{-ind}\, X = -1$ if and only if $X \in \mathscr{P}$; and $\mathscr{P}\text{-ind}\, X \leq n$, where n is an integer ≥ 0, if for every point $x \in X$ and every closed subset A of X with $x \notin A$ there exists a partition C in X between x and A such that $\mathscr{P}\text{-ind}\, C < n$. (If we replace the point x by any closed set B disjoint from A we will obtain the definition of $\mathscr{P}$-Ind).

Throughout the present paper, considered classes $\mathscr{P}$ are assumed to contain the empty space $\emptyset$ and every space homeomorphic to a closed subspace of each space which belongs to $\mathscr{P}$.

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The functions $\mathscr{P}$-ind and $\mathscr{P}$-Ind are natural generalizations of the well known *small* (*large*) *inductive dimension* ind (Ind), i.e. the case of $\mathscr{P} = \{\emptyset\}$, and *the small* (*large*) *inductive compactness degree* cmp ($\mathscr{K}_0$-Ind) due to de Groot (cf. [**2**]), i.e. the case of $\mathscr{P}$ being the class of compact spaces $\mathscr{K}_0$. Note that $\mathscr{P}$-ind and $\mathscr{P}$-Ind are monotone with respect to closed subsets, and the inequality $\mathscr{P}\text{-ind} \le \mathscr{P}\text{-Ind}$ holds. Moreover, if $X = X_1 \oplus X_2$ is the topological sum of spaces X_1 and X_2 then $\mathscr{P}\text{-d}\, X = \max\{\mathscr{P}\text{-d}\, X_1, \mathscr{P}\text{-d}\, X_2\}$, where d is either ind or Ind, provided that the topological sum of any two elements of $\mathscr{P}$ is in $\mathscr{P}$.

Recall ([**2**, Chapter II.9]) that every absolutely additive Borel class $A(\alpha)$ and every absolutely multiplicative Borel class $M(\alpha)$, where $0 \le \alpha < \omega_1$, satisfy the conditions mentioned above. Moreover, the following hierarchy of these classes holds (a diagram in which a class $\mathscr{P}_1$ is contained in a class $\mathscr{P}_2$ iff $\mathscr{P}_2$ is to the right of $\mathscr{P}_1$, and the arrows indicate inclusions):

$$\begin{tikzcd}
 & & \mathrm{A}(1) \arrow[r] \arrow[ddr] & \mathrm{A}(2) \arrow[r] \arrow[ddr] & \cdots \\
\mathrm{A}(0) \arrow[r] & \mathrm{M}(0) \arrow[ur] \arrow[dr] & & & \\
 & & \mathrm{M}(1) \arrow[r] \arrow[uur] & \mathrm{M}(2) \arrow[r] \arrow[uur] & \cdots
\end{tikzcd} \tag{*}$$

Observe that if $\mathscr{P}_2 \subset \mathscr{P}_1$ then $\mathscr{P}_1\text{-d} \le \mathscr{P}_2\text{-d}$, where d is either ind or Ind. Using this fact and (*) we get the following inequalities concerning the inductive dimensions modulo absolute Borel classes:

(1.1) $A(0)\text{-d} \ge M(0)\text{-d} \ge \max\{A(1)\text{-d}, M(1)\text{-d}\}$,
(1.2) $\min\{A(\alpha)\text{-d}, M(\alpha)\text{-d}\} \ge \max\{A(\beta)\text{-d}, M(\beta)\text{-d}\}$, whenever $1 \le \alpha < \beta < \omega_1$.

Recall (cf. [**2**]) that $A(0) = \{\emptyset\}$, $M(0) = \mathscr{K}_0$, $A(1) = \mathscr{S}_0$, $M(1) = \mathscr{C}_0$, where $\mathscr{C}_0$ and $\mathscr{S}_0$ are the classes of completely metrizable spaces and σ-compact spaces, respectively, and the following notations are used in the literature:

$A(0)\text{-ind} = \text{ind}$, $A(0)\text{-Ind} = \text{Ind}$, $M(0)\text{-ind} = \text{cmp}$, $M(0)\text{-Ind} = \mathscr{K}_0\text{-Ind}$,
$A(1)\text{-ind} = \mathscr{S}\text{-ind}$, $A(1)\text{-Ind} = \mathscr{S}\text{-Ind}$, $M(1)\text{-ind} = \text{icd}$, $M(1)\text{-Ind} = \text{Icd}$.

Let us recall some facts about these functions. It is well known (see [**2**, Chapter II.10]) that for every space X we have

- $A(\alpha)\text{-ind}\, X = A(\alpha)\text{-Ind}\, X$ for each $\alpha \ge 0$,
- $M(\alpha)\text{-ind}\, X = M(\alpha)\text{-Ind}\, X$ for each $\alpha \ge 1$,
- $\text{cmp}\, \boldsymbol{R}^n = 0$, and
- $\mathscr{K}_0\text{-Ind}\, \boldsymbol{R}^n = n$ for each integer $n \ge 1$ ([**2**, Example II.6.12 (a)]).

Moreover, for each integer $n \geq 1$ we have $\operatorname{icd}(Q_1 \times \boldsymbol{I}^n) = n$ ([**2**, Example I.7.12]) and $\mathscr{S}$-ind$\,(P_1 \times \boldsymbol{I}^n) = n$ ([**2**, Example I.10.6]), where Q_1 (resp. P_1) is the space of rational (resp. irrational) numbers in the closed interval $\boldsymbol{I} = [0,1]$. Hence $\operatorname{cmp}(Q_1 \times \boldsymbol{I}^n) = \operatorname{cmp}(P_1 \times \boldsymbol{I}^n) = n$. In addition, for each integer $n \geq 0$ there is a subset X_n of $\boldsymbol{I}^{n+1}$ such that $A(\alpha)$-ind $X_n = M(\alpha)$-ind $X_n = n$ for each ordinal $0 \leq \alpha < \omega_1$ ([**2**, Example II.10.5]). Notice that $\operatorname{Ind} X_n = n$. We adopt the following notations: $X_{-1} = \emptyset$ and D is the countable discrete space. For any space Z let Z^0 be the one-point space and $Z^{-1} = \emptyset$. For arbitrary integers $k \geq l \geq \max\{m,n\} \geq \min\{m,n\} \geq p \geq -1$ we put

$$X = \begin{cases} \boldsymbol{I}^k \oplus \boldsymbol{R}^l \oplus (Q_1 \times \boldsymbol{I}^m) \oplus (P_1 \times \boldsymbol{I}^n) \oplus X_p, \text{ if } l \geq 1; \\ \boldsymbol{I}^k \oplus D \oplus (Q_1 \times \boldsymbol{I}^m) \oplus (P_1 \times \boldsymbol{I}^n) \oplus X_p, \text{ if } l = 0; \\ \boldsymbol{I}^k, \text{ if } l = -1. \end{cases}$$

Taking into account all facts mentioned above it is easy to see that $\operatorname{ind} X = k$, $\mathscr{K}_0$-Ind $X = l$, $\operatorname{icd} X = m$, $\mathscr{S}$-ind $X = n$ and $\mathscr{P}$-ind $X = p$, where $\mathscr{P}$ is either $A(\alpha)$ or $M(\alpha)$ for each $\alpha \geq 2$. Furthermore, if $l \geq 1$, then $\operatorname{cmp} X = \max\{0, m, n\}$, and if $l \leq 0$, then $\operatorname{cmp} X = l$.

Problem 1.1. Let d be either ind or Ind, and $a(\alpha)$, $m(\alpha)$, where $0 \leq \alpha < \omega_1$, either integers ≥ -1 or ∞ such that

(i) $a(0) \geq m(0) \geq \max\{a(1), m(1)\}$ and
(ii) $\min\{a(\alpha), m(\alpha)\} \geq \max\{a(\beta), m(\beta)\}$, if $1 \leq \alpha < \beta < \omega_1$.

Does there exist a space X such that $A(\alpha)$-d $X = a(\alpha)$ and $M(\alpha)$-d $X = m(\alpha)$ for each $0 \leq \alpha < \omega_1$?

Observe that inequalities (1.1), (1.2) and Problem 1.1 for Ind and ind differ only in the case of $M(0)$. In [**10**] Smirnov introduced the large transfinite inductive dimension trInd and presented for each ordinal $\alpha < \omega_1$, a compact space S^α such that $\operatorname{trInd} S^\alpha = \alpha$. Some years later Levshenko [**7**] proved that $\operatorname{trInd} S^\alpha \leq \omega_0 \cdot \operatorname{trind} S^\alpha$, where trind is a natural transfinite extension of ind due to Hurewicz (cf. [**5**]). These results together with the inductive character of the function trind implies, for each ordinal $\alpha < \omega_1$, the existence of a compact space L_α such that $\operatorname{trind} L_\alpha = \alpha \leq \operatorname{trInd} L_\alpha \neq \infty$.

In [**9**] R. Pol showed that for each $\alpha < \omega_1$ there exists a completely metrizable σ-compact space C_α such that $\alpha \leq \operatorname{trcmp} C_\alpha \leq \operatorname{trInd} C_\alpha \neq \infty$. From this result he obtained that for each $\alpha < \omega_1$ there exists a completely metrizable σ-compact space R_α such that $\operatorname{trcmp} R_\alpha = \alpha$ and $\operatorname{trInd} R_\alpha \neq \infty$ (here trcmp is a natural transfinite extension of cmp). It is also easy to see that for each $\alpha < \omega_1$ there exists a completely metrizable σ-compact space X_α such that $\mathscr{K}_0$-trInd $X_\alpha = \alpha$ and

$\mathrm{trInd}\, X_\alpha \neq \infty$ (where $\mathscr{K}_0$-trInd is a natural transfinite extension of $\mathscr{K}_0$-Ind). In addition, R. Pol observed that the reasoning of Aarts [**1**] in the proof of equality $\mathrm{cmp}\,(Q_1 \times \boldsymbol{I}^n) = n$ yields that for every compact space K_α with $\mathrm{trind}\, K_\alpha = \alpha \geq \omega_0$, $\mathrm{trcmp}\,(Q_1 \times K_\alpha) = \alpha$, but $\mathrm{trInd}\,(Q_1 \times K_\alpha) = \infty$ and $Q_1 \times K_\alpha$ is not completely metrizable. Let us also note the reasoning in the proof of equality $\mathrm{icd}\,(Q_1 \times \boldsymbol{I}^n) = n$ yields that $\mathrm{tricd}\,(Q_1 \times K_\alpha) = \alpha$, where tricd is a natural transfinite extension of icd.

In ([**3**]) Charalambous considered the small and large transfinite inductive dimensions modulo a class $\mathscr{P}$, $\mathscr{P}$-trind and $\mathscr{P}$-trInd, which are natural transfinite extensions of $\mathscr{P}$-ind and $\mathscr{P}$-Ind, respectively, such that $\{\emptyset\}$-trind $=$ trind, $\mathscr{K}_0$-trind $=$ trcmp, $\mathscr{C}_0$-trind $=$ tricd and so on. Moreover he demonstrated for each given ordinal $\alpha < \omega_1$ the existence of a space $C^\alpha_{\mathscr{T}}$ such that $\mathscr{T}$-$\mathrm{trind}\, C^\alpha_{\mathscr{T}} = \alpha$ (but $\mathscr{T}$-$\mathrm{trInd}\, C^\alpha_{\mathscr{T}} = \infty$ if $\alpha > \omega_0$), where the letter $\mathscr{T}$ denotes a class of spaces which, like the classes $M(\beta), A(\beta)$ are Borel sets of any space that contains them.

Note that inequalities (1.1) and (1.2) are also valid for $d = \mathrm{trInd}$ and $d = \mathrm{trind}$. In [**4**] we presented for each class $\mathscr{P}$ from the diagram $(*)$ a space $X_{\mathscr{P}}$ such that $\mathscr{P}$-$\mathrm{trind}\, X_{\mathscr{P}} = \infty$ and $\mathscr{Q}$-$\mathrm{trInd}\, X_{\mathscr{P}} = -1$ for any other class $\mathscr{Q}$ from the diagram $(*)$ which is not contained in $\mathscr{P}$. (Recall that in [**8**] E. Pol constructed a completely metrizable σ-compact space P such that $\mathrm{trcmp}\, P = \infty$.) Then the following generalization of Problem 1.1 arises.

Problem 1.2. Let d be either trind or trInd, and $a(\alpha)$, $m(\alpha)$, where $0 \leq \alpha < \omega_1$, either countable ordinals, -1 or ∞ such that

(i) $a(0) \geq m(0) \geq \max\{a(1), m(1)\}$, and
(ii) $\min\{a(\alpha), m(\alpha)\} \geq \max\{a(\beta), m(\beta)\}$, if $1 \leq \alpha < \beta < \omega_1$.

Does there exist a space X such that $A(\alpha)$-$\mathrm{d}\, X = a(\alpha)$ and $M(\alpha)$-$\mathrm{d}\, X = m(\alpha)$ for each $0 \leq \alpha < \omega_1$?

Observe that inequalities (1.1), (1.2) and Problem 1.2 for d $=$ trInd and d $=$ trind differ even for $A(0)$ because there are compact spaces X such that $\mathrm{trind}\, X < \mathrm{trInd}\, X$ ([**5**, Problem 7.1 G (e)]).

In this paper we solve Problem 1.1 for d $=$ Ind (see Corollary 4.2) and Problem 1.2 for d $=$ trInd (see Theorem 4.1) as well. Our solutions are based on a generalization of the Smirnov's construction. In particular (see Theorem 3.1), for each class $\mathscr{P}$ from the diagram (*) and each $\alpha < \omega_1$ we present a space $S^\alpha_{\mathscr{P}}$ such that $\mathscr{P}$-$\mathrm{trInd}\, S^\alpha_{\mathscr{P}} = \mathrm{trInd}\, S^\alpha_{\mathscr{P}} = \alpha$ and $\mathscr{Q}$-$\mathrm{trInd}\, S^\alpha_{\mathscr{P}} = -1$ for any other class $\mathscr{Q}$ from the diagram (*) which is not contained in $\mathscr{P}$. Moreover, $S^\alpha_{\mathscr{P}}$ is a subset of the cube $\boldsymbol{I}^{\alpha+1}$ if $\alpha < \omega_0$, and $S^\alpha_{\mathscr{P}}$ is a subset of Smirnov's space S^α otherwise. Using the results obtained here, the inductive character of the function $\mathscr{P}$-trind and an analog of the Levshenko's result for the pair $\mathscr{P}$-trind and $\mathscr{P}$-trInd due to Charalambous ([**3**]) we show (see Corollary 3.3) for each class $\mathscr{P}$ from the diagram (*) and each $\alpha < \omega_1$ the existence of a space $X^\alpha_{\mathscr{P}}$ such that $\alpha = \mathscr{P}\text{-trind}\, X^\alpha_{\mathscr{P}} \le \operatorname{trInd} X^\alpha_{\mathscr{P}} \neq \infty$ and $\mathscr{Q} - \operatorname{trInd} X^\alpha_{\mathscr{P}} = -1$ for any other class $\mathscr{Q}$ from the diagram (*) which is not contained in $\mathscr{P}$. Note that Problem 1.1 for $\mathrm{d} = \operatorname{ind}$ and Problem 1.2 for $\mathrm{d} = \operatorname{trind}$ still remain open. In particular, we do not know if there is a completely metrizable and σ-compact space C_n such that $\operatorname{cmp} C_n = n = \operatorname{ind} C_n$ for some (each) integer $n \ge 3$.

2. Preliminaries.

Recall that a subset C of a space X is *a partition* between two disjoint sets A and B in X if there are disjoint open subsets U and V of X such that $A \subset U$, $B \subset V$ and $C = X \setminus (U \cup V)$.

Let X be a space, $\mathscr{P}$ a class of spaces and α an ordinal number ≥ 0. Then the *small transfinite dimension modulo a class $\mathscr{P}$, $\mathscr{P}$-trind*, is defined as follows.

(i) $\mathscr{P}\text{-trind}\, X = -1$ if and only if $X \in \mathscr{P}$,

(ii) $\mathscr{P}\text{-trind}\, X \le \alpha$ (≥ 0) if for every point $x \in X$ and every closed subset A of X with $x \notin A$ there exists a partition C in X between x and A such that $\mathscr{P}\text{-trind}\, C < \alpha$.

(iii) $\mathscr{P}\text{-trind}\, X = \alpha$ if $\mathscr{P}\text{-trind}\, X \le \alpha$ and $\mathscr{P}\text{-trind}\, X > \beta$ for each ordinal $\beta < \alpha$,

(iv) $\mathscr{P}\text{-trind}\, X = \infty$ if $\mathscr{P}\text{-trind}\, X > \alpha$ for each ordinal α.

(If we replace the point x by any closed set B disjoint from A we obtain the definition of the *large transfinite dimension modulo a class $\mathscr{P}$, $\mathscr{P}$-trInd*).
It is obvious that $\mathscr{P}\text{-trind}\, X = -1$ if and only if $\mathscr{P}\text{-trInd}\, X = -1$, and $\mathscr{P}\text{-trind}\, X \le \mathscr{P}\text{-trInd}\, X$. Moreover, the following easy statements hold, where $\mathscr{P}$-trd is either $\mathscr{P}$-trind or $\mathscr{P}$-trInd:

- $\mathscr{P}_1\text{-trd} = \mathscr{P}_2\text{-trd}$ if and only if $\mathscr{P}_1 = \mathscr{P}_2$ (and hence $\operatorname{trcmp} \neq \operatorname{trind}$ and $\mathscr{K}_0\text{-trInd} \neq \operatorname{trInd}$).
- If $\mathscr{P}_2 \subset \mathscr{P}_1$, then $\mathscr{P}_1\text{-trd} \le \mathscr{P}_2\text{-trd}$ (in particular, $\operatorname{trcmp} \le \operatorname{trind}$ and $\mathscr{K}_0\text{-trInd} \le \operatorname{trInd}$).
- $\mathscr{P}$-trd is monotone with respect to closed subsets, that is if A is a closed subset of a space X then $\mathscr{P}\text{-trd}\, A \le \mathscr{P}\text{-trd}\, X$.
- If $X = X_1 \oplus X_2$ is the topological sum of spaces X_1 and X_2, then $\mathscr{P}\text{-trd}\, X = \max\{\mathscr{P}\text{-trd}\, X_1, \mathscr{P}\text{-trd}\, X_2\}$ provided that the topological sum of any two elements of $\mathscr{P}$ is in $\mathscr{P}$. Note that $\operatorname{trInd}(\oplus_{n=1}^{\infty} \boldsymbol{I}^n) = \infty$.

We will denote by $\mathscr{B}(X)$ the family of Borel sets of a space X and by $\prod^0_\alpha(X)$

(resp. $\sum_\alpha^0(X)$) the multiplicative (resp. additive) Borel class α of X, where $0 \le \alpha < \omega_1$. The following statement is known.

PROPOSITION 2.1 ([**11**, Theorem 5.2.11]). *Let X, Y be compact metric spaces and $f : X \to Y$ a continuous onto mapping. Suppose that $A \subset Y$ and $0 \le \alpha < \omega_1$. Then $A \in \prod_\alpha^0(Y)$ if and only if $f^{-1}(A) \in \prod_\alpha^0(X)$.*

Recall (cf. [**2**]) that a space X is said to be *absolutely of the multiplicative* (resp. *the additive*) *class* α, in brief $X \in M(\alpha)$ (resp. $X \in A(\alpha)$), where $0 \le \alpha < \omega_1$, if X is a member of the multiplicative (resp. additive) Borel class α in Y whenever X is a subspace of a space Y (that is for any homeomorphic embedding $h : X \to Y$ of X into Y the image $h(X)$ is an element of the multiplicative (resp. additive) class α in Y). Put $\mathscr{AB} = \cup\{A(\alpha) : \alpha < \omega_1\}$ $(= \cup\{M(\alpha) : \alpha < \omega_1\})$. It is well known that $A(0) = \{\emptyset\}$, $M(0) = \mathscr{K}_0$, $A(1) = \mathscr{S}_0$, $M(1) = \mathscr{C}_0$, and for every $2 \le \alpha < \omega_1$ we have $X \in M(\alpha)$ (resp. $X \in A(\alpha)$) if and only if there is a homeomorphic embedding $h : X \to Y$ of X in a space $Y \in \mathscr{C}_0$ such that the image $h(X)$ is an element of the multiplicative (resp. the additive) class α in Y. So if $X \in \mathscr{P}$, where $\mathscr{P}$ is either an absolutely additive or multiplicative Borel class, then $X \times K \in \mathscr{P}$ for every compact space K.

Let P_0 be a one-point space, $Q_0 = \{1/n : n = 1, 2, \dots\}$ the subspace of $\boldsymbol{I}$, P_1 (resp. Q_1) the space of irrational (resp. rational) numbers in $\boldsymbol{I}$. Note that $P_0 \in \mathscr{K}_0$, $Q_0 \in (\mathscr{S}_0 \cap \mathscr{C}_0) \setminus \mathscr{K}_0$, $P_1 \in \mathscr{C}_0 \setminus \mathscr{S}_0$ and $Q_1 \in \mathscr{S}_0 \setminus \mathscr{C}_0$. Moreover (see [**4**]) for every α with $2 \le \alpha < \omega_1$ there are subspaces P_α and Q_α of $\boldsymbol{I}$ such that $P_\alpha \in M(\alpha) \setminus A(\alpha)$ and $Q_\alpha \in A(\alpha) \setminus M(\alpha)$. All spaces P_α and Q_α, where $0 \le \alpha < \omega_1$, can be assumed zero-dimensional. Recall [**3**] that a subset A of a space X is a *Bernstein set* if $|A \cap B| = |(X \setminus A) \cap B| = c$ for every uncountable Borel set B of X. Let us denote by $Brn(X)$ the family of all Berstein sets of a space X. Note that $Brn(X) \neq \emptyset$ if X is uncountable and completely metrizable. From Proposition 2.1 we get easily the following.

PROPOSITION 2.2. *Let X be a compact metrizable space and $f : X \to \boldsymbol{I}$ a continuous onto mapping. Then we have the following.*

(i) $f^{-1}(Q_0) \in (\mathscr{C}_0 \cap \mathscr{S}_0) \setminus \mathscr{K}_0$.
(ii) $f^{-1}(P_\alpha) \in M(\alpha) \setminus A(\alpha)$ *and* $f^{-1}(Q_\alpha) \in A(\alpha) \setminus M(\alpha)$, *whenever* $1 \le \alpha < \omega_1$.
(iii) $f^{-1}(J) \notin \mathscr{B}(X)$, *and hence* $f^{-1}(J) \notin \mathscr{AB}$ *if* $J \in Brn(\boldsymbol{I})$.

The following proposition is a natural generalization of [**2**, Corollary I. 4.7], and this can be shown similarly.

PROPOSITION 2.3 ([**2**, Corollary I. 4.7] for $\mathscr{P} = \{\emptyset\}$). *Suppose that X is a hereditarily normal space and Y is a subspace of X with $\mathscr{P}$-$\operatorname{Ind} Y \le n$, where n is* *an integer ≥ 0. For each collection of $n+1$ pairs (F_i, G_i) of disjoint closed subsets of X, $i = 0, 1, \dots, n$, there are partitions T_i between F_i and G_i in X for every i such that $Y \cap (\cap_{i=0}^n T_i) \in \mathscr{P}$.*

Let m be an integer ≥ 1. For each positive integer $i \leq m$ we put

$$A_i^m = \{(x_1, \dots, x_m) \in \boldsymbol{I}^m : x_i = 0\}, \quad B_i^m = \{(x_1, \dots, x_m) \in \boldsymbol{I}^m : x_i = 1\},$$

$$\overline{A}_i^m = \left\{(x_1, \dots, x_m) \in \boldsymbol{I}^m : 0 \leq x_i \leq \frac{1}{3}\right\},$$
$$\overline{B}_i^m = \left\{(x_1, \dots, x_m) \in \boldsymbol{I}^m : \frac{2}{3} \leq x_i \leq 1\right\}.$$

Note that the set $\overline{A}_i^m$ (resp. $\overline{B}_i^m$) is a closed neighborhood of A_i^m (resp. B_i^m) in $\boldsymbol{I}^m$.

Proposition 2.4 ([**12**, Lemma 5.2]). *Let L_{i_j}, $j = 1, \dots p$, be partitions between the opposite faces $A_{i_j}^n$ and $B_{i_j}^n$ in $\boldsymbol{I}^n$, where $1 \leq i_1 < i_2 < \dots < i_p \leq n$ and $1 \leq p < n$. Then for each $k \in \{1, \dots n\} - \{i_1, \dots i_p\}$, there is a continuum $C \subset \cap_{j=1}^p L_{i_j}$ meeting the faces A_k^n and B_k^n.*

Let J be a subset of $\boldsymbol{I}$. Put $M_J = J \times \boldsymbol{I}^n \subset \boldsymbol{I}^{n+1}$, where $n \geq 0$. Propositions 2.2 and 2.4 easily imply the following.

Proposition 2.5 ([**4**, Proposition 4.5]). *Let L_i be a partition in $\boldsymbol{I}^{n+1}$ between A_i^{n+1} and B_i^{n+1}, where $2 \leq i \leq k$ and $k \leq n+1$. Then, we have the following.*

- (i) $M_{Q_0} \cap (\cap_{i=2}^k L_i) \notin \mathscr{K}_0$.
- (ii) $M_{Q_\alpha} \cap (\cap_{i=2}^k L_i) \notin M(\alpha)$ *and* $M_{P_\alpha} \cap (\cap_{i=2}^k L_i) \notin A(\alpha)$ *for each* α *with* $1 \leq \alpha < \omega_1$.
- (iii) $M_J \cap (\cap_{i=2}^k L_i) \notin \mathscr{AB}$, *where* $J \in Brn(\boldsymbol{I})$.

Now we are ready to prove the following theorem.

Theorem 2.1.

- (i) $\mathscr{K}_0$-Ind $M_{Q_0} = n$ *and* $M_{Q_0} \in \mathscr{S}_0 \cap \mathscr{C}_0$ (i.e. $\mathscr{S}_0$-Ind $M_{Q_0} = \mathscr{C}_0$-Ind $M_{Q_0} = -1$).
- (ii) *Let $1 \leq \alpha < \omega_1$. Then we have*
  - (a) $M(\alpha)$-Ind $M_{Q_\alpha} = n$ *and* $M_{Q_\alpha} \in A(\alpha)$ (i.e. $A(\alpha)$-Ind $M_{Q_\alpha} = -1$),
  - (b) $A(\alpha)$-Ind $M_{P_\alpha} = n$ *and* $M_{P_\alpha} \in M(\alpha)$ (i.e. $M(\alpha)$-Ind $M_{P_\alpha} = -1$).

(iii) $\mathscr{AB}$-$\operatorname{Ind} M_J = n$ *if* $J \in Brn(\boldsymbol{I})$.

Furthermore, it follows that $\operatorname{Ind} M_J = n$ *for all considered above cases.*

Proof. We show (i)-(iii) simultaneously. If $n = 0$ then $M_J = J$ and the theorem is evidently valid. Suppose that $n \geq 1$. It follows from Propositions 2.3 and 2.5 that $\mathscr{P}$-$\operatorname{Ind} M_J \geq n$, where $\mathscr{P}$ is $\mathscr{K}_0$ for (i), $M(\alpha)$ for (ii a), $A(\alpha)$ for (ii b) and $\mathscr{AB}$ for (iii). Observe that all sets J considered here are zero-dimensional. Hence $\mathscr{P}$-$\operatorname{Ind} M_J \leq \operatorname{Ind} M_J = n$ for each case (i)-(iii). □

Remark 2.1. Observe that (i) of Theorem 2.1, (ii a) of the case of $\alpha = 1$ and (ii b) of the case of $\alpha = 1$ can be obtained from [**2**, Example II.4.11 (a)], [**2**, Example II.4.11 (c)] and [**2**, Example II.4.11 (b)] respectively.

Remark 2.2. Because of the monotonicity of dimensions modulo classes $\mathscr{P}$ with respect to closed subsets the integer n in Theorem 2.1 can be substituted by ∞.

Remark 2.3. For any integers $0 \leq m \leq n$ there exists a space $X(m,n)$ such that $\operatorname{cmp} X(m,n) = m$ and $\mathscr{K}_0$-$\operatorname{Ind} X(m,n) = n$. Indeed, recall that $\mathscr{K}_0$-Ind $\boldsymbol{R}^n = n$ ([**2**, Example II.6.12 (a)]) for each $n \geq 1$ and $\operatorname{cmp}(Q_1 \times \boldsymbol{I}^m) = m$ ([**2**, Example I.7.12]) for each $m \geq 0$. Put $X(m,n) = \boldsymbol{R}^n \oplus (Q_1 \times \boldsymbol{I}^m)$.

For an isolated ordinal number α we denote by α^- the predecessor of α.

3. Counterparts of Smirnov's compacta for inductive functions $\mathscr{P}$-trInd.

Let $X = \oplus_{i=1}^{\infty} X_i$ be the topological sum of spaces X_i, $i = 1, 2 \ldots$. The *one-point extension* X_+ of the space X is the union $\{x_\infty\} \cup X$ of the set X and a point $x_\infty \notin X$ (we will call this point *the extension point of* X_+) with the topology defined as follows: A set $U \subset X_+$ is open if and only if either U is an open subset of the space X or $X_+ \setminus U$ is a closed subset of X and there exists an integer n such that $\oplus_{i=n}^{\infty} X_i \subset U$.

Henceforth, $X \hookrightarrow Y$ denotes an embedding of a space X into a space Y.

Proposition 3.1.

(i) *The space* X_+ *is separable metrizable.*
(ii) *If* $X_i \hookrightarrow Y_i$ *for each* $i = 1, 2, \ldots$, *then* $X_+ \hookrightarrow Y_+$.
(iii) *If* X_i *is compact for each* i, *then* X_+ *is the Alexandroff compactification of* $X = \oplus_{i=1}^{\infty} X_i$.
(iv) *Let* $\alpha \geq 1$ *and* $\mathscr{P}$ *be either the absolutely multiplicative class* $M(\alpha)$ *or the*

absolutely additive class $A(\alpha)$. If $X_i \in \mathscr{P}$ for each $i = 1, 2, \dots,$ then $X_+ \in \mathscr{P}$.

PROOF. (i)-(iii) are evident. We show (iv). Choose for each $i = 1, 2, \dots$ a compact space Y_i such that $X_i \subset Y_i$. Recall that $X_+ \hookrightarrow Y_+$, the class $\sum^0_\alpha(\cdot)$ is countably additive and $\neg \sum^0_\alpha(\cdot) = \prod^0_\alpha(\cdot)$. $\square$

We will suggest a generalization of Smirnov's construction.

DEFINITION 3.1. Let X be a space. For each $0 \leq \alpha < \omega_1$ we define by induction the space S^α_X as follows.

(i) If $\alpha < \omega_0$, then $S^\alpha_X = X \times \boldsymbol{I}^\alpha$.
(ii) If α is a limit number, then S^α_X is the one-point extension of the topological sum $\oplus_{\beta<\alpha} S^\beta_X$.
(iii) If $\alpha \geq \omega_0$ and α is not limit, then $S^\alpha_X = S^{\alpha-1}_X \times \boldsymbol{I}$.

One can easily show the following elementary properties on S^α_X.

PROPOSITION 3.2. *Let $\alpha < \omega_1$. Then we have the following.*

(i) *If X is a singleton, then S^α_X is the Smirnov's compactum S^α.*
(ii) *If $X_1 \hookrightarrow X_2$ then $S^\alpha_{X_1} \hookrightarrow S^\alpha_{X_2}$.*
(iii) *If $\dim X < \infty$ and $\omega_0 \leq \alpha$ then $S^\alpha_X \hookrightarrow S^\alpha$.*
(iv) *$S^\alpha_{Q_0} \in \mathscr{C}_0 \cap \mathscr{S}_0$, and for each β with $1 \leq \beta < \omega_1$ we have $S^\alpha_{Q_\beta} \in A(\beta)$ and $S^\alpha_{P_\beta} \in M(\beta)$.*

Let $\alpha = \lambda(\alpha) + n(\alpha)$ be the natural decomposition of an ordinal number $\alpha \geq 0$ into the sum of the limit number $\lambda(\alpha)$ and the finite number $n(\alpha)$ (if $\alpha < \omega_0$ we adopt $\lambda(\alpha) = 0$).

PROPOSITION 3.3. *For every space X with $\dim X < \infty$, each countable ordinal number α and every compactum K with $\dim K \leq n(\alpha)$ we have*

$$\operatorname{trInd}(S^{\lambda(\alpha)}_X \times K) \leq \begin{cases} \dim X + \alpha, & \text{if} \quad \alpha < \omega_0, \\ \alpha, & \text{if} \quad \omega_0 \leq \alpha < \omega_1. \end{cases}$$

PROOF. Observe that if $\alpha < \omega_0$, then $S^{\lambda(\alpha)}_X = X$ and so $S^{\lambda(\alpha)}_X \times K = X \times K$. Hence for such α we have $\operatorname{trInd}(S^{\lambda(\alpha)}_X \times K) \leq \dim X + \alpha$. We shall prove $\operatorname{trInd}(S^{\lambda(\alpha)}_X \times K) \leq \alpha$ for $\omega_0 \leq \alpha < \omega_1$ by transfinite induction on α. Let $\omega_0 \leq \alpha < \omega_1$, and x_∞ the extension point of the space $S^{\lambda(\alpha)}_X$. Note that for any closed subset F of $S^{\lambda(\alpha)}_X \times K$ which does not meet $\{x_\infty\} \times K$, there are finitely many ordinals $\beta_1, \dots, \beta_n < \lambda(\alpha)$ such that $F \subset \oplus_{i=1}^n S_X^{\beta_i}$. Let $\alpha = \omega_0$. Then $\lambda(\alpha) = \omega_0$, $n(\alpha) = 0$ and $\dim K \leq 0$. Consider disjoint closed subsets A and B in $S_X^{\omega_0} \times K$. We can assume that $A' = A \cap (\{x_\infty\} \times K) \neq \emptyset$ and $B' = B \cap (\{x_\infty\} \times K) \neq \emptyset$. Since $\dim K = 0$, the empty set separates A' and B' in $\{x_\infty\} \times K$. Hence, there exits a partition L between A and B in $S_X^{\omega_0} \times K$ which extends the empty partition. It is clear that L is contained in the topological sum of finitely many finite-dimensional sets. Hence $\operatorname{Ind} L < \omega_0$ and $\operatorname{trInd}(S_X^{\omega_0} \times K) \leq \omega_0$. Hence the statement is valid for $\alpha = \omega_0$.

Let $\beta > \omega_0$ and assume that the inequality holds for all α with $\omega_0 \leq \alpha < \beta < \omega_1$. If β is limit then the statement is valid by inductive assumption and a similar argument as in the case of $\alpha = \omega_0$. Then we suppose that $\beta = \beta^- + 1$. Consider disjoint closed subsets A and B in $S_X^{\lambda(\beta)} \times K$. We can assume that $A' = A \cap (\{x_\infty\} \times K) \neq \emptyset$ and $B' = B \cap (\{x_\infty\} \times K) \neq \emptyset$. Choose open subsets O_A, O_B in K and a clopen neighborhood V of x_∞ in $S_X^{\lambda(\beta)}$ such that

(i) $A' \subset O_A$, $B' \subset O_B$ and $\operatorname{Cl} O_A \cap \operatorname{Cl} O_B = \emptyset$, and

(ii) $A \cap (V \times K) \subset V \times \operatorname{Cl} O_A$ and $B \cap (V \times K) \subset V \times \operatorname{Cl} O_B$.

By our assumption, we can find a partition L' between $\operatorname{Cl} O_A$ and $\operatorname{Cl} O_B$ in K such that $\dim L' \leq n(\beta^-) < n(\beta)$. It is evident that the set $L'' = V \times L'$ is a partition between $A \cap (V \times K)$ and $B \cap (V \times K)$ in $V \times K$, and $V \times K$ is a clopen subset of $S_X^{\lambda(\beta)} \times K$. By the inductive assumption it follows that $\operatorname{trInd} L'' \leq \beta^- < \beta$. Extend the partition L'' to a partition L between A and B in $S_X^{\lambda(\beta)} \times K$. Evidently, the set $L''' = L \setminus L''$ is the topological sum of finitely many sets with $\operatorname{trInd} < \lambda(\beta)$. Note also that the partition $L = L'' \oplus L'''$ is the topological sum of L'' and L'''. So $\operatorname{trInd} L \leq \beta^- < \beta$ and hence $\operatorname{trInd}(S_X^{\lambda(\beta)} \times K) \leq \beta$. $\square$

Proposition 3.4. *Let J be a subspace of $\boldsymbol{I}$. For each countable ordinal α, each integer $n \geq 1$ and each partition L'_i in $S_J^\alpha \times \boldsymbol{I}^n$ between $S_J^\alpha \times \overline{A}_i^n$ and $S_J^\alpha \times \overline{B}_i^n$, $i = 1, \dots, n$, we have*

$$\alpha \leq \begin{cases} \mathscr{K}_0\text{-}\operatorname{trInd}(\cap_{i=1}^n L'_i), & \text{if } J = Q_0, \\ M(\beta)\text{-}\operatorname{trInd}(\cap_{i=1}^n L'_i), & \text{if } J = Q_\beta \text{ and } 1 \leq \beta < \omega_1, \\ A(\beta)\text{-}\operatorname{trInd}(\cap_{i=1}^n L'_i), & \text{if } J = P_\beta \text{ and } 1 \leq \beta < \omega_1, \\ \mathscr{A}\mathscr{B}\text{-}\operatorname{trInd}(\cap_{i=1}^n L'_i), & \text{if } J \in Brn(\boldsymbol{I}). \end{cases} \qquad (3.1)_\alpha$$

Proof. Apply induction on α. If $\alpha = 0$ then $S_J^\alpha \times \boldsymbol{I}^n = J \times \boldsymbol{I}^n = M_J \subset \boldsymbol{I}^{n+1}$ and $S_J^\alpha \times \overline{A}_k^n = M_J \cap \overline{A}_{k+1}^{n+1}$, $S_J^\alpha \times \overline{B}_k^n = M_J \cap \overline{B}_{k+1}^{n+1}$ for every k. For each i with $2 \leq i \leq n+1$, there is a partition L_i in $\boldsymbol{I}^{n+1}$ between A_i^{n+1} and B_i^{n+1} such that $L_i \cap M_J = L'_{i-1}$. Since $(\cap_{i=2}^{n+1} L_i) \cap M_J = \cap_{i=1}^n L'_i$, by Proposition 2.5, we have the inequality $(3.1)_0$. Let $\mu > 0$ be a countable ordinal and assume that $(3.1)_\alpha$ holds for all α with $\alpha < \mu$. Let $\mathscr{P}$ be either $\mathscr{K}_0$ if $J = Q_0$, $M(\beta)$ if $J = Q_\beta$, $A(\beta)$ if $J = P_\beta$, or $\mathscr{AB}$ if $J \in Brn(\boldsymbol{I})$. Consider an integer $n \geq 1$ and suppose that for each $i = 1, 2, \ldots, n$, there exists a partition L'_i in $S^\mu_J \times \boldsymbol{I}^n$ between $S^\mu_J \times \overline{A}^n_i$ and $S^\mu_J \times \overline{B}^n_i$ such that $\mathscr{P}\text{-trInd}(\cap^n_{i=1} L'_i) = \gamma < \mu$. If μ is a limit number, then $\gamma + 1 < \mu$. Note that for each $i = 1, 2, \ldots, n$, the set $L''_i = L'_i \cap (S^{\gamma+1}_J \times \boldsymbol{I}^n)$ is a partition between $S^{\gamma+1}_J \times \overline{A}^n_i$ and $S^{\gamma+1}_J \times \overline{B}^n_i$ in the clopen subset $S^{\gamma+1}_J \times \boldsymbol{I}^n$ of $S^\mu_J \times \boldsymbol{I}^n$. On the other hand, $\mathscr{P}\text{-trInd}\,(\cap^n_{i=1} L''_i) \leq \mathscr{P}\text{-trInd}\,(\cap^n_{i=1} L'_i) = \gamma < \gamma + 1$. This is a contradiction with the inductive assumption. If $\mu = \mu^- + 1$, then we have $S^\mu_J \times \boldsymbol{I}^n = S^{\mu^-}_J \times \boldsymbol{I}^{n+1}$ and $\gamma \leq \mu^-$. We put $F = \cap^n_{i=1} L'_i$. By our assumption, $\mathscr{P}\text{-trInd}\, F = \gamma < \mu$. Hence, there exists a partition L''_0 between $F \cap A$ and $F \cap B$ in F, where $A = S^{\mu^-}_J \times [0, 1/3] \times \boldsymbol{I}^n$ and $B = S^{\mu^-}_J \times [2/3, 1] \times \boldsymbol{I}^n$, such that $\mathscr{P}\text{-trInd}\, L''_0 < \gamma \leq \mu^-$. There exists a partition L'_0 between A and B in $S^\mu_J \times \boldsymbol{I}^n = S^{\mu^-}_J \times \boldsymbol{I}^{n+1}$ such that $F \cap L'_0 \subset L''_0$ (see [**5**, Lemma 1.2.9 and Remark 1.2.10]). Hence we have $\mathscr{P}\text{-trInd}\,(\cap^n_{i=0} L'_i) \leq \mathscr{P}\text{-trInd}\, L''_0 < \gamma \leq \mu^-$, which also contradicts the inductive assumption. □

Now we are ready to extend Theorem 2.1 to transfinite dimensions.

THEOREM 3.1. *For every countable ordinal α and every $J \subset \boldsymbol{I}$ with* $\dim J = 0$ *we have* $\operatorname{trInd} S^\alpha_J = \alpha$. *Moreover, we have the following.*

(i) *$\mathscr{K}_0\text{-trInd}\, S^\alpha_J = \alpha$ and $\mathscr{C}_0\text{-trInd}\, S^\alpha_J = \mathscr{S}_0\text{-trInd}\, S^\alpha_J = -1$ if $J = Q_0$.*
(ii) *If $1 \leq \beta < \omega_1$, then*
  (a) *$M(\beta)\text{-trInd}\, S^\alpha_J = \alpha$ and $A(\beta)\text{-trInd}\, S^\alpha_J = -1$ if $J = Q_\beta$,*
  (b) *$A(\beta)\text{-trInd}\, S^\alpha_J = \alpha$ and $M(\beta)\text{-trInd}\, S^\alpha_J = -1$ if $J = P_\beta$.*
(iii) *$\mathscr{AB}\text{-trInd}\, S^\alpha_J = \alpha$ if $J \in Brn(\boldsymbol{I})$.*

PROOF. It follows from Proposition 3.3 that $\operatorname{trInd} S^\alpha_J \leq \alpha$. Let $\mathscr{P}$ be either $\mathscr{K}_0$ if $J = Q_0$, $M(\beta)$ if $J = Q_\beta$, $A(\beta)$ if $J = P_\beta$ or $\mathscr{AB}$ if $J \in Brn(\boldsymbol{I})$. It suffices to show that $\mathscr{P}\text{-trInd}\, S^\alpha_J \geq \alpha$, because $\alpha \geq \operatorname{trInd} S^\alpha_J \geq \mathscr{P}\text{-trInd}\, S^\alpha_J$. We notice that, by Proposition 3.4, for every ordinal γ and any partition L' in $S^\gamma_J \times \boldsymbol{I} = S^{\gamma+1}_J$ between $S^\gamma_J \times [0, 1/3]$ and $S^\gamma_J \times [2/3, 1]$ we have $\mathscr{P}\text{-trInd}\, L' \geq \gamma$, hence $\mathscr{P}\text{-trInd}\, S^{\gamma+1}_J > \gamma$. Thus if $\alpha = \gamma + 1$ we have $\mathscr{P}\text{-trInd}\, S^\alpha_J \geq \alpha$ and if α is a limit number then for every $\gamma < \alpha$ we have $\mathscr{P}\text{-trInd}\, S^\alpha_J \geq \mathscr{P}\text{-trInd}\, S^{\gamma+1}_J > \gamma$, because $S^{\gamma+1}_J$ is a clopen subspace of S^α_J. Hence also in this case $\mathscr{P}\text{-trInd}\, S^\alpha_J \geq \alpha$. □

COROLLARY 3.1. *Let α be a countable limit ordinal, $\{\beta_j\}^\infty_{j=1}$ a sequence of ordinals such that $\beta_j < \beta_{j+1}$, for $j \geq 1$, and $\sup \beta_j = \alpha$. Let μ be a countable ordinal number and $X = (\oplus^\infty_{j=1} S^\mu_{P_{\beta_j}})_+$. Then $A(\gamma)\text{-trInd}\, X = M(\gamma)\text{-trInd}\, X = \mu$ for each $\gamma < \alpha$, and $A(\nu)\text{-trInd}\, X = M(\nu)\text{-trInd}\, X = -1$ for each $\nu \geq \alpha$.*

PROOF. Let $\gamma < \alpha$. Then there is β_j such that $\gamma < \beta_j < \alpha$. By Theorem 3.1, we have $M(\gamma)\text{-trInd}\, S^{\mu}_{P_{\beta_j}} = A(\gamma)\text{-trInd}\, S^{\mu}_{P_{\beta_j}} = A(\beta_j)\text{-trInd}\, S^{\mu}_{P_{\beta_j}} = \mu$. Hence $A(\gamma)$-trInd $X \geq \mu$ and $M(\gamma)\text{-trInd}\, X \geq \mu$ by the monotonicity of the inductive dimensions modulo classes. In order to show that $A(\gamma)\text{-trInd}\, X \leq \mu$ let us consider disjoint closed sets F and G of X. It is easy to see that there is a partition L in X between F and G such that L is the topological sum of finitely many sets with $A(\gamma)$-trInd $< \mu$. Hence $A(\gamma)\text{-trInd}\, L < \mu$ and $A(\gamma)\text{-trInd}\, X \leq \mu$. Similarly we get $M(\gamma)\text{-trInd}\, X \leq \mu$. The equalities $A(\nu)\text{-trInd}\, X = M(\nu)\text{-trInd}\, X = -1$ for each $\nu \geq \alpha$ is a direct consequence of Proposition 3.1 (iv). $\square$

REMARK 3.1. Note that $\mathscr{K}_0\text{-trind}\, S^{\omega_0}_{Q_0} = 0$ and $\mathscr{K}_0\text{-trind}\, S^{\omega_0+1}_{Q_0} = 1$. The first equality and the inequality $\mathscr{K}_0\text{-trind}\, S^{\omega_0+1}_{Q_0} \leq 1$ are evident. The inequality $\mathscr{K}_0\text{-trind}\, S^{\omega_0+1}_{Q_0} \geq 1$ can be proved with the help of Proposition 3.5 below due to Charalambous. Indeed, $S^{\omega_0+1}_{Q_0}$ is contained in the class Δ of spaces in Proposition 3.5 below, because every space X with $\text{trInd}\, X \neq \infty$ has a compact subspace $S(X)$ such that for each closed subset $F \subset X$ disjoint from $S(X)$ we have $\dim F < \infty$ ([**5**, Theorem 7.1.23]).

PROPOSITION 3.5 ([**3**]). *Let Δ be the class of all spaces X that contain a compact subspace X_∞ such that every closed set of X disjoint from X_∞ has arbitrary small neighborhoods V with $\dim \mathrm{Bd}\, V < \infty$. Then for each X in Δ we have $\mathscr{P}\text{-trInd}\, X \leq \omega_0 \cdot (\mathscr{P}\text{-trind}\, X + 1)$, where $\mathscr{P}$ is a class of spaces such that if $X = Y \cup Z$, where Y and Z are closed in X and $Y, Z \in \mathscr{P}$, then $X \in \mathscr{P}$.*

Theorem 3.1 and Proposition 3.5 easily imply the following.

COROLLARY 3.2 (cf. [**5**, Example 7.2.12] for trind). *For each β with $0 \leq \beta < \omega_1$ and each $J \in Brn(\boldsymbol{I})$, we have*

$$\sup_{\alpha<\omega_1} M(\beta)\text{-trind}\, S^{\alpha}_{Q_\beta} = \sup_{\alpha<\omega_1} A(\beta)\text{-trind}\, S^{\alpha}_{P_\beta} = \sup_{\alpha<\omega_1} \mathscr{AB}\text{-trind}\, S^{\alpha}_{J} = \omega_1.$$

Furthermore, by the inductive character of the function $\mathscr{P}$-trind, we get the following statement which answers [**4**, Problem 4.1].

COROLLARY 3.3. *For every countable ordinal number α there exist spaces H_α and T_α such that*

(i) $\text{trcmp}\, H_\alpha = \alpha \leq \text{trInd}\, H_\alpha \neq \infty$ *and* $\mathscr{C}_0\text{-trInd}\, H_\alpha = \mathscr{S}_0\text{-trInd}\, H_\alpha = -1$, *and*
(ii) $\mathscr{AB}\text{-trind}\, T_\alpha = \alpha \leq \text{trInd}\, T_\alpha \neq \infty$.

Moreover, for each β with $1 \leq \beta < \omega_1$ there exist spaces $Y_\alpha(\beta)$ and $Z_\alpha(\beta)$ such that

(iii) $M(\beta)$-trind $Y_\alpha(\beta) = \alpha \leq \operatorname{trInd} Y_\alpha(\beta) \neq \infty$ *and* $A(\beta)$-trInd $Y_\alpha(\beta) = -1$,
(iv) $A(\beta)$-trind $Z_\alpha(\beta) = \alpha \leq \operatorname{trInd} Z_\alpha(\beta) \neq \infty$ *and* $M(\beta)$-trInd $Z_\alpha(\beta) = -1$.

REMARK 3.2. Observe that a similar result as in Corollary 3.3 (i) can be found in [**9**]. In [**3**, Example 17] Charalambous demonstrated the existence of a space $C^\alpha_{\mathscr{T}}$ such that $\mathscr{T}$-trind $C^\alpha_{\mathscr{T}} = \alpha$ for each α with $\omega_0 < \alpha < \omega_1$ and each class $\mathscr{T}$ consisting of spaces which are Borel sets of any space that contains them. Note that the space $C^\alpha_{\mathscr{T}}$, unlike to the spaces T_α from Corollary 3.1, has $\mathscr{T}$-trInd $C^\alpha_{\mathscr{T}} = \infty$ for each $\alpha > \omega_0$. Indeed, each space $C^\alpha_{\mathscr{T}}$ is a Bernstein set of a space by the construction. Recall [**3**, Proposition 13] that if A is a Bernstein set of a space X with $\omega_0 \leq \mathscr{T}$-trInd $A < \infty$ then $\mathscr{T}$-trInd $A = \operatorname{trInd} X = \omega_0$.

A complement to Theorem 3.1 is the following.

PROPOSITION 3.6 ([**4**]). *For every ordinal number with* $1 \leq \alpha < \omega_1$ *there exist spaces* X_α *and* Y_α *such that*

(i) $A(\alpha)$-trind $X_\alpha = \infty$ *and* $M(\alpha)$-trind $X_\alpha = -1$,
(ii) $A(\alpha)$-trind $Y_\alpha = -1$ *and* $M(\alpha)$-trind $Y_\alpha = \infty$.

We notice that $A(\alpha)$-trInd $X_\alpha = \infty$, $M(\alpha)$-trInd $X_\alpha = -1$ and $A(\alpha)$-trInd $Y_\alpha = -1$, $M(\alpha)$-trInd $Y_\alpha = \infty$ for spaces X_α and Y_α in Proposition 3.6.

4. Main results.

Let $\Omega = \{\alpha : \alpha < \omega_1\}$ and $\mathscr{F}$ be the set of functions $f : \Omega \to \{-1\} \cup \Omega \cup \{\infty\}$ such that $f(\alpha) \geq f(\beta)$ whenever $0 \leq \alpha < \beta < \omega_1$. Note that if $f \in \mathscr{F}$ then for each countable limit ordinal α there exists an ordinal $\beta < \alpha$ such that $f(\gamma) = f(\beta)$ for each $\beta \leq \gamma < \alpha$. Put $f_L(\alpha) = f(\beta)$. An ordinal α, $1 \leq \alpha < \omega_1$, is said to be *a lowered point* of $f \in \mathscr{F}$ if $f(\alpha) < \min\{f(\gamma) : \gamma < \alpha\}$. Denote by $Low(f)$ the set of all lowered points of f. It is evident that the cardinality of $Low(f)$ is finite for each $f \in \mathscr{F}$. An ordered pair (f_1, f_2) of functions from $\mathscr{F}$ is said to be *admissible* if

(i) $f_1(0) \geq f_2(0) \geq \max\{f_1(1), f_2(1)\}$, and
(ii) $\min\{f_1(\alpha), f_2(\alpha)\} \geq \max\{f_1(\beta), f_2(\beta)\}$, if $1 \leq \alpha < \beta < \omega_1$.

For every admissible pair (f_1, f_2) put $Low(f_1, f_2) = Low(f_1) \cup Low(f_2)$.

PROPOSITION 4.1. *Let* (f_1, f_2) *be admissible and* $0 \leq \alpha < \beta < \omega_1$. *If* $f_i(\alpha) = f_i(\beta) = \mu_i \geq -1$ *for each* $i = 1, 2$, *then* $\mu_1 = \mu_2 = \mu$ *and for each ordinal* γ *with* $\alpha \leq \gamma \leq \beta$ *we have* $f_1(\gamma) = f_2(\gamma) = \mu$.

PROOF. Note that $\min\{f_1(\alpha), f_2(\alpha)\} = \min\{\mu_1, \mu_2\} \geq \max\{f_1(\beta), f_2(\beta)\} =$

$\max\{\mu_1, \mu_2\}$ and $f_i(\alpha) \geq f_i(\gamma) \geq f_i(\beta)$ for each ordinal γ with $\alpha \leq \gamma \leq \beta$. The rest is evident. □

The following is a direct consequence of Proposition 4.1.

COROLLARY 4.1. *Let (f_1, f_2) be an admissible pair. Then we have the following.*

(i) *If $Low(f_1, f_2) = \emptyset$, then f_1 and f_2 are constant maps and $f_1 = f_2$.*

(ii) *Let $Low(f_1, f_2) = \{\alpha_1, \ldots, \alpha_k\} \neq \emptyset$, where $\alpha_i < \alpha_j$ if $i < j$, $\alpha_0 = 0$, $\alpha_{k+1} = \omega_1$ and $1 \leq p \leq k+1$. Then the following is valid.*

  (a) *If α_p is limit, then there is $\mu_p \in \{-1\} \cup \Omega \cup \{\infty\}$ such that $f_1(\gamma) = f_2(\gamma) = \mu_p$ for each γ with $\alpha_{p-1} \leq \gamma < \alpha_p$ and hence, $\mu_p = (f_1)_L(\alpha_p) = (f_2)_L(\alpha_p)$.*

  (b) *If $\alpha_p = \alpha_p^- + 1$ and $\alpha_{p-1} < \alpha_p^-$, then there is $\mu_p \in \{-1\} \cup \Omega \cup \{\infty\}$ such that $f_1(\gamma) = f_2(\gamma) = \mu_p$ for each γ with $\alpha_{p-1} \leq \gamma \leq \alpha_p^-$, moreover*

$$\mu_p = \begin{cases} f_2(\alpha_p) \text{ if } \alpha_p \in Low(f_1) \setminus Low(f_2), \\ f_1(\alpha_p) \text{ if } \alpha_p \in Low(f_2) \setminus Low(f_1). \end{cases}$$

We are ready for our main result.

THEOREM 4.1. *Let (f_1, f_2) be admissible. Then there exists a space X such that $A(\alpha)$-$\mathrm{trInd}\, X = f_1(\alpha)$ and $M(\alpha)$-$\mathrm{trInd}\, X = f_2(\alpha)$ for each α with $0 \leq \alpha < \omega_1$. Moreover, we have $\mathscr{AB}$-$\mathrm{trInd}\, X = (f_1)_L(\omega_1) = (f_2)_L(\omega_1)$.*

PROOF. Let $Low(f_1, f_2) = \{\alpha_1, \ldots, \alpha_k\}$, where $\alpha_i < \alpha_j$ if $i < j$, $\alpha_0 = 0$, $\alpha_{k+1} = \omega_1$ and $1 \leq p \leq k+1$ (if $Low(f_1, f_2) = \emptyset$ we put $k = 0$). If α_p is a limit ordinal, then we fix a sequence $\{\beta_j^p\}_{j=1}^\infty$ of ordinals such that $\alpha_{p-1} < \beta_j^p < \beta_{j+1}^p$ for each j and $\sup \beta_j^p = \alpha_p$. For each $i = 1, \ldots, k+1$, we put

$$X_i = \begin{cases} S_{P_{\alpha_i^-}}^{f_1(\alpha_i^-)}, \text{ if } \alpha_i = \alpha_i^- + 1 \text{ and } \alpha_i \in Low(f_1) \setminus Low(f_2), \\ S_{Q_{\alpha_i^-}}^{f_2(\alpha_i^-)}, \text{ if } \alpha_i = \alpha_i^- + 1 \text{ and } \alpha_i \in Low(f_2) \setminus Low(f_1), \\ S_{Q_{\alpha_i^-}}^{f_2(\alpha_i^-)} \oplus S_{P_{\alpha_i^-}}^{f_1(\alpha_i^-)}, \text{ if } \alpha_i = \alpha_i^- + 1 \text{ and } \alpha_i \in Low(f_1) \cap Low(f_2), \\ (\oplus_{j=1}^\infty S_{P_{\beta_j^i}}^{(f_1)_L(\alpha_i)})_+, \text{ if } \alpha_i \text{ is a limit ordinal}, \\ S_J^{(f_1)_L(\omega_1)}, \text{ where } J \in Brn(\boldsymbol{I}), \text{ if } i = k+1, \end{cases}$$

where S^α_J is the space defined in the previous section. Furthermore, we put $X = \oplus_{i=1}^{k+1} X_i$. Then it follows from Theorem 3.1 and Corollary 3.1 that for each $i = 1, \dots, k+1$ we have the values of $A(\alpha)$-trInd X_i and $M(\alpha)$-trInd X_i as follows.
(a) If $1 \le i \le k$, α_i is not a limit ordinal and $\alpha_i \in Low(f_1) \setminus Low(f_2)$, then

$$\begin{cases} A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = f_1(\alpha_i^-), \text{ if } 0 \le \gamma < \alpha_i^-, \\ A(\alpha_i^-)\text{-trInd}\, X_i = f_1(\alpha_i^-), \\ M(\alpha_i^-)\text{-trInd}\, X_i = -1, \\ A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = -1, \text{ if } \alpha_i \le \gamma < \omega_1. \end{cases}$$

(b) If $1 \le i \le k$, α_i is not a limit ordinal and $\alpha_i \in Low(f_2) \setminus Low(f_1)$, then

$$\begin{cases} A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = f_2(\alpha_i^-), \text{ if } 0 \le \gamma < \alpha_i^-, \\ A(\alpha_i^-)\text{-trInd}\, X_i = -1, \\ M(\alpha_i^-)\text{-trInd}\, X_i = f_2(\alpha_i^-), \\ A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = -1 \text{ if } \alpha_i \le \gamma < \omega_1. \end{cases}$$

(c) If $1 \le i \le k$, α_i is not a limit ordinal and $\alpha_i \in Low(f_1) \cap Low(f_2)$, then

$$\begin{cases} A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = \max\{f_1(\alpha_i^-), f_2(\alpha_i^-)\}, \text{ if } 0 \le \gamma < \alpha_i^-, \\ A(\alpha_i^-)\text{-trInd}\, X_i = f_1(\alpha_i^-), \\ M(\alpha_i^-)\text{-trInd}\, X_i = f_2(\alpha_i^-), \\ A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = -1, \text{ if } \alpha_i \le \gamma < \omega_1. \end{cases}$$

(d) If $1 \le i \le k$ and α_i is a limit ordinal, then

$$\begin{cases} A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = (f_1)_L(\alpha_i) = (f_2)_L(\alpha_i), \text{ if } 0 \le \gamma < \alpha_i, \\ A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = -1 \text{ if } \alpha_i \le \gamma < \omega_1. \end{cases}$$

(e) If $i = k+1$, then

$$A(\gamma)\text{-trInd}\, X_i = M(\gamma)\text{-trInd}\, X_i = (f_1)_L(\omega_1) = (f_2)_L(\omega_1), \text{ if } 0 \le \gamma < \omega_1.$$

Furthermore, we have the following.

(4.1) If $0 \le i \le k$ and $\alpha_i \le \gamma < \omega_1$, then $A(\gamma)$-trInd $X = A(\gamma)$-trInd $(\cup_{p=i+1}^{k+1} X_p)$ and $M(\gamma)$-trInd $X = M(\gamma)$-trInd $(\cup_{p=i+1}^{k+1} X_p)$.

(4.2) If $0 < i \leq k$, $p \geq i+1$ and $0 \leq \gamma < \alpha_i$, then
$\max\{A(\gamma)\text{-trInd}\, X_p, M(\gamma)\text{-trInd}\, X_p\} \leq \max\{f_1(\alpha_i), f_2(\alpha_i)\}$.

Indeed, let $0 \leq i \leq k$, $\alpha_i \leq \gamma < \omega_1$ and $\mathscr{P}$ either $A(\gamma)$ or $M(\gamma)$. Then it follows from the above estimations (a), (b), (c) and (d) that

$$\begin{aligned}
\mathscr{P}\text{-trInd}\, X &= \max\{\mathscr{P}\text{-trInd}\, X_1, \ldots, \mathscr{P}\text{-trInd}\, X_i, \mathscr{P}\text{-trInd}\,(\cup_{p=i+1}^{k+1} X_p)\} \\
&= \max\{-1, \mathscr{P}\text{-trInd}\,(\cup_{p=i+1}^{k+1} X_p)\} \\
&= \mathscr{P}\text{-trInd}\,(\cup_{p=i+1}^{k+1} X_p).
\end{aligned}$$

This implies (4.1). Next, we shall show (4.2). Let $0 < i \leq k$, $p \geq i+1$ and $0 \leq \gamma < \alpha_i$. If $i = k$ we have $\max\{f_1(\alpha_k), f_2(\alpha_k)\} \geq (f_1)_L(\omega_1) = (f_2)_L(\omega_1) = \mathscr{P}\text{-trInd}\, X_{k+1}$. If $0 < i < k$, then we have

$$\max\{f_1(\alpha_i), f_2(\alpha_i)\} \geq \left\{ \begin{array}{l} \max\{f_1(\alpha_p^-), f_2(\alpha_p^-)\},\ \text{if } \alpha_p \text{ is not limit} \\ (f_1)_L(\alpha_p) = (f_2)_L(\alpha_p),\ \text{if } \alpha_p \text{ is limit} \end{array} \right\} \geq \mathscr{P}\text{-trInd}\, X_p.$$

Let us continue the proof of the theorem. Assume first that $Low(f_1, f_2) = \emptyset$ (the case of $k = 0$). Then, by Corollary 4.1 (i), f_1 and f_2 are constant maps and $f_1 = f_2$. It follows from Theorem 3.1 (iii) that $f_1(\alpha) = f_2(\alpha) = (f_1)_L(\omega_1) = \mathscr{P}\text{-trInd}\, X_{k+1} = \mathscr{P}\text{-trInd}\, X$ for each ordinal α and each class $\mathscr{P}$ from (*).

Assume now that $Low(f_1, f_2) \neq \emptyset$ (the case of $k \geq 1$). We consider the following condition $(\#)_i$ for each i with $0 \leq i \leq k$.

$(\#)_i$ For each ordinal γ with $\alpha_i \leq \gamma < \omega_1$, $A(\gamma)\text{-trInd}\, X = f_1(\gamma)$ and
$M(\gamma)\text{-trInd}\, X = f_2(\gamma)$.

It suffices to show that $(\#)_0$ holds and we shall show inductively $(\#)_i$ for every i. At first, we consider $(\#)_k$. By Corollary 4.1 (ii) (a), there is an ordinal $\mu_{k+1} \geq -1$ such that for each $\alpha_k \leq \gamma < \omega_1$ we have $f_1(\gamma) = f_2(\gamma) = \mu_{k+1}$. Note that $\mu_{k+1} = (f_1)_L(\omega_1)$. Let γ be an ordinal such that $\alpha_k \leq \gamma < \omega_1$ and $\mathscr{P}$ be either $A(\gamma)$ or $M(\gamma)$. It follows from (4.1) and (e) that $\mathscr{P}\text{-trInd}\, X = \mathscr{P}\text{-trInd}\, X_{k+1} = (f_1)_L(\omega_1) = f_1(\gamma) = f_2(\gamma)$. Hence $(\#)_k$ holds.

Assume that $(\#)_i$ holds for some $i \leq k$. We will show $(\#)_{i-1}$. If α_i is limit, then, by Corollary 4.1 (ii) (a), there is $\mu_i \in \{-1\} \cup \Omega \cup \{\infty\}$ such that $f_1(\gamma) = f_2(\gamma) = \mu_i$ for each γ with $\alpha_{i-1} \leq \gamma < \alpha_i$. Let $\alpha_{i-1} \leq \gamma < \alpha_i$ and $\mathscr{P}$ be either $A(\gamma)$ or $M(\gamma)$. Note that $\mu_i = (f_1)_L(\alpha_i) = (f_2)_L(\alpha_i) = \mathscr{P}\text{-trInd}\, X_i$ by (d). Moreover, by (4.2), we have $\mathscr{P}\text{-trInd}\, X_p \leq \max\{f_1(\alpha_i), f_2(\alpha_i)\} \leq \mu_i$ for each p with $i+1 \leq p \leq k+1$. Hence, by (4.1), we get $\mathscr{P}\text{-trInd}\, X = \mathscr{P}\text{-trInd}\,(\cup_{p=i}^{k+1} X_p) = \max\{\mathscr{P}\text{-trInd}$

$X_p : i \le p \le k+1\} = \max\{\mu_i, \max\{\mathscr{P}\text{-trInd}\, X_p : i+1 \le p \le k+1\}\} = \mu_i$ that precisely as we needed.

If α_i is not limit, then we consider three cases separately.

CASE (1). Suppose that $\alpha_i \in \mathit{Low}(f_1) \setminus \mathit{Low}(f_2)$. Then, $f_1(\alpha_i^-) > f_1(\alpha_i)$ and $f_2(\alpha_i^-) = f_2(\alpha_i)$. Since (f_1, f_2) is admissible, it follows that $f_2(\alpha_i^-) \ge \min\{f_1(\alpha_i^-), f_2(\alpha_i^-)\} \ge \max\{f_1(\alpha_i), f_2(\alpha_i)\} \ge f_2(\alpha_i)$, and hence $f_2(\alpha_i^-) = \min\{f_1(\alpha_i^-), f_2(\alpha_i^-)\} = \max\{f_1(\alpha_i), f_2(\alpha_i)\} = f_2(\alpha_i)$. By (a), we notice that $A(\alpha_i^-)\text{-trInd}\, X_i = f_1(\alpha_i^-)$ and $M(\alpha_i^-)\text{-trInd}\, X_i = -1$. It follows from (4.2) that if $\mathscr{P} = A(\alpha_i^-)$ or $M(\alpha_i^-)$ and $i+1 \le p \le k+1$, then we have $\mathscr{P}\text{-trInd}\, X_p \le \max\{f_1(\alpha_i), f_2(\alpha_i)\} = f_2(\alpha_i) = f_2(\alpha_i^-) \le f_1(\alpha_i^-)$. Hence, by (4.1), it follows that $A(\alpha_i^-)\text{-trInd}\, X = A(\alpha_i^-)\text{-trInd}\,(\cup_{p=i}^{k+1} X_p) = \max\{A(\alpha_i^-)\text{-trInd}\, X_p : i \le p \le k+1\} = f_1(\alpha_i^-)$, and $M(\alpha_i^-)\text{-trInd}\, X = M(\alpha_i^-)\text{-trInd}\,(\cup_{p=i}^{k+1} X_p) = \max\{M(\alpha_i^-)\text{-trInd}\, X_p : i \le p \le k+1\} = \max\{-1, \max\{M(\alpha_i^-)\text{-trInd}\, X_p : i+1 \le p \le k+1\}\} \le f_2(\alpha_i)$. On the other hand, by the inductive assumption $(\#)_i$, we have $M(\alpha_i^-)\text{-trInd}\, X \ge M(\alpha_i)\text{-trInd}\, X = f_2(\alpha_i)$. Hence $M(\alpha_i^-)\text{-trInd}\, X = f_2(\alpha_i) = f_2(\alpha_i^-)$. Thus if $\alpha_i^- = \alpha_{i-1}$, we get $(\#)_{i-1}$.

Now, we assume that $\alpha_{i-1} < \alpha_i^-$. Then, by Corollary 4.1 (ii) (b), there is $\mu_i \in \{-1\} \cup \Omega \cup \{\infty\}$ such that $f_1(\gamma) = f_2(\gamma) = \mu_i$ for each γ with $\alpha_{i-1} \le \gamma \le \alpha_i^-$. Let $\alpha_{i-1} \le \gamma < \alpha_i^-$ and $\mathscr{P}$ be either $A(\gamma)$ or $M(\gamma)$. By (a) again, it follows that $\mathscr{P}\text{-trInd}\, X_i = f_1(\alpha_i^-) = \mu_i$. Note that, by (4.2), we have $\mathscr{P}\text{-trInd}\, X_p \le \max\{f_1(\alpha_i), f_2(\alpha_i)\} = f_2(\alpha_i) = f_2(\alpha_i^-) = \mu_i$. Hence, by (4.1), we get $\mathscr{P}\text{-trInd}\, X = \mathscr{P}\text{-trInd}\,(\cup_{p=i}^{k+1} X_p) = \max\{\mathscr{P}\text{-trInd}\, X_p : i \le p \le k+1\} = \mu_i = f_1(\gamma) = f_2(\gamma)$ precisely as we needed.

CASE (2). If $\alpha_i \in \mathit{Low}(f_2) \setminus \mathit{Low}(f_1)$, then we can prove $(\#)_{i-1}$ similar to the case (1).

CASE (3). Suppose that $\alpha_i \in \mathit{Low}(f_1) \cap \mathit{Low}(f_2)$. It follows from (c) that $A(\alpha_i^-)\text{-trInd}\, X_i = f_1(\alpha_i^-)$ and $M(\alpha_i^-)\text{-trInd}\, X_i = f_2(\alpha_i^-)$. Note that by (4.2) for $\mathscr{P}$ is either $A(\alpha_i^-)$ or $M(\alpha_i^-)$ we have $\mathscr{P}\text{-trInd}\, X_p \le \max\{f_1(\alpha_i), f_2(\alpha_i)\} \le \min\{f_1(\alpha_i^-), f_2(\alpha_i^-)\}$ for each p with $i+1 \le p \le k+1$. Hence, by (4.1), $A(\alpha_i^-)\text{-trInd}\, X = A(\alpha_i^-)\text{-trInd}\,(\cup_{p=i}^{k+1} X_p) = \max\{A(\alpha_i^-)\text{-trInd}\, X_p : i \le p \le k+1\} = f_1(\alpha_i^-)$, and $M(\alpha_i^-)\text{-trInd}\, X = M(\alpha_i^-)\text{-trInd}\,(\cup_{p=i}^{k+1} X_p) = \max\{M(\alpha_i^-)\text{-trInd}\, X_p : i \le p \le k+1\} = f_2(\alpha_i^-)$. Hence, we get $(\#)_{i-1}$ if $\alpha_{i-1} = \alpha_i^-$. Now, we assume that $\alpha_{i-1} < \alpha_i^-$. Then, by Corollary 4.1 (ii) (b), there is $\mu_i \in \{-1\} \cup \Omega \cup \{\infty\}$ such that $f_1(\gamma) = f_2(\gamma) = \mu_i$ for each γ with $\alpha_{i-1} \le \gamma \le \alpha_i^-$. Let $\alpha_{i-1} \le \gamma < \alpha_i^-$ and $\mathscr{P}$ is either $A(\gamma)$ or $M(\gamma)$. Then, by (c) again, it follows that $\mathscr{P}\text{-trInd}\, X_i = \mu_i$. Note that by (4.2) we have $\mathscr{P}\text{-trInd}\, X_p \le \max\{f_1(\alpha_i), f_2(\alpha_i)\} \le \min\{f_1(\alpha_i^-), f_2(\alpha_i^-)\} = \mu_i$. Hence, we get $\mathscr{P}\text{-trInd}\, X = \mathscr{P}\text{-trInd}\,(\cup_{p=i}^{k+1} X_p) = \max\{\mathscr{P}\text{-trInd}\, X_p : i \le p \le k+1\} = \mu_i$, and hence $(\#)_{i-1}$ holds. $\square$

Question 4.1. Is the counterpart of Theorem 4.1 for the small transfinite inductive dimensions modulo $\mathscr{P}$ valid ?

Let $\mathscr{F}_1 = \{f \in \mathscr{F} : f(\Omega) \subset \{-1\} \cup \{\alpha : \alpha < \omega_0\} \cup \{\infty\}\}$.

Corollary 4.2. *Let $(f_1, f_2) \in \mathscr{F}_1 \times \mathscr{F}_1$ be admissible. Then there exists a space X such that $A(\alpha)$-$\operatorname{Ind} X = f_1(\alpha)$, $M(\alpha)$-$\operatorname{Ind} X = f_2(\alpha)$ for each α with $0 \leq \alpha < \omega_1$ and $\mathscr{AB}$-$\operatorname{Ind} X = (f_1)_L(\omega_1) = (f_2)_L(\omega_1)$.*

Question 4.2. Is Corollary 4.2 valid for the small inductive dimensions modulo $\mathscr{P}$?

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