J. Math. Soc. Japan
Vol. 51, No. 2, 1999

Abelian varieties and the class invariant homomorphism

By Gerald E. Welters

(Received June 23, 1997)

Abstract. We discuss the problem of a generalization to higher dimensions of an earlier result of ours on a geometrical description of Selmer groups of elliptic curves over global fields of positive characteristic.

0. Introduction.

Let $\mathscr{G}$ be a finite, flat, commutative group scheme over a scheme B, and let $\mathscr{G}^D$ be its Cartier dual. In the late sixties it was observed by S. U. Chase, S. S. Shatz and W. C. Waterhouse (cf [**8**], [**9**]) that the local-to-global spectral sequence for Ext in the flat topology, $H^i(B, \mathscr{E}xt^j(\mathscr{G}, \boldsymbol{G}_m)) \Rightarrow \mathrm{Ext}^{i+j}(\mathscr{G}, \boldsymbol{G}_m)$, yields in particular an isomorphism $H^1(B, \mathscr{G}^D) \xrightarrow{\simeq} \mathrm{Ext}^1(\mathscr{G}, \boldsymbol{G}_m)$. Composing this with the natural map to $\mathrm{Pic}(\mathscr{G})$, one obtains a morphism $H^1(B, \mathscr{G}^D) \to \mathrm{Pic}(\mathscr{G})$, which associates to every principal homogeneous space or torsor for $\mathscr{G}^D$ its so-called Picard invariant or class invariant in $\mathrm{Pic}(\mathscr{G})$. The image of this map is contained in the subgroup $\mathrm{Pic}(\mathscr{G})^{inv} \subset \mathrm{Pic}(\mathscr{G})$ consisting of those elements $\xi \in \mathrm{Pic}(\mathscr{G})$ such that $s^*(\xi) = p_1^*(\xi) + p_2^*(\xi)$ holds in $\mathrm{Pic}(\mathscr{G} \times_B \mathscr{G})$, where $s, p_1, p_2 : \mathscr{G} \times_B \mathscr{G} \to \mathscr{G}$ denote the addition and the two projection maps, respectively. In analogy with [**6**], p. 182, Théorème 5, Waterhouse raised the question whether the image of the class invariant homomorphism (c.i. homomorphism, for short) actually equals $\mathrm{Pic}(\mathscr{G})^{inv}$, and, for a while, some attention was paid to this problem (cf. [**3**]).

In more recent times (cf. [**1**] and references therein) the c.i. homomorphism appeared in connection with the study of the Galois module structure of certain Kummer orders associated to abelian varieties over global fields. For an abelian variety A over a number field K, with everywhere good reduction, call $B = \mathrm{Spec}\, O_K$ and let $\mathscr{A}, \hat{\mathscr{A}}$ be the Néron models of A and its dual $\hat{A}$, respectively. Given $n \in \boldsymbol{N}$, the exact sequence $0 \to {}_n\hat{\mathscr{A}} \to \hat{\mathscr{A}} \xrightarrow{n} \hat{\mathscr{A}} \to 0$ yields a morphism $\hat{A}(K)/n\hat{A}(K) \to H^1(B, {}_n\hat{\mathscr{A}})$. Taking then in the above $\mathscr{G} = {}_n\mathscr{A}$, one has $\mathscr{G}^D = {}_n\hat{\mathscr{A}}$, and whence a c.i. homomorphism $H^1(B, {}_n\hat{\mathscr{A}}) \to \mathrm{Pic}({}_n\mathscr{A})^{inv}$ which, composed with the preceding one, gives a map $\hat{A}(K)/n\hat{A}(K) \to \mathrm{Pic}({}_n\mathscr{A})^{inv}$. It is this map which plays then a main role in the aforementioned context.

Thirdly, unaware of the preceding, the author himself came across a natural generalization of the c.i. homomorphism in [**10**], while searching for a geometric interpretation for Selmer groups of (non twisted-constant) elliptic curves over global fields of positive characteristics. In this paper, which is a footnote to [**10**], we aim at clarifying this viewpoint, raising a natural issue for abelian varieties of arbitrary

---

1991 Mathematics Subject Classification. Primary 14K15; Secondary 11G05, 14G10.
Key Words and Phrases. Abelian varieties over global (function) fields, Néron model, Selmer group.
Partially supported by the project "AGE-Algebraic Geometry in Europe", contract No. ERB940557.

dimension over such fields, and pointing out some basic aspects of it. If A is an abelian variety over a global field K of positive characteristic p, and $\mathscr{A} \to B$ is the Néron model of A, one has, for most integers $n \in \boldsymbol{N}$ a class invariant homomorphism for the Selmer group of the dual abelian variety $\hat{A}$ of $A : S_n(\hat{A}/K) \to \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}$, where ${}_n\mathscr{A}^c$ is the regular completion of the regular curve ${}_n\mathscr{A}$. For $\dim(A) = 1$ and A not twisted-constant, we showed in [**10**] that, again for most integers n, this map is an isomorphism. We ask whether this result continues to hold for arbitrary abelian varieties, and discuss this question (cf. Proposition 2.6, and Remark 2.7). In particular, the result in [**10**] may be completed to the effect that the answer is yes for abelian varieties which are isogenous to twisted forms of products of elliptic curves and a constant abelian variety, whence including in particular CM-varieties (cf. Theorem 2.8).

Our interest in this question is, as mentioned above, the geometric understanding of Selmer groups in the function field case (cf. also Section 3, in this respect). We point out, moreover, that the framework of the present paper would not be suited, as it stands, for abelian varieties over number fields.

1. Neron models of etale group schemes.

Throughout in this paper, cohomology means etale cohomology. Let K be a global field of positive characteristic p. Call $k \subset K$ its field of constants, and call B the smooth, projective, geometrically irreducible curve over k such that $K = k(B)$. Let G be a finite commutative group scheme over K, of order prime to p. Thus, as a scheme, $G = \bigsqcup_i \mathrm{Spec}(L_i)$, with $K \subset L_i$ finite separable extensions. Writing K^s for a separable closure of K, the group scheme $G \otimes_K K^s$ yields an abstract abelian group, endowed with a continuous $\mathrm{Gal}(K^s/K)$-action. Conversely, by descent theory, every discrete $\mathrm{Gal}(K^s/K)$-module of finite order, prime to p, is obtained in this way. We shall write $\pi^c : \mathscr{G}^c \to B$ for the normalization of B in the ring $\Gamma\mathscr{O}_G = \prod_i L_i$. This is a complete (i.e. proper over $\boldsymbol{Z}$) regular curve, non connected if $G \neq 0$. Let $\pi : \mathscr{G} \to B$ be the restriction of π^c to the open subscheme of $\mathscr{G}^c$ where the map π^c is unramified. This is a quasifinite commutative etale group scheme, the Néron model of G (cf. [**2**], p. 12). And the curve $\mathscr{G}^c$ is the regular completion of $\mathscr{G}$.

Let $H = G^D$ be the Cartier dual of G, and call $\mathscr{H} \to B$ its Néron model. The duality pairing $G \times_K H \to \boldsymbol{G}_{m,K}$ extends uniquely to a bilinear map $\mathscr{G} \times_B \mathscr{H} \to \boldsymbol{G}_{m,B}$. If $B' \subset B$ denotes the open subset complementary to the discriminant locus of π^c (one might say: the open subset where G has good reduction) the restrictions $\mathscr{G}|B'$ and $\mathscr{H}|B'$ are mutually dual finite etale commutative group schemes.

Definition 1.1. In the above notations, we define a natural morphism (the class invariant homomorphism)

$$\gamma_G : H^1(B, \mathscr{H}) \to \mathrm{Pic}(\mathscr{G}^c)^{inv},$$

where $\mathrm{Pic}(\mathscr{G}^c)^{inv} \subset \mathrm{Pic}(\mathscr{G}^c)$ is the following subgroup. Call $(\mathscr{G} \times_B \mathscr{G})^c$ the regular completion of the regular curve $\mathscr{G} \times_B \mathscr{G}$, and call also $s, p_1, p_2 : (\mathscr{G} \times_B \mathscr{G})^c \to \mathscr{G}^c$ the maps which extend the addition and the projection maps $s, p_1, p_2 : \mathscr{G} \times_B \mathscr{G} \to \mathscr{G}$, respectively. Then $\mathrm{Pic}(\mathscr{G}^c)^{inv}$ is the subgroup of $\mathrm{Pic}(\mathscr{G}^c)$ of those elements ξ such that $s^*(\xi) = p_1^*(\xi) + p_2^*(\xi)$ holds, in $\mathrm{Pic}((\mathscr{G} \times_B \mathscr{G})^c)$.

For $\tilde{B} \to B$ a finite map from another complete regular curve to B, we shall denote by $\mathcal{G}_{\tilde{B}}$, $\mathcal{H}_{\tilde{B}}$ the pullbacks of $\mathcal{G}$ and $\mathcal{H}$ to $\tilde{B}$, and by $\mathcal{G}_{(\tilde{B})}$, $\mathcal{H}_{(\tilde{B})}$ the Néron models of the pullbacks of G and H to the generic points of the components of $\tilde{B}$.

We have, as before, a bilinear mapping $e : \mathcal{G}_{(\mathcal{G}^c)} \times_{\mathcal{G}^c} \mathcal{H}_{(\mathcal{G}^c)} \to \boldsymbol{G}_{m,\mathcal{G}^c}$. On the other hand, the tautological section of $\mathcal{G}_{(\mathcal{G}^c)}|\mathcal{G} = \mathcal{G}_{\mathcal{G}}$ extends uniquely to a global section $\omega \in H^0(\mathcal{G}^c, \mathcal{G}_{(\mathcal{G}^c)})$, and cup product with this section gives a map $e(\omega, \ ) : H^1(\mathcal{G}^c, \mathcal{H}_{(\mathcal{G}^c)}) \to H^1(\mathcal{G}^c, \boldsymbol{G}_m) = \mathrm{Pic}(\mathcal{G}^c)$. Then γ_G is obtained by composing this with the pullback map $H^1(B, \mathcal{H}) \to H^1(\mathcal{G}^c, \mathcal{H}_{\mathcal{G}^c})$ followed by the map $H^1(\mathcal{G}^c, \mathcal{H}_{\mathcal{G}^c}) \to H^1(\mathcal{G}^c, \mathcal{H}_{(\mathcal{G}^c)})$. Given $\zeta \in H^1(B, \mathcal{H})$, we shall write $\zeta_{\mathcal{G}^c}$ and $\zeta_{(\mathcal{G}^c)}$ for the images of this element in $H^1(\mathcal{G}^c, \mathcal{H}_{\mathcal{G}^c})$ and $H^1(\mathcal{G}^c, \mathcal{H}_{(\mathcal{G}^c)})$, respectively. In these notations: $\gamma_G(\zeta) = e(\omega, \zeta_{(\mathcal{G}^c)})$.

It remains to show that the image of γ_G lies in $\mathrm{Pic}(\mathcal{G}^c)^{inv}$. To simplify the notations, we put $\tilde{\mathcal{G}} = \mathcal{G} \times_B \mathcal{G}$ and $\tilde{\mathcal{G}}^c = (\mathcal{G} \times_B \mathcal{G})^c$. Call $\tilde{e} : \mathcal{G}_{(\tilde{\mathcal{G}}^c)} \times_{\tilde{\mathcal{G}}^c} \mathcal{H}_{(\tilde{\mathcal{G}}^c)} \to \boldsymbol{G}_{m,\tilde{\mathcal{G}}^c}$ the corresponding bilinear mapping. For all morphisms of B-schemes $f : \tilde{\mathcal{G}}^c \to \mathcal{G}^c$, the morphisms of group schemes $f^*\mathcal{G}_{(\mathcal{G}^c)} \to \mathcal{G}_{(\tilde{\mathcal{G}}^c)}$ and $f^*\mathcal{H}_{(\mathcal{G}^c)} \to \mathcal{H}_{(\tilde{\mathcal{G}}^c)}$ which they canonically induce are compatible with e and $\tilde{e}$, whence the maps $f^* : H^i(\mathcal{G}^c, \mathcal{G}_{(\mathcal{G}^c)}) \to H^i(\tilde{\mathcal{G}}^c, \mathcal{G}_{(\tilde{\mathcal{G}}^c)})$ and $f^* : H^i(\mathcal{G}^c, \mathcal{H}_{(\mathcal{G}^c)}) \to H^i(\tilde{\mathcal{G}}^c, \mathcal{H}_{(\tilde{\mathcal{G}}^c)})$ defined by these satisfy, for all $\alpha \in H^i(\mathcal{G}^c, \mathcal{G}_{(\mathcal{G}^c)})$, $\beta \in H^i(\mathcal{G}^c, \mathcal{H}_{(\mathcal{G}^c)}) : \tilde{e}(f^*\alpha, f^*\beta) = f^*e(\alpha, \beta)$. Applying this to $f = s$, p_1 and p_2 respectively, one has, for all $\zeta \in H^1(B, \mathcal{H}) : s^*e(\omega, \zeta_{(\mathcal{G}^c)}) = \tilde{e}(s^*\omega, s^*\zeta_{(\mathcal{G}^c)}) = \tilde{e}(s^*\omega, \zeta_{(\tilde{\mathcal{G}}^c)}) = \tilde{e}(p_1^*\omega + p_2^*\omega, \zeta_{(\tilde{\mathcal{G}}^c)}) = \tilde{e}(p_1^*\omega, p_1^*\zeta_{(\mathcal{G}^c)}) + \tilde{e}(p_2^*\omega, p_2^*\zeta_{(\mathcal{G}^c)}) = p_1^*e(\omega, \zeta_{(\mathcal{G}^c)}) + p_2^*e(\omega, \zeta_{(\mathcal{G}^c)})$, whence $\gamma_G(\zeta) \in \mathrm{Pic}(\mathcal{G}^c)^{inv}$.

If G has everywhere good reduction, that is, if $\mathcal{G}^c = \mathcal{G}$, then γ_G is the usual c.i. homomorphism (cf. the Introduction). This is the case in the following example, which we recall for later use.

(1.2) Constant group schemes. Let $G = G_0 \otimes_k K$, where G_0 is a finite commutative group scheme, of order prime to p, over k. Writing $H_0 = G_0^D$, one has $H = H_0 \otimes_k K$, and $\mathcal{G}^c = \mathcal{G} = B \times_k G_0$, $\mathcal{H} = B \times_k H_0$. We write $\bar{k}$ for the algebraic closure of k and denote by a bar the effect of base change from k to $\bar{k}$. One has a commutative diagram

$$\begin{tikzcd}
H^1(B, H_0) \arrow[r, "\gamma_G"] \arrow[d] & \mathrm{Pic}(B \times_k G_0)^{inv} \arrow[r, "\simeq"] & \mathrm{Hom}(G_0, \underline{\mathrm{Pic}}_{B/k}) \arrow[d, "\simeq"] \\
H^1(\bar{B}, \bar{H}_0)^{Gal(\bar{k}/k)} \arrow[rr] & & \mathrm{Hom}(\bar{G}_0, \mathrm{Pic}(\bar{B}))^{\mathrm{Gal}(\bar{k}/k)},
\end{tikzcd}$$

where the bottom row is induced by the map $H^1(\bar{B}, \bar{H}_0) \to \mathrm{Hom}(\bar{G}_0, \mathrm{Pic}(\bar{B}))$, given by cup product ($\bar{G}_0 = H^0(\bar{B}, \bar{G}_0)$, $\mathrm{Pic}(\bar{B}) = H^1(\bar{B}, \boldsymbol{G}_m)$). We claim that this map is an isomorphism (whence so is the bottom row above, too). By writing $\bar{G}_0$ as a direct sum of cyclic groups, we may asssume that $\bar{G}_0 = \boldsymbol{Z}/n\boldsymbol{Z}$ and $\bar{H}_0 = \mu_n$. In this case, taking cohomology in the exact sequence $0 \to \mu_{n,\bar{B}} \to \boldsymbol{G}_{m,\bar{B}} \xrightarrow{n} \boldsymbol{G}_{m,\bar{B}} \to 0$, one obtains $H^1(\bar{B}, \mu_n) \xrightarrow{\simeq} {}_n\mathrm{Pic}(\bar{B}) = \mathrm{Hom}(\boldsymbol{Z}/n\boldsymbol{Z}, \mathrm{Pic}(\bar{B}))$.

Secondly, the left hand side vertical arrow in the above diagram comes from the Lyndon-Hochschild-Serre spectral sequence $H^i(\mathrm{Gal}(\bar{k}/k), H^j(\bar{B}, \bar{H}_0)) \Rightarrow H^{i+j}(B, H_0)$, yielding, in low degrees, an exact sequence: $0 \to H^1(\mathrm{Gal}(\bar{k}/k), \bar{H}_0) \to H^1(B, H_0) \to H^1(\bar{B}, \bar{H}_0)^{\mathrm{Gal}(\bar{k}/k)} \to H^2(\mathrm{Gal}(\bar{k}/k), \bar{H}_0)$. The last written term vanishes, because $\bar{H}_0$ is a torsion module. Putting it all together, we obtain an exact sequence

$$0 \longrightarrow H^1(k, H_0) \longrightarrow H^1(B, H_0) \xrightarrow{\gamma_G} \mathrm{Pic}(B \times_k G_0)^{inv} \longrightarrow 0.$$

Moreover, writing σ for the Frobenius automorphism of $\overline{k}/k$, the exact sequence $0 \to H^0(k, H_0) \to \overline{H}_0 \xrightarrow{1-\sigma} \overline{H}_0 \to H^1(k, H_0) \longrightarrow 0$ shows that the group $H^1(k, H_0)$ vanishes if and only if $H_0(k)$ does.

(1.3) Functoriality. Let G_i, $i = 1, 2$, be group schemes over K, as before. We shall write $H_i, \mathscr{G}_i, \mathscr{H}_i$, $i = 1, 2$ for the corresponding objects, as defined above. Let $\varphi : G_1 \to G_2$ be a morphism of K-group schemes. Call $\psi = \varphi^D : H_2 \to H_1$ its dual, and $\Phi : \mathscr{G}_1 \to \mathscr{G}_2$, $\Psi : \mathscr{H}_2 \to \mathscr{H}_1$ the morphisms of B-group schemes which they induce. Call moreover $\Phi^c : \mathscr{G}_1^c \to \mathscr{G}_2^c$ the morphism of B-schemes which uniquely extends Φ. Then $(\Phi^c)^* : \mathrm{Pic}(\mathscr{G}_2^c) \to \mathrm{Pic}(\mathscr{G}_1^c)$ sends $\mathrm{Pic}(\mathscr{G}_2^c)^{inv}$ into $\mathrm{Pic}(\mathscr{G}_1^c)^{inv}$ and the following diagram is commutative:

$$\begin{array}{ccc} H^1(B, \mathscr{H}_2) & \xrightarrow{\gamma_{G_2}} & \mathrm{Pic}(\mathscr{G}_2^c)^{inv} \\ \Big\downarrow \Psi & & \Big\downarrow (\Phi^c)^* \\ H^1(B, \mathscr{H}_1) & \xrightarrow{\gamma_{G_1}} & \mathrm{Pic}(\mathscr{G}_1^c)^{inv}. \end{array}$$

The proof is in the style of (1.1), and we leave it to the reader. A useful application of this is the following:

(1.4) Additivity. Let G_i, $i = 1, 2$ be as before, and call $G = G_1 \times_K G_2$. Keeping the preceding notations, one has $H = H_1 \times_K H_2$, $\mathscr{H} = \mathscr{H}_1 \times_B \mathscr{H}_2$, and a commutative diagram

$$\begin{array}{ccc} H^1(B, \mathscr{H}) & \xrightarrow{\gamma_G} & \mathrm{Pic}(\mathscr{G}^c)^{inv} \\ \Big\downarrow \simeq & & \Big\downarrow \simeq \\ H^1(B, \mathscr{H}_1) \oplus H^1(B, \mathscr{H}_2) & \xrightarrow{\gamma_{G_1} \oplus \gamma_{G_2}} & \mathrm{Pic}(\mathscr{G}_1^c)^{inv} \oplus \mathrm{Pic}(\mathscr{G}_2^c)^{inv}. \end{array}$$

The diagram is obtained by applying twice (1.3), to each of the inclusions $G_i \hookrightarrow G$, $i = 1, 2$. It remains to explain the isomorphism on the right hand side. To this end, applying (1.3) to the projection maps $G \to G_i$, $i = 1, 2$, we obtain a morphism $\mathrm{Pic}(\mathscr{G}_1^c)^{inv} \oplus \mathrm{Pic}(\mathscr{G}_2^c)^{inv} \to \mathrm{Pic}(\mathscr{G}^c)^{inv}$ which, followed by the given one, yields the identity. And, to see that the composition in the reverse order also equals the identity, it suffices, taking $\xi \in \mathrm{Pic}(\mathscr{G}^c)^{inv}$, to restrict the equality $s^*(\xi) = p_1^*(\xi) + p_2^*(\xi)$ to the open subscheme $\mathscr{G}^c = (\mathscr{G}_1 \times_B \mathscr{G}_2)^c \hookrightarrow \tilde{\mathscr{G}}^c = (\mathscr{G} \times_B \mathscr{G})^c$.

Properties (1.3) and (1.4) may be restated by saying that γ is a natural transformation between additive functors. Note in particular that the additivity of the functor which assigns to G the group $\mathrm{Pic}(\mathscr{G}^c)^{inv}$ implies that $\mathrm{Pic}(\mathscr{G}^c)^{inv} \subset {}_n\mathrm{Pic}(\mathscr{G}^c)$, if n is such that $nG = 0$.

As a consequence of (1.4), in the study of the c.i. homomorphism we may restrict ourselves, whenever wanted, to the case of l-primary group schemes G over K, for primes $l \neq p$.

(1.5) Base change. Let $K \subset K_1$ be a finite field extension, and let G be a group scheme over K, as at the beginning of this section. Put now $G_1 = G \otimes_K K_1$. Then $H_1 = G_1^D = H \otimes_K K_1$. Call B_1 the complete regular curve which has K_1 as its function field, and let $\mathscr{G}_1$ and $\mathscr{H}_1$ be the Néron models of G_1 and H_1, respectively. The natural maps $\mathscr{H}_{B_1} \to \mathscr{H}_1$ and $\mathscr{G}_1^c \to \mathscr{G}^c$ then provide the vertical arrows of a commutative diagram:

$$\begin{array}{ccc} H^1(B_1, \mathscr{H}_1) & \xrightarrow{\gamma_{G_1}} & \mathrm{Pic}(\mathscr{G}_1^c)^{inv} \\ \uparrow & & \uparrow \\ H^1(B, \mathscr{H}) & \xrightarrow{\gamma_G} & \mathrm{Pic}(\mathscr{G}^c)^{inv} \end{array}$$

(details again are left to the reader).

2. Abelian varieties and Selmer groups.

Let K, B, k be as in the previous section. Let A be an abelian variety over K, and call $\hat{A}$ the dual abelian variety. Call $\mathscr{A}$ and $\hat{\mathscr{A}}$ the Néron models of A and $\hat{A}$, respectively. In the notation of Section 1, we take now $G = {}_nA = \ker(A \xrightarrow{n} A)$, for n prime to p, whence $H = {}_n\hat{A}$ and, moreover, $\mathscr{G} = {}_n\mathscr{A} = \ker(\mathscr{A} \xrightarrow{n} \mathscr{A})$ and $\mathscr{H} = {}_n\hat{\mathscr{A}}$. The class invariant homomorphism reads here:

$$\gamma_{nA} : H^1(B, {}_n\hat{\mathscr{A}}) \to \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}.$$

Motivated by the results in [**10**], we consider the following

(2.1) Question. Does there exist $N \in \boldsymbol{N}$ such that, for all $n \in \boldsymbol{N}$ with $(n, N) = 1$ the map γ_{nA} is an isomorphism?

We express this by saying that γ_{nA} is an isomorphism "for almost all" n. A first and obvious remark is that the answer depends only on the isogeny class of A, for, if A and A' are K-isogenous abelian varieties, then, for almost all $n : {}_nA \simeq {}_nA'$.

Relation with Selmer groups. Consider the standard exact sequence of group schemes $0 \to \hat{\mathscr{A}}^o \to \hat{\mathscr{A}} \to \hat{\Phi} \to 0$, where $\hat{\mathscr{A}}^o$ is the open subgroup scheme of $\hat{\mathscr{A}}$ intersecting every fibre along its neutral component. One has an inclusion $H^1(B, \hat{\mathscr{A}}) \hookrightarrow H^1(K, \hat{A})$, and the image of the morphism $H^1(B, \hat{\mathscr{A}}^o) \to H^1(B, \hat{\mathscr{A}}))$ is the Shafarevich-Tate group $\text{Ш}(\hat{A}/K) = \ker(H^1(K, \hat{A}) \to \bigoplus_{x \in B} H^1(K_x, \hat{A}))$ (cf. [**5**], pp. 368–369). Secondly, given $n \in \boldsymbol{N}$ satisfying the condition (R_0) of being prime to p, the Selmer group $S_n(\hat{A}/K)$ is defined by the requirement that the following diagram be exact, where the bottom row is the cohomology sequence of the exact sequence $0 \to {}_n\hat{A} \to \hat{A} \xrightarrow{n}$

$\hat{A} \to 0$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{A}(K)/n\hat{A}(K) & \longrightarrow & S_n(\hat{A}/K) & \longrightarrow & {}_n\text{Ш}(\hat{A}/K) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{A}(K)/n\hat{A}(K) & \longrightarrow & H^1(K, {}_n\hat{A}) & \longrightarrow & {}_nH^1(K, \hat{A}) & \longrightarrow & 0. \end{array}$$

Lemma 2.2. *Let $n \in \boldsymbol{N}$ satisfy condition* (R_0) *plus the additional condition* (R_1) *of being prime to the order of the finite group scheme $\hat{\Phi}$. Then one has a natural isomorphism $S_n(\hat{A}/K) \xrightarrow{\simeq} H^1(B, {}_n\hat{\mathscr{A}})$.*

Proof. Conditions (R_0) and (R_1) together imply the existence of an exact sequence $0 \to {}_n\hat{\mathscr{A}} \to \hat{\mathscr{A}} \xrightarrow{n} \hat{\mathscr{A}} \to 0$, whose cohomology sequence $0 \to \hat{A}(K)/n\hat{A}(K) \to H^1(B, {}_n\hat{\mathscr{A}}) \to {}_nH^1(B, \hat{\mathscr{A}}) \to 0$ can be inserted between the two rows in the above diagram, yielding an inclusion $S_n(\hat{A}/K) \hookrightarrow H^1(B, {}_n\hat{\mathscr{A}})$. Secondly, again by these conditions and by the exact sequence defining $\hat{\Phi}$, we have ${}_n\text{Ш}(\hat{A}/K) = {}_nH^1(B, \hat{\mathscr{A}})$, and therefore $S_n(\hat{A}/K) = H^1(B, {}_n\hat{\mathscr{A}})$.

For n satisfying (R_0) and (R_1), the c.i. homomorphism can be written then as a map $\gamma_n : S_n(\hat{A}/K) \to \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}$, and Question (2.1) is restated accordingly.

Proposition 2.3 (Base change). *Let $K \subset K_1$ be a finite normal field extension, and write $A_1 = A \otimes_K K_1$. Then, if Question* (2.1) *has an affirmative answer for the abelian variety A_1 over K_1, the same holds for A.*

Proof. For almost all n, the base change diagram from (1.5) can be written here as

$$\begin{array}{ccc} S_n(\hat{A}_1/K_1) & \xrightarrow{\gamma_n} & \mathrm{Pic}({}_n\mathscr{A}_1^c)^{inv} \\ \uparrow & & \uparrow \\ S_n(\hat{A}/K) & \xrightarrow{\gamma_n} & \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}. \end{array}$$

It suffices then to consider the case of a Galois extension and the case of a purely inseparable extension, separatedly. We note that, in either case, for almost all n the right hand side vertical map is injective. Namely, writing $d := [K_1 : K]$, the map ${}_n\mathscr{A}_1^c \to {}_n\mathscr{A}^c$ is finite of degree d. Whence an element in the kernel of $\mathrm{Pic}({}_n\mathscr{A}^c) \to \mathrm{Pic}({}_n\mathscr{A}_1^c)$ has d-torsion. On the other side, the group $\mathrm{Pic}({}_n\mathscr{A}^c)^{inv}$ has n-torsion (cf. (1.4)), and it suffices to require that n be prime to d, in order to achieve the claimed injectiveness.

Suppose first that $K \subset K_1$ is Galois, and put $\Gamma = \mathrm{Gal}(K_1/K)$. The morphism $S_n(\hat{A}/K) \to S_n(\hat{A}_1/K_1)$ is induced by the base change map $H^1(K, {}_n\hat{A}) \to H^1(K_1, {}_n\hat{A}_1)$. And, if $x \in B$ is a point of good reduction for A, and $B_1 \to B$ is unramified at $x_1 \in B_1$ above x, then (cf. [**5**], p. 57, Proposition 3.8) the map $H^1(K_x, \hat{A}) \to H^1(K_{1,x_1}, \hat{A}_1)$ is injective. It follows that, choosing n such that ${}_nH^1(K_x, \hat{A}) = 0$ for all $x \in B$ such that either $\hat{A}$ has bad reduction at x or $B_1 \to B$ is ramified above x, the group $S_n(\hat{A}/K)$

coincides with the inverse image of $S_n(\hat{A}_1/K_1)$. Now, for any $x \in B$, the group ${}_nH^1(K_x, \hat{A})$ is dual to $A(K_x)/nA(K_x) = \mathscr{A}(k(x))/n\mathscr{A}(k(x))$, and so it vanishes if and only if ${}_n\mathscr{A}(k(x))$ does. Therefore it suffices to choose n prime with the orders of the finite groups $\mathscr{A}(k(x))$ for $x \in B$ as above, to obtain the vanishing of these groups. Secondly, by the LHS spectral sequence $H^i(\Gamma, H^j(K_1, {}_n\hat{A}_1)) \Rightarrow H^{i+j}(K, {}_n\hat{A})$, one has an isomorphism $H^1(K, {}_n\hat{A}) \xrightarrow{\simeq} H^1(K_1, {}_n\hat{A}_1)^\Gamma$, provided that $H^i(\Gamma, H^0(K_1, {}_n\hat{A}_1)) = 0$, $i = 1, 2$ holds. But, by the Mordell-Weil theorem, for almost all n the group $H^0(K_1, {}_n\hat{A}_1) = {}_n\hat{A}_1(K_1)$ is zero, whence this condition is satisfied. Putting the two remarks together, the above diagram gives, for almost all n, a commutative diagram

$$\begin{array}{ccc} S_n(\hat{A}_1/K_1)^\Gamma & \xrightarrow{\gamma_n} & \mathrm{Pic}({}_n\mathscr{A}_1^c)^{inv,\Gamma} \\ \uparrow \simeq & & \uparrow \\ S_n(\hat{A}/K) & \xrightarrow{\gamma_n} & \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}, \end{array}$$

in which the top row is an isomorphism. Thus the bottom row is an isomorphism, too.

Suppose now that $K \subset K_1$ is purely inseparable. We claim that, in this case, for n prime to p, one has an isomorphism $S_n(\hat{A}/K) \xrightarrow{\simeq} S_n(\hat{A}_1/K_1)$. The result then follows, as before. By decomposing the extension into a tower of subextensions, we may assume that $K = K_1^p$ or, what is the same, that our extension is $K \hookrightarrow K$, given by $a \mapsto a^p$. Then $\hat{A}_1 = \hat{A}^{(p)}$ and the map $S_n(\hat{A}/K) \to S_n(\hat{A}^{(p)}/K)$ coincides with the morphism induced by the relative Frobenius homomorphism $F : \hat{A} \to \hat{A}^{(p)}$ over K. Writing $V : \hat{A}^{(p)} \to \hat{A}$ for the Verschiebung homomorphism, one has $VF = p_{\hat{A}}$ (multiplication by p map on $\hat{A}$) and $FV = p_{\hat{A}^{(p)}}$. For n prime to p, multiplication by p is an isomorphism on the n-Selmer group, and therefore the same holds for F (and V). □

Corollary 2.4. *In answering Question* (2.1), *it is sufficient to consider geometrically simple abelian varieties and integers n which are powers of prime numbers. Moreover, it can be further assumed that these are abelian varieties with either good or semistable reduction.*

Proof. This follows from Proposition 2.3, the additivity property (1.4) and the fact, already noticed, that the question depends only on the isogeny class of the abelian variety.

Corollary 2.5. *Question* (2.1) *has a positive answer for twisted constant abelian varieties.*

Proof. By Proposition 2.3, it suffices to consider the case of constant abelian varieties. Let A_0 be an abelian variety over k, and put $A = A_0 \otimes_k K$. Applying (1.2) to $G_0 = {}_nA_0$ and $H_0 = {}_n\hat{A}_0$ we find that the c.i. homomorphism is surjective, and that its kernel is zero if and only if ${}_n\hat{A}_0(k) = 0$. This happens for almost all n, and the result follows.

Let now A be again an arbitrary abelian variety over K, and let $n \in \mathbf{N}$ satisfy conditions (R_0) and (R_1). We prove a conditional result, subjected to two hypotheses which we discuss subsequently. Although this is not essential here, we shall suppose also for simplicity that n satisfies the additional condition (R_2) of being prime to the degree of a given polarization of A over K. This implies that ${}_nA$ and ${}_n\hat{A}$ are isomorphic group schemes over K, hence that they behave similarly under base field extension (we shall not identify them, however). Consider the base change situation (1.5) for the Galois extension $K \subset K_n$ given by $K_n = K({}_nA)$. We shall write accordingly B_n for the curve corresponding to this field, and $\Gamma_n = \mathrm{Gal}(K_n/K)$. The diagram from (1.5) leads to a commutative diagram:

$$\begin{array}{ccc} H^1(B_n, {}_n\hat{A}(K_n))^{\Gamma_n} & \longrightarrow & \mathrm{Hom}_{\Gamma_n}({}_nA(K_n), \mathrm{Pic}(B_n)) \\ \Big\uparrow & & \Big\uparrow \lambda_n \\ S_n(\hat{A}/K) & \xrightarrow{\gamma_n} & \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}. \end{array}$$

The top row is restriction to the Γ_n-invariant parts of the c.i. homomorphism for the constant (in fact, discrete) group scheme $B_n \times_n \hat{A}(K_n)$. Let us call (R_3) the condition for n of being prime to the orders of the groups $\mathscr{A}(k(x))$ for all $x \in B$ where A (or, equivalently, $\hat{A}$) has bad reduction. Then one has

Proposition 2.6. *Let $n \in \mathbf{N}$ satisfy the conditions (R_i), $i = 0, 1, 2, 3$. Suppose that the following hypotheses hold*:

(1) $H^i(\Gamma_n, {}_nA(K_n)) = 0$, $i = 0, 1, 2$.
(2) *The map λ_n is injective.*

Then the c.i. homomorphism $\gamma_n : S_n(\hat{A}/K) \to \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}$ is an isomorphism.

Proof. By (1.2), we have an exact sequence of Γ_n-modules

$$0 \to {}_n\hat{A}(K_n) \to H^1(B_n, {}_n\hat{A}(K_n)) \to \mathrm{Hom}({}_nA(K_n), \mathrm{Pic}(B_n)) \to 0,$$

whose cohomology sequence, together with the hypothesis (1), implies that the top arrow in the diagram is an isomorphism. Secondly, $H^1(B_n, {}_n\hat{A}(K_n)) = \mathrm{Hom}(\pi_1(B_n), {}_n\hat{A}(K_n)) = \mathrm{Hom}(\mathrm{Gal}(K_n^{un}/K_n), {}_n\hat{A}(K_n))$, where K_n^{un} denotes the maximal unramified extension of K_n, and Hom means continuous homomorphisms. By similar arguments as in the proof of Proposition (2.3), and using the restriction (R_3) imposed to n, one shows that $S_n(\hat{A}/K)$ equals the inverse image of $\mathrm{Hom}(\mathrm{Gal}(K_n^{un}/K_n), {}_n\hat{A}(K_n))$ by the base change morphism $H^1(K, {}_n\hat{A}) \to H^1(K_n, {}_n\hat{A} \otimes_K K_n) = \mathrm{Hom}(\mathrm{Gal}(K_n^s/K_n), {}_n\hat{A}(K_n))$ (cf. [**10**], Lemma (2.15), whose proof applies verbatim here). By the hypothesis (1) and the LHS spectral sequence, this morphism yields an isomorphism $H^1(K, {}_n\hat{A}) \xrightarrow{\simeq} H^1(K_n, {}_n\hat{A} \otimes_K K_n)^{\Gamma_n}$, and so the left hand side vertical arrow in the diagram is an isomorphism, too. Then the hypothesis (2) implies the desired result.

Remark 2.7. (1) As shown in [**10**] ((2.13), (3.1), (3.8) and (3.11)), both hypotheses hold, for almost all n, for A a non twisted-constant abelian variety of dimension 1 (cf Remarks (2) and (3) here below). It follows that Question (2.1) has a positive answer for these varieties (loc. cit., Theorem (0.4)).

(2) As it seems, the theory of Galois groups of torsion points of abelian varieties is more developed for number fields than in the function field case. The group Γ_n

embeds into $GL({}_nA(K_n))$ and, for number fields, Serre has announced in [**7**] strong results which imply, in particular, the hypothesis (1) in that case: According to Serre, there exists an integer $c \in \boldsymbol{N}$ such that, for all $n \in \boldsymbol{N} : ((\boldsymbol{Z}/n\boldsymbol{Z})^*)^c \subset \Gamma_n$ (here $(\boldsymbol{Z}/n\boldsymbol{Z})^*$ is thought inside $GL({}_nA(K_n))$ as the group of homotheties). Then, taking n such that all its prime factors are $\geq c+2$, one has $H^i(\Gamma_n, {}_nA(K_n)) = 0$ for all i. We recall briefly a proof of this: Suppose first that $n = l^r$, with l prime. Then $0 \neq \Xi := ((\boldsymbol{Z}/l\boldsymbol{Z})^*)^c \subset \Gamma_n$ is a normal subgroup, of order prime to n. So $H^j(\Xi, {}_nA(K_n)) = 0$ for $j > 0$ and, by direct inspection, $H^0(\Xi, {}_nA(K_n)) = 0$. Thus all the terms in the spectral sequence $H^i(\Gamma_n/\Xi, H^j(\Xi, {}_nA(K_n))) \Rightarrow H^{i+j}(\Gamma_n, {}_nA(K_n))$ vanish, and the result follows in this case. In the general situation we decompose $n = \prod l^a$ into prime factors and write ${}_nA(K_n) = \bigoplus_{l^a} A(K_n)$. Taking every summand separately, the inclusion $((\boldsymbol{Z}/l\boldsymbol{Z})^*)^c \subset \Gamma_n$ implies, as before, that $H^i(\Gamma_n, {}_{l^a}A(K_n)) = 0$ for all i. Back now to function fields, one might hope to find sufficiently many homotheties in this case, too, so as to make this proof go through. In [**10**] this was made possible by the good understanding of Galois groups of torsion points of abelian varieties of dimension one, thanks to the work of Igusa (cf. [**4**]). Needless to say, the case $i = 0$ of hypothesis (1) is already covered, for almost all n, by the Mordell-Weil theorem.

(3) The morphism λ_n is pullback along the map $B_n \times {}_nA(K_n) \to {}_n\mathscr{A}^c = (B_n \times {}_nA(K_n))/\Gamma_n$. In [**10**], the study of the ramification of this map, arising from the bad reduction of A, showed the impossibility of factoring it through a non trivial etale covering, and hence the injectivity of λ_n. This could work perhaps more generally, in the semistable case, provided one has -again- sufficient information on the monodromy. The case of everywhere good reduction remains however more mysterious. Denoting by $k_n \subset K_n$ the field of constants of K_n, the kernel of λ_n is given, in this case, by $H^1(\Gamma_n, \bigoplus_{{}_nA(K_n)} k_n^*)^{inv}$.

We collect into a single statement the results obtained so far on Question (2.1) (cf. also [**10**]):

Theorem 2.8. *Let A be an abelian variety over a global field K of positive characteristic, which is isogenous to a twisted form of a product of elliptic curves and a constant abelian variety. Then, for almost all $n \in \boldsymbol{N}$, the class invariant homomorphism $\gamma_n : S_n(\hat{A}/K) \to \mathrm{Pic}({}_n\mathscr{A}^c)^{inv}$ is an isomorphism.*

3. A relation with other questions.

We shall need a supplement to Section 1. In the notations of that section, let G be a group scheme over K, of the usual type. Suppose that $n_1, n_2 \in \boldsymbol{N}$ are such that the sequence $G \xrightarrow{n_1} G \xrightarrow{n_2} G$ is exact. Then one obtains the following commutative diagram from (1.3), by bearing in mind that the dual map of ${}_{n_1}G \hookrightarrow G$ is $H \xrightarrow{n_2} {}_{n_1}H$ (here $H = G^D$, as before):

$$\begin{array}{ccc} H^1(B, \mathscr{H}) & \xrightarrow{\gamma_G} & \mathrm{Pic}(\mathscr{G}^c)^{inv} \\ \Big\downarrow{\scriptstyle n_2} & & \Big\downarrow{\scriptstyle rest.} \\ H^1(B, {}_{n_1}\mathscr{H}) & \xrightarrow{\gamma_{n_1 G}} & \mathrm{Pic}({}_{n_1}\mathscr{G}^c)^{inv}, \end{array}$$

the right hand side vertical arrow being the restriction map.

We apply this to an abelian variety A over K, taking $G = {}_{l^{r+1}}A$, with l a prime integer satisfying conditions (R_0) and (R_1) from Section 2, and obtain a commutative diagram

$$\begin{array}{ccc} S_{l^{r+1}}(\hat{A}/K) & \xrightarrow{\gamma_{l^{r+1}}} & \mathrm{Pic}({}_{l^{r+1}}\mathscr{A}^c)^{inv} \\ \Big\downarrow l & & \Big\downarrow rest. \\ S_{l^r}(\hat{A}/K) & \xrightarrow{\gamma_{l^r}} & \mathrm{Pic}({}_{l^r}\mathscr{A}^c)^{inv}. \end{array}$$

Taking projective limits as r varies, we obtain a map $T_l(\gamma) : T_lS(\hat{A}/K) \to \mathrm{Pic}(\mathscr{A}(l)^c)^{inv}$, where $\mathscr{A}(l) = \bigcup_{r \geq 0} {}_{l^r}\mathscr{A}$ and $\mathscr{A}(l)^c = \bigcup_{r \geq 0} {}_{l^r}\mathscr{A}^c$. On the other hand, the maps $\hat{A}(K)/l^r\hat{A}(K) \to S_{l^r}(\hat{A}/K)$ yield, in the limit, a map $\hat{A}(K) \otimes \boldsymbol{Z}_l \to T_lS(\hat{A}/K)$, and the Shafarevich-Tate conjecture on the finiteness of the group $\text{Ш}(\hat{A}/K)$ can be expressed by saying that this map is an isomorphism (for all, resp. for one l). The composite map

$$\hat{A}(K) \otimes \boldsymbol{Z}_l \to \mathrm{Pic}(\mathscr{A}(l)^c)^{inv}$$

has a natural interpretation, at least when A has everywhere good reduction (cf. [**1**], cf. also [**10**]): In this case $\hat{\mathscr{A}} = \underline{\mathrm{Pic}}^o_{\mathscr{A}/B}$, and $\mathscr{A}(l)^c = \mathscr{A}(l)$. An element ζ of $\hat{A}(K)$ defines in a canonical way an element of $\mathrm{Pic}(\mathscr{A})$, trivialized along the zero section. The restriction of this element to $\mathscr{A}(l) \subset \mathscr{A}$ gives then the image of ζ in $\mathrm{Pic}(\mathscr{A}(l)^c)^{inv}$.

Together with the Shafarevich-Tate conjecture, a positive answer to Question (2.1) would imply then the bijectivity of this map, for all but a finite set of prime numbers l.

References

[1] A. Agboola, A geometric description of the class invariant homomorphism, J. de Théorie des Nombres de Bordeaux **6** (1994), 273–280.

[2] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron models, Erg. der Math. 21, Springer Verlag, Berlin, 1990.

[3] L. N. Childs, A. R. Magid, The Picard invariant of a principal homogeneous space, J. of Pure and Applied Algebra **4** (1974), 273–286.

[4] J. I. Igusa, Fibre systems of jacobian varieties III, Amer. J. of Math. **81** (1959), 453–476.

[5] J. S. Milne, Arithmetic duality theorems, Academic Press, New York, 1986.

[6] J. P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959.

[7] J. P. Serre, Résumé des cours au Collège de France 1985/86, Annuaire du Collège de France 1986, pp. 95–99.

[8] S. Shatz, Principal homogeneous spaces for finite group schemes, Proc. AMS **22** (1969), 678–680.

[9] W. C. Waterhouse, Principal homogeneous spaces and group scheme extensions, Trans. AMS **153** (1971), 181–189.

[10] G. E. Welters, Selmer groups and Picard Groups, Comment. Math. Univ. St. Pauli, to appear.

Gerald E. WELTERS
Facultad de Matemáticas
Universidad de Barcelona
Gran Vía, 585
08007 Barcelona, Spain