J. Math. Soc. Japan
Vol. 49, No. 4, 1997

Smooth plane curves with one place at infinity

By Yuji NAKAZAWA and Mutsuo OKA

(Received Aug. 7, 1995)

1. Introduction and statement of the result.

We consider a smooth affine curve $C^a=\{f(x, y)=0\}\subset \boldsymbol{C}^2$ of degree n with one place at infinity, say at $\rho=[1;0;0]\in \boldsymbol{P}^2$ and let g be the genus of the smooth compactification of C^a. By the assumption that C^a has one place at infinity, the Newton diagram of the polynomial $f(x, y)$ has only one outside boundary and the corresponding face function has only one factor. As this place is assumed to be at ρ, $f(x, y)$ is written as

$$(1.1)\qquad f(x, y)=(y^{a_1}+\xi_1 x^{c_1})^{A_2}+(\text{lower terms}),\quad \xi_1\in \boldsymbol{C}^*,\ c_1<a_1,\ n=a_1A_2$$

where a_1, c_1, A_2 are integers and $\gcd(a_1, c_1)=1$.

If $c_1=1$, we can take the change of affine coordinates: $x'=y^{a_1}+\xi_1 x$, $y'=y$ so that the degree of $\deg f'(x', y'):=f(\xi_1^{-1}(x'-y'^{a_1}), y')$ is strictly less than n. We say C^a is *minimal* if $c_1\geqq 2$.

The purpose of this note is to classify the possible normal forms of $f(x, y)$ for a minimal smooth curve with one place at infinity of a given genus g, which we call the *generalized Abhyankar-Moh problem* or *G.A.M-problem*. Abhyankar-Moh and Suzuki independently studied the case $g=0$ ([**3**], [**12**]) and they showed that C^a is isomorphic to a line. The case $g\leqq 3$ is studied by A'Campo-Oka in [**4**] as an application of a Tschirnhausen resolution tower and we essentially follow their treatment. There also exists a work of D.W. Neumann ([**9**]) for $g\leqq 4$ from the viewpoint of the link at infinity.

Let C be the closure of C^a in $\boldsymbol{P}^2$. Recall that the homogeneous polynomial $F(X, Y, Z):=f(X/Z, Y/Z)Z^n$ defines the projective curve C in $\boldsymbol{P}^2$ and $F(X, Y, Z)$ is written as

$$F(X, Y, Z)=(Y^{a_1}+\xi_1 X^{c_1}Z^{b_1})^{A_2}+(\text{lower terms}),\quad b_1=a_1-c_1,\ n=a_1A_2$$

In the affine space $U_0:=\boldsymbol{P}^2-\{X=0\}$ with the affine coordinates $u=Z/X$, $v=Y/X$, $C\cap U_0$ is defined by $\{(u, v)\in \boldsymbol{C}^2;\ h(u, v)=0\}$ where

The second author is partially supported by Inamori foundation.

(#) $$h(u, v) := F(1, v, u) = u^n f(u^{-1}, vu^{-1}),$$
$$f(x, y) = F(x, y, 1) = x^n h(x^{-1}, yx^{-1})$$

and ρ corresponds to the origin in this coordinate chart U_0. Note that

(1.1)′ $$h(u, v) = (v^{a_1}+\xi_1 u^{b_1})^{A_2}+(\text{higher terms}).$$

Observe that $h(u, v)$ is a monic polynomial of degree n in v and the degree of $h(u, v)$ is also n. We call $h(u, v)$ *the polynomial at infinity* of $f(x, y)$. The determination of $f(x, y)$ is thus equivalent to that of $h(u, v)$. For the determination of $h(u, v)$, we use the method developed in [**4**] and [**11**].

(1.2) **Tschirnhausen resolution tower.** Let

$$\mathcal{T} = \{X_k \xrightarrow{p_k} X_{k-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{p_1} X_0 = \boldsymbol{C}^2\}$$

be a Tschirnhausen resolution tower of $(C\cap U_0, \rho)$ with the corresponding weight vectors $P_i={}^t(a_i, b_i)$ for $i=1, \cdots, k$ ([**4**]). Starting with an admissible toric modification $p_1: X_1 \to X_0$ of $h(u, v)$, the weight sequence $P_1, \cdots, P_k$ are uniquely determined as long as $a_i \geqq 2$ (see [**4**]). $P_1={}^t(a_1, b_1)$ is already determined by the expression (1.1)′. Let $A_{i,i+j}=a_i a_{i+1}\cdots a_{i+j}$ and let $h_i(u, v)$ (resp. $F_i(X, Y, Z)$) be the $A_{i+1,k}$-th Tschirnhausen approximate polynomial of $h(u, v)$ as a polynomial of v (resp. of $F(X, Y, Z)$ as a polynomial of Y) and let

$$C_i = \{(X; Y; Z) \in \boldsymbol{P}^2; F_i(X, Y, Z)=0\} \subset \boldsymbol{P}^2, \quad i=1, \cdots, k.$$

Recall that Tschirnhausen approximate polynomials $h_i(u, v)$ and $F_i(X, Y, Z)$ are the unique monic polynomials of degree $A_{1,i}$ in v and Y respectively such that $\deg_v(h(u, v)-h_i(u, v)^{A_{i+1,k}}) < A_{1,k}-A_{1,i}$ and $\deg_Y(F(X, Y, Z)-F_i(X, Y, Z)^{A_{i+1,k}}) < A_{1,k}-A_{1,i}$. It is easy to observe that $F_i(1, v, u)=h_i(u, v)$, $\deg C_i=A_{1,i}$, $h_k=h$ and $C_k=C$.

In a Tschirnhausen resolution tower, (C_i, ρ) is resolved exactly at the i-th stage X_i of the tower $\mathcal{T}$. More explicitly, the pull backs of the Tschirnhausen polynomial h_i, $i=0, \cdots, k$ and u to X_i are written by [**4**] as

(a) $$\Phi_i^* h_l(u_i, v_i) = \begin{cases} u_i^{m_i(h_l)}(v_i^{a_{i+1}}+\xi_{i+1}u_i^{b_{i+1}})^{A_{i+2,l}}+(\text{higher terms}), & i<l \\ u_i^{m_i(h_l)} v_i, & i=l \\ u_i^{m_i(h_l)} U_{i,l}, \quad U_{i,l}: \text{a unit}, & i>l \end{cases}$$

(b) $$\Phi_i^* u(u_i, v_i) = u_i^{m_i(u)} U_{u,i}, \quad U_{u,i}: \text{a unit}$$

for some $\xi_{i+1} \in \boldsymbol{C}^*$. Here (u_i, v_i) are so called *the Tschirnhausen coordinates* centered at $\Xi_i := \{u_i=0\} \cap C^{(i)}$, $C^{(i)}$ being the strict transform of C to X_i. When an admissible toric modification $p_i: X_i \to X_{i-1}$ with center Ξ_{i-1} for $\Phi_{i-1}^* h(u_{i-1}, v_{i-1})$ is given, the Tschirnhausen coordinates (u_i, v_i) on X_i at Ξ_i are uniquely determined by the following properties. (1) The divisor $\{u_i=0\}$ is the unique irreducible component of the exceptional divisor of $p_i^{-1}(\Xi_{i-1})$ which intersects with the strict transform $C^{(i)}$ of C to X_i and $\Xi_i=(0, 0)$. This divisor corresponds to the weight vector $P_i={}^t(a_i, b_i)$ and this characterizes u_i. (2) v_i is characterized by the property $\Phi_i^* h_i(u_i, v_i)=u_i^{m_i(h_i)}v_i$.

(1.3) **Multiplicity.** The multiplicities $m_i(h_l)$ and $m_i(u)$ along the divisor $\{u_i=0\}$ are very important in our argument and they are determined inductively by the equalities:

$$\text{(c)} \qquad m_0(h_l)=0, \ m_i(h_l)=\begin{cases} a_i m_{i-1}(h_l)+a_i b_i A_{i+1,l}, & i\leqq l \\ a_{l+1}m_l(h_l)+b_{l+1}, & i=l+1, \\ a_i m_{i-1}(h_l), & i>l+1 \end{cases}$$

and $m_0(u)=1$, $m_i(u)=a_i m_{i-1}(u)$ where $A_{l+1,l}=1$ by definition. In particular, we use

$$\text{(c)}' \qquad \begin{cases} m_l(h_l)=a_1b_1A_{2,l}^2+\cdots+a_lb_lA_{l+1,l}^2 \\ m_{l+1}(h_l)=a_{l+1}m_l(h_l)+b_{l+1}, \quad m_{l+j}=m_{l+1}(h_l)A_{l+2,l+j}. \end{cases}$$

Define $m_i(\nu_i, \cdots, \nu_0, \mu)$ by

$$\text{(d)} \qquad m_i(\nu_i, \cdots, \nu_0, \mu):=\mu m_i(u)+\sum_{j=0}^{i}\nu_j m_i(h_j).$$

This is the multiplicity of the Tschirnhausen monomial $M=h_i^{\nu_i}\cdots h_0^{\nu_0}u^{\mu}$ along $\{u_i=0\}\subset X_i$. Here a polynomial $h_{l-1}^{\nu_{l-1}}\cdots h_0^{\nu_0}u^{\mu}$ is called a *Tschirnhausen monomial* if $\nu_i<a_{i+1}$ for any $i=0, \cdots, l-1$.

(1.4) **The Tschirnhausen expansion of a polynomial.** Any polynomial $g(u, v)$ with $\deg_v g(u, v)<A_{1,i+1}$ for some $i\leqq k-1$ can be uniquely expanded as a linear combination of the Tschirnhausen monomials in $h_i, \cdots, h_0, u$ ([**2**], Proposition (2.1), [**4**]):

$$g=\sum_{\nu_i,\cdots,\nu_0,\mu} C_{\nu_i,\cdots,\nu_0,\mu}h_i^{\nu_i}\cdots h_0^{\nu_0}u^{\mu}, \quad C_{\nu_i,\cdots,\nu_0,\mu}\in \boldsymbol{C}, \quad \nu_j<a_{j+1}, \quad j=0, \cdots, i.$$

In particular, by the definition of the Tschirnhausen polynomial, we have the expression:

$$\text{(e)} \qquad h_{i+1}=h_i^{a_{i+1}}+R_{i+1}, \quad R_{i+1}=\sum C_{\nu_i,\cdots,\nu_0,\mu}h_i^{\nu_i}\cdots h_0^{\nu_0}u^{\mu}, \quad C_{\nu_i,\cdots,\nu_0,\mu}\in \boldsymbol{C}$$

where the Tschirnhausen monomials $h_i^{\nu_i}\cdots h_0^{\nu_0}u^{\mu}$ satisfy $\nu_i<a_{i+1}-1$ and $\nu_j<a_{j+1}$ for $j=0, \cdots, i-1$. The first inequality $\nu_i<a_{i+1}-1$ results from the definition of the Tschirnhausen polynomial. In Lemma (2.7) of [**11**], it is proved that

(f) $$m_i(\nu_i, \cdots, \nu_0, \mu) = m_i(\nu'_i, \cdots, \nu'_0, \mu'), \quad \nu_i = \nu'_i \Longrightarrow \nu_j = \nu'_j, \quad j = 0, \cdots, i, \quad \mu = \mu'.$$

This implies that in the pull-back of h_{i+1}:

$$\begin{aligned}\Phi_i^* h_{i+1}(u_i, v_i) &= u_i^{a_{i+1} m_i(h_i)} v_i^{a_{i+1}} + \sum C_{\nu_i, \cdots, \nu_0, \mu} \Phi_i^*(h_i^{\nu_i} \cdots h_0^{\nu_0} u^\mu) \\ &= u_i^{a_{i+1} m_i(h_i)} v_i^{a_{i+1}} + \sum C_{\nu_i, \cdots, \nu_0, \mu} u_i^{m_i(\nu_i, \cdots, \nu_0, \mu)} v_i^{\nu_i} U_{\nu_i, \cdots, \nu_0, \mu}\end{aligned}$$

($U_{\nu_i, \cdots, \nu_0, \mu}$ is a unit), the leading terms of the second sum do not cancel each other. In particular, comparing the expression of $\Phi_i^* h_{i+1}(u_i, v_i)$ given by (a), we have

(g) $$a_{i+1} m_i(\nu_i, \cdots, \nu_0, \mu) + b_{i+1} \nu_i \geqq a_{i+1}^2 m_i(h_i) + a_{i+1} b_{i+1}$$

for any $M = h_i^{\nu_i} \cdots h_0^{\nu_0} u^\mu$ with $C_M \neq 0$ and there exist a unique Tschirnhausen monomial $M = \mathcal{M}_{i+1}$ in the above sum with $C_{\mathcal{M}_{i+1}} \neq 0$ which gives the term $\xi_{i+1} u_i^{m_i(h_{i+1}) + b_{i+1}}$ in (a) and satisfies the equality in (g). Note that $\nu_i = 0$ for such $\mathcal{M}_{i+1}$. We consider a subset $\mathcal{N}_i(\mathcal{W})$ by

$$\begin{aligned} M = h_{i-1}^{\nu_{i-1}} \cdots h_0^{\nu_0} u^\mu \in \mathcal{N}_i(\mathcal{W}) \Longleftrightarrow \\ a_i m_{i-1}(\nu_{i-1}, \cdots, \nu_0, \mu) + b_i \nu_{i-1} \geqq a_i^2 m_{i-1}(h_{i-1}) + a_i b_i, \\ \nu_j < a_{j+1}, \quad j = 0, \cdots, i-1.\end{aligned}$$

Note that $\mathcal{N}_i(\mathcal{W})$ depends only on $\mathcal{W}_i := \{P_1, \cdots, P_i\}$ ([**11**]).

(1.5) **Strategy of the classification.** Now we explain our strategy to classify smooth affine curves with one place at infinity.

(1) For a given g, we first determine the possible weight sequence $\mathcal{W} = \{P_1, \cdots, P_k\}$. For this purpose, we use the following (§8, [**4**]):

(P-g) $$\sum_{i=1}^{k} (A_{i,k} - 1) b_i A_{i+1,k} = (A_{1,k} - 1)^2 - 2g$$

(B) $$\sum_{i=1}^{k} a_i b_i A_{i+1,k}^2 \leqq A_{1,k}^2$$

where (P-g) follows from the modified Pücker's formula for the topological Euler Poincaré characteristics and the assumption that the corresponding affine curve is smooth of genus g with one place at infinity. The second inequality (B) is most essential and follows from a formula for the intersection multiplicity $I(C, C_{k-1}; \rho)$ of C and C_{k-1} at ρ (Theorem (4.5) of [**4**]) and the Bezout Theorem. A set of primitive integral vectors $\mathcal{W} = \{P_1, \cdots, P_k\}$ with $a_i \geqq 2$, $b_i \geqq 1$ and $a_1 - b_1 \geqq 2$ which satisfies (P-g) and (B) is called *a minimal numerical G.A.M-solution.* The minimal numerical G.A.M-solutions can be computed easily.

(2) When we have a minimal numerical G. A. M-solution $\mathcal{W}=\{P_1, \cdots, P_k\}$, we have to check if there exists a minimal smooth curve with one place at infinity whose sequence of the weight vectors at infinity is $\mathcal{W}$. If there exists such a curve, the corresponding polynomial $h(u, v)$ is called a *geometric G. A. M-solution* corresponding to the numerical G. A. M-solution $\mathcal{W}$.

The following theorem guarantees the existence of a monic polynomial having a given weight sequence. A similar assertion is proved by Jaworski [**5**] but we need a more explicit formulation for our purpose.

THEOREM (1.6) (Theorem (3.1), [**11**]). *Let $P_i={}^t(a_i, b_i)$, $i=1, \cdots, k$ be a given sequence of weight vectors and $\mathcal{W}=\{P_1, \cdots, P_k\}$. There exists polynomial $h(u, v)$ which is a monic polynomial of degree $A_{1,k}$ in v such that the curve $C:=\{(u, v)\in \boldsymbol{C}^2; h(u, v)=0\}$ is locally irreducible at the origin and $\mathcal{W}:=\{P_1, \cdots, P_k\}$ is the sequence of the weight vectors of the Tschirnhausen resolution tower of C at the origin. Let $h_j(u, v)$ be the $A_{j+1,k}$-th Tschirnhausen approximate polynomial of $h(u, v)$ for $j=0, \cdots, k$, with $h_k(u, v)=h(u, v)$. There exists a unique Tschirnhausen monomial $\mathcal{M}_j(\mathcal{W}):=h_{j-2}^{\nu_{j-2}^{(j)}} \cdots h_0^{\nu_0^{(j)}} u^{\mu^{(j)}}$, $\nu_i^{(j)}<a_{i+1}$, $i=0, \cdots, j-2$, such that h_j is written as:*

$$(1.6.1) \qquad h_j(u, v) = h_{j-1}^{a_j}+\lambda_j\mathcal{M}_j(\mathcal{W})+R_j', \quad R_j' = \sum_{M\in\mathcal{N}_j(\mathcal{W}),\, M\neq\mathcal{M}_j(\mathcal{W})} C_M M$$

$$(1.6.2) \qquad a_j m_{j-1}(\nu_{j-1}^{(j)}, \cdots, \nu_0^{(j)}, \mu^{(j)})+b_j \nu_{j-1}^{(j)} = a_j^2 m_{j-1}(h_{j-1})+a_j b_j$$

The integers $\nu_{j-2}^{(j)}, \cdots, \nu_0^{(j)}$ and $\mu^{(j)}$ depend only on $\mathcal{W}_j:=\{P_1, \cdots, P_j\}$, where $\lambda_j\in \boldsymbol{C}^*$, $C_{\nu_{j-1}, \cdots, \nu_0, \mu}\in \boldsymbol{C}$ and $j=0, \cdots, k$.

Here $\mathcal{M}_j(\mathcal{W}):=h_{j-2}^{\nu_{j-2}^{(j)}} \cdots h_0^{\nu_0^{(j)}} u^{\mu^{(j)}}$ is unique as a monomial of variables $\{h_{j-1}, \cdots, h_0, u\}$ but as a polynomial of variables $\{u, v\}$, $\mathcal{M}_j(\mathcal{W})$ depends on the choice of $h_{j-1}, \cdots, h_0$.

Here is another problem. Let $h(u, v)$ be as in Theorem (1.6). If $\deg_{(u,v)} h(u, v)>A_k$, the corresponding polynomial $f(x, y)$ defines an affine curve with other intersection at infinity. For example, as a numerical G. A. M-solution for $g=3$, we have $\mathcal{W}=\{P_1={}^t(3, 1), P_2={}^t(3, 17)\}$ $(=\mathcal{W}(3, 2; 2)$ in § 2) and this can be given by $h(u, v)=(v^3+\xi_1 u)^3+\xi_2 v^2 u^8+tu^{10}$, $\xi_1, \xi_2\neq 0$. The corresponding affine curve is defined by $C^a=\{(x, y)\in \boldsymbol{C}^2; (y^3+\xi_1 x^2)^3 x+\xi_2 y^2+t=0\}$ and C^a is smooth for a generic $t\neq 0$ but $C\cap\{Z=0\}=\{[1;0;0], [0;1;0]\}$. Thus there is no geometric solution for $\mathcal{W}$.

(1.7) **Characteristic Tschirnhausen approximate polynomials.** The characteristic Tschirnhausen approximate polynomials $\bar{h}_1(u, v), \cdots, \bar{h}_k(u, v)$ of $\mathcal{W}$ are defined by

$$\bar{h}_0 = v,$$
$$\bar{h}_1 = \bar{h}_0^{a_1} + \lambda_1 u^{b_1} = v^{a_1} + \lambda_1 u^{b_1}, \quad \lambda_1 \in \boldsymbol{C}^*$$
$$\bar{h}_2 = \bar{h}_1^{a_2} + \lambda_2 \mathcal{M}_2(\mathcal{W}) = \bar{h}_1^{a_2} + \lambda_2 \bar{h}_0^{\nu_0^{(2)}} u^{\mu^{(2)}}, \quad \lambda_2 \in \boldsymbol{C}^*$$
$$\cdots\cdots\cdots$$
$$\bar{h}_k = \bar{h}_{k-1}^{a_k} + \lambda_k \mathcal{M}_k(\mathcal{W}) = \bar{h}_{k-1}^{a_k} + \lambda_k \bar{h}_{k-2}^{\nu_{k-2}^{(k)}} \cdots \bar{h}_0^{\nu_0^{(k)}} u^{\mu^{(k)}}, \quad \lambda_k \in \boldsymbol{C}^*$$

where $\mathcal{M}_i(\mathcal{W})$ is characterized in Theorem (1.6). *The characteristic Tschirnhausen approximate polynomials* $\bar{h}_1, \cdots, \bar{h}_k$ *are inductively determined while* $\bar{h}_1, \cdots, \bar{h}_{k-1}$ *are also characterized as the Tschirnhausen polynomials of* $\bar{h}_k$.

Let $\mathcal{P}_k(\mathcal{W})$ be the set of polynomials $h(u, v)$ satisfying the assertion of Theorem (1.6) and let $\mathcal{P}_k(\mathcal{W}; A_{1,k})$ be the subset of $\mathcal{P}_k(\mathcal{W})$ which is defined by

$$\mathcal{P}_k(\mathcal{W}; A_{1,k}) := \{h(u, v) \in \mathcal{P}_k(\mathcal{W}); \deg_{(u,v)} h(u, v) = A_{1,k}\}.$$

If $h(u, v)$ gives a geometric G.A.M-solution, we have $h(u, v) \in \mathcal{P}_k(\mathcal{W}; A_{1,k})$ by definition. We also observe that if $h(u, v) \in \mathcal{P}_k(\mathcal{W}; A_{1,k})$, $h_i(u, v) \in \mathcal{P}_i(\mathcal{W}_i; A_{1,i})$ where $\mathcal{W}_i = \{P_1, \cdots, P_i\}$.

Let $\mathcal{N}_i(\mathcal{W}; A_{1,i})$ be the subset of the Tschirnhausen monomial $M = h_{i-1}^{\nu_{i-1}} \cdots h_0^{\nu_0} u^{\mu}$ in $\mathcal{N}_i(\mathcal{W})$ such that $\deg_{(u,v)} M \leqq A_{1,i}$. Put $\mathcal{N}_i(\mathcal{W}; A_{1,i})' = \mathcal{N}_i(\mathcal{W}; A_{1,i}) - \{\mathcal{M}_i(\mathcal{W})\}$. The following property is satisfied ([**11**]).

(1) Assume that there exists such a polynomial $h(u, v) \in \mathcal{P}_k(\mathcal{W}; A_{1,k})$ and let $h_0, \cdots, h_{k-1}$ be the corresponding Tschirnhausen approximate polynomials. Then $\deg_{(u,v)} h_i(u, v) = A_{1,i}$ i.e., $h_i(u, v) \in \mathcal{P}_i(\mathcal{W}_i; A_{1,i})$ for $i = 1, \cdots, k-1$.

(2) Assume that $h_j \in \mathcal{P}_j(\mathcal{W}_j; A_{1,j})$ for $j = 0, \cdots, i-1$ and choose arbitrary coefficients $\lambda_i' \neq 0$ and $C_M' \in \boldsymbol{C}$ for $M \in \mathcal{N}_i(\mathcal{W})$. Consider the polynomial

$$h_i(u, v)' := h_{i-1}^{a_i} + \lambda_i' \mathcal{M}_i(\mathcal{W}) + \sum_{M \in \mathcal{N}_i(\mathcal{W}),\, M \neq \mathcal{M}_i(\mathcal{W})} C_M' M$$

Then $h_i(u, v)' \in \mathcal{P}_i(\mathcal{W})$, $\deg_{(u,v)}(h_{i-1}^{\nu_{i-1}} \cdots h_0^{\nu_0} u^{\mu}) = \mu + \sum_{j=1}^{i-1} \nu_j A_{1,j}$ and

$$\deg_{(u,v)} h_i(u, v)' = \max\{A_{1,i}, \deg_{(u,v)} \mathcal{M}_i(\mathcal{W}), \max_{M;\, C_M \neq 0} \deg_{(u,v)} M\}$$

This gives the following criterion:

$$h_i(u, v)' \in \mathcal{P}_i(\mathcal{W}_i; A_{1,i}) \Longleftrightarrow \deg_{(u,v)} \mathcal{M}_i(\mathcal{W}) \leqq A_{1,i},$$
$$\deg_{(u,v)} M \leqq A_{1,i}, \quad \forall M, \quad C_M \neq 0.$$

(3) In particular, assume that there exists such a polynomial $h(u, v) \in \mathcal{P}_k(\mathcal{W}; A_{1,k})$ and let $h_0, \cdots, h_{k-1}$ be the corresponding Tschirnhausen approximate polynomials. Then h_i is written as

$$h_i(u, v) = h_{i-1}^{a_i} + \lambda_i \mathcal{M}_i(\mathcal{W}) + R_i', \quad R_i' = \sum_{M \in \mathcal{N}_i(\mathcal{W}; A_{1,i})'} C_M M \tag{1.7.1}$$

for some $\lambda_i \in \boldsymbol{C}^*$ and $C_M \in \boldsymbol{C}$, $M \in \mathcal{N}_i(\mathcal{W}; A_{1,i})'$. Let $\pi_i: \mathcal{P}_i(\mathcal{W}_i; A_{1,i}) \to \mathcal{P}_{i-1}(\mathcal{W}_{i-1}; A_{1,i-1})$ be the canonical projection. Then the inverse image $\pi_i^{-1}(h_{i-1})$ is isomorphic to $\boldsymbol{C}^* \times \boldsymbol{C}^{|\mathcal{N}_i(\mathcal{W}; A_{1,i})'|}$. The first factor $\boldsymbol{C}^*$ corresponds to $\lambda_i \in \boldsymbol{C}^*$. So the fiber is $|\mathcal{N}_i(\mathcal{W}; A_{1,i})|$-dimensional. See [11] for further detail.

For the determination of the existence of the G. A. M-solution corresponding to a numerical solution $\mathcal{W}$, we need to know if $\mathcal{P}_k(\mathcal{W}; A_{1,k}) \neq \emptyset$. To see this, we use the following which results from the above observation.

THEOREM (1.8) (Corollary (4.3), [**11**]). *The following two conditions are equivalent.*

(i) *There exists a monic polynomial* $h(u, v) \in \mathcal{P}_k(\mathcal{W}; A_{1,k})$.

(ii) *The characteristic Tschirnhansen approximate polynomials* $\bar{h}_1(u, v), \cdots, \bar{h}_k(u, v)$ *of* $\mathcal{W}$ *satisfy the equalities* $\deg_{(u,v)} \bar{h}_i(u, v) = A_{1,i}$ *for* $i = 1, \cdots, k$.

For any $h \in \mathcal{P}_k(\mathcal{W}; A_{1,k})$, the corresponding affine polynomial $f(x, y)$ defines an affine curve C which has one place at infinity with the corresponding weight sequence $\mathcal{W}$. However C may have some other singularity. Thus $\mathcal{P}_k(\mathcal{W}; A_{1,k})$ is not moduli space of geometric G. A. M-solutions but the moduli space of monic polynomials with one place at infinity with $\mathcal{W}$ as the weight sequence at infinity.

Note that the monomial $u^{A_{1,k}}$ corresponds to the constant term of $f(x, y)$ by (#). Thus if $u^{A_{1,k}} \in \mathcal{N}_k(\mathcal{W}; A_{1,k})$, C is smooth in $\boldsymbol{C}^2$ for a generic choice of the coefficient of $u^{A_{1,k}}$ by Bertini's theorem. Thus it gives a geometric G. A. M-solution. In fact, we will show that $u^{A_{1,k}} \in \mathcal{N}_k(\mathcal{W}; A_{1,k})$ for any $\mathcal{W}$ such that $\mathcal{P}_k(\mathcal{W}; A_{1,k}) \neq \emptyset$. See Appendix in § 3. Thus the subset corresponding to geometric G. A. M-solutions is a Zariski open dense set of $\mathcal{P}_k(\mathcal{W}; A_{1,k})$. The following is our main result. The classification for $g \geqq 8$ can be computed in a similar way.

MAIN THEOREM (1.9). *Let* $C^a = \{f(x, y) = 0\}$ *be a minimal smooth curve with one place at infinity, say* $\rho = [1; 0; 0]$, *of genus* $g = 3, 4, 5, 6, 7$. *Then the polynomial at infinity* $h(u, v) = h_k(u, v)$ *of* $f(x, y)$ *takes one of the following forms. In the list,* $\xi_i \in \boldsymbol{C}^*$ *and* $h_0(u, v) = v + tu$ *with* $t \in \boldsymbol{C}$. *The coefficients in* R_1, R_2, R_3 *are generically chosen so that the corresponding affine curve is smooth.*

$g=3$ (*This case is done in* [4] *without proof.*):

(1) $k=1$, $P_1 = {}^t(4, 1)$. $h(u, v) = h_1(u, v) = h_0^4 + \xi_1 u + R_1'(u, v)$, $\dim R_1' = 8$ *and* $\mathcal{N}_1(\mathcal{W}; 4)' = \{u^2, u^3, u^4, h_0 u, h_0 u^2, h_0 u^3, h_0^2 u, h_0^2 u^2\}$

(2) $k=1$, $P_1 = {}^t(7, 5)$. $h(u, v) = h_1(u, v) = h_0^7 + \xi_2 u^5 + R_1'(u, v)$, $\dim R_1' = 10$ *and* $\mathcal{N}_1(\mathcal{W}; 7)' = \{u^6, u^7, h_0 u^5, h_0 u^6, h_0^2 u^4, h_0^2 u^5, h_0^3 u^3, h_0^3 u^4, h_0^4 u^3, h_0^5 u^2\}$.

(3) $k=2$, $P_1 = {}^t(3, 1)$, $P_2 = {}^t(2, 9)$. $h_1(u, v) = h_0(u, v)^3 + \xi_1 u + R_1'(u, v)$, $h_2(u, v) = h_1(u, v)^2 + \xi_2 u^5 + R_2'(u, v)$, $\dim R_1' = 4$, $\mathcal{N}_1(\mathcal{W}; 3)' = \{u^2, u^3, h_0 u, h_0 u^2\}$, $\dim R_2' = 2$, $\mathcal{N}_2(\mathcal{W}; 6)' = \{u^6, h_0 u^5\}$.

$g=4$:

(1) $k=1,\ P_1={}^t(5,2).\ h(u,v)=h_1(u,v)=h_0^5+\xi_1u^2+R_1'(u,v),\ \dim R_1'=10$ *and* $\mathcal{N}_1(\mathcal{W};5)'=\{u^3,u^4,u^5,h_0u^2,h_0u^3,h_0u^4,h_0^2u^2,h_0^2u^3,h_0^3u,h_0^3u^2\}$.

(2) $k=1,\ P_1={}^t(9,7).\ h(u,v)=h_1(u,v)=h_0(u,v)^9+\xi_1u^7+R_1'(u,v),\ \dim R_1'=13$ *and* $\mathcal{N}_1(\mathcal{W};9)'=\{h_0^au^b;\ a+b\leqq 9,\ 9b+7a>63,\ a<8\}$.

(3) $k=2,\ P_1={}^t(3,1),\ P_2={}^t(2,7),\ h_1(u,v)$ *is as* (3) *of* $g=3$, $h(u,v)=h_1(u,v)^2+\xi_2u^4v+R_2'(u,v),\ \dim R_2'=4$ *and* $\mathcal{N}_2(\mathcal{W};6)'=\{u^5,u^6,h_0u^5,h_0^2u^4\}$.

(4) $k=2,\ P_1={}^t(3,1),\ P_2={}^t(3,16),\ h_1(u,v)$ *is as* (3) *of* $g=3$, $h(u,v)=h_1(u,v)^3+\xi_2u^8v+R_2'(u,v),\ \dim R_2'=2$ *and* $\mathcal{N}_2(\mathcal{W};9)'=\{u^9,h_1u^6\}$.

$g=5$:

(1) $k=1,\ P_1={}^t(11,9),\ h(u,v)=h_1(u,v)=h_0^{11}+\xi_1u^9+R_1'(u,v),\ \dim R_1'=16$ *and* $\mathcal{N}_1(\mathcal{W};11)'=\{h_0^au^b;\ a+b\leqq 11,\ 11b+9a>99,\ a<10\}$.

(2) $k=2,\ P_1={}^t(3,1),\ P_2={}^t(2,5),\ h_1$ *is as* (3) *of* $g=3$, $h(u,v)=h_1(u,v)^2+\xi_2u^3h_0^2+R_2'(u,v),\ \dim R_2'=6$ *and* $\mathcal{N}_2(\mathcal{W};6)'=\{u^4,u^5,u^6,h_0u^4,h_0u^5,h_0^2u^4\}$.

$g=6$:

(1) $k=1,\ P_1={}^t(5,1),\ h(u,v)=h_1(u,v)=h_0^5+\xi_1v+R_1'(u,v),\ \dim R_1'=13$ *and* $\mathcal{N}_1(\mathcal{W};5)'=\{h_0^au^b;\ a+b\leqq 5,\ 5b+a>5,\ a<4\}$.

(2) $k=1,\ P_1={}^t(7,4),\ h(u,v)=h_1(u,v)=h_0^7+\xi_1u^4+R_1'(u,v),\ \dim R_1'=14$ *and* $\mathcal{N}_1(\mathcal{W};7)'=\{h_0^au^b;\ a+b\leqq 7,\ 7b+4a>28,\ a<6\}$.

(3) $k=1,\ P_1={}^t(13,11),\ h(u,v)=h_1(u,v)=h_0^{13}+\xi_1u^{11}+R_1'(u,v),\ \dim R_1'=19$ *and* $\mathcal{N}_1(\mathcal{W};13)'=\{h_0^au^b;\ a+b\leqq 13,\ 13b+11a>143,\ a<12\}$.

(4) $k=2,\ P_1={}^t(3,1),\ P_2={}^t(2,3),\ h_1$ *is as* (3) *of* $g=3$, $h(u,v)=h_1(u,v)^2+\xi_2u^3+R_2'(u,v),\ \dim R_2'=8$ *and* $\mathcal{N}_2(\mathcal{W};6)'=\{u^4,u^5,u^6,h_0u^3,h_0u^4,h_0u^5,h_0^2u^3,h_0^2u^4\}$.

(5) $k=2,\ P_1={}^t(5,3),\ P_2={}^t(2,15),\ h_1(u,v)=h_0^5+\xi_1u^3+R_1'(u,v)$, $\dim R_1'=7,\ \mathcal{N}_1(\mathcal{W};5)'=\{u^4,u^5,h_0u^3,h_0u^4,h_0^2u^2,h_0^2u^3,h_0^3u^2\}$, $h(u,v)=h_1(u,v)^2+\xi_2u^9+R_2'(u,v),\ \dim R_2'=3$ *and* $\mathcal{N}_2(\mathcal{W};10)'=\{u^{10},h_0u^9,h_0^2u^8\}$.

(6) $k=2,\ P_1={}^t(3,1),\ P_2={}^t(3,14),\ h_1(u,v)$ *is as* (3) *of* $g=3$, $h(u,v)=h_1(u,v)^3+\xi_2h_0^2u^7+R_2'(u,v),\ \dim R_2'=5$ *and* $\mathcal{N}_2(\mathcal{W};9)'=\{u^8,u^9,h_0u^8,h_1u^6,h_1h_0u^5\}$.

$g=7$:

(1) $k=1,\ P_1={}^t(15,13),\ h(u,v)=h_0^{15}+\xi_1u^{13}+R_1'(u,v),\ \dim R_1'=22$ *and* $\mathcal{N}_1(\mathcal{W};15)'=\{h_0^au^b;\ a+b\leqq 15,\ 15b+13a>195,\ a<14\}$.

(2) $k=1,\ P_1={}^t(8,5),\ h(u,v)=h_1(u,v)=h_0^8+\xi_1u^5+R_1'(u,v),\ \dim R_1'=16$ *and* $\mathcal{N}_1(\mathcal{W};8)'=\{h_0^au^b;\ a+b\leqq 8,\ 8b+5a>40,\ a<7\}$.

(3) $k=2$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 1)$, h_1 *is as* (3) *of* $g=3$,
$h(u, v)=h_1(u, v)^2+\xi_2 u^2 v+R_2'(u, v)$, $\dim R_2'=10$ *and*
$\mathcal{N}_2(\mathcal{W}; 6)'=\{u^3, u^4, u^5, u^6, h_0u^3, h_0u^4, h_0u^5, h_0^2u^2, h_0^2u^3, h_0^2u^4\}$.

(4) $k=2$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 13)$, $h_1(u, v)$ *is as* (3) *of* $g=3$,
$h(u, v)=h_1(u, v)^3+\xi_2 u^7 v+R_2'(u, v)$, $\dim R_2'=7$ *and*
$\mathcal{N}_2(\mathcal{W}; 9)'=\{u^8, u^9, h_0u^8, h_0^2u^7, h_1u^6, h_1u^5, h_1h_0u^5\}$.

(5) $k=2$, $P_1={}^t(3, 1)$, $P_2={}^t(4, 21)$, $h_1(u, v)$ *is as* (3) *of* $g=3$,
$h(u, v)=h_1(u, v)^4+\xi_2 u^{11}+R_2'(u, v)$, $\dim R_2'=5$ *and*
$\mathcal{N}_2(\mathcal{W}; 12)'=\{u^{12}, h_0u^{11}, h_1u^9, h_1h_0u^8, h_1^2u^6\}$.

(6) $k=2$, $P_1={}^t(3, 1)$, $P_2={}^t(5, 28)$, $h_1(u, v)$ *is as* (3) *of* $g=3$,
$h(u, v)=h_1(u, v)^5+\xi_2 u^{14}h_0+R_2'(u, v)$, $\dim R_2'=4$ *and*
$\mathcal{N}_2(\mathcal{W}; 15)'=\{u^{15}, h_1u^{12}, h_1^2u^9, h_1^3u^6\}$.

(7) $k=2$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 13)$, $h_1(u, v)$ *is as* (5) *of* $g=6$,
$h_2(u, v)=h_1(u, v)^2+\xi_2 u^8 v+R_2'(u, v)$, $\dim R_2'=5$ *and*
$\mathcal{N}_2(\mathcal{W}; 10)'=\{u^9, u^{10}, h_0u^9, h_0^2u^8, h_0^3u^7\}$.

(8) $k=2$, $P_1={}^t(5, 3)$, $P_2={}^t(3, 28)$, $h_1(u, v)$ *is as* (5) *of* $g=6$,
$h(u, v)=h_1(u, v)^3+\xi_2 u^{14} v+R_2'(u, v)$, $\dim R_2'=2$ *and*
$\mathcal{N}_2(\mathcal{W}; 15)'=\{u^{15}, h_1u^{10}\}$.

(9) $k=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 21)$, $h_1(u, v)$ *is as* (1) *of* $g=3$,
$h_2(u, v)=h_1(u, v)^2+\xi_2 u^7 h_0+R_2'(u, v)$, $\dim R_2'=1$ *and* $\mathcal{N}_2(\mathcal{W}; 8)'=\{u^8\}$.

(10) $k=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(2, 9)$, h_1 *is as* (3) *of* $g=3$,
$h_2(u, v)=h_1(u, v)^2+\xi_2 u^5+R_2'$, $\dim R_2'=2$, $\mathcal{N}_2(\mathcal{W}; 6)'=\{u^6, h_0u^5\}$,
$h(u, v)=h_3(u, v)=h_2(u, v)^2+\xi_3 h_1(u, v)u^9+R_3'(u, v)$, $\dim R_3'=1$
and $\mathcal{N}_3(\mathcal{W}; 12)=\{u^{12}\}$.

REMARK (1.9.1). For the case $g=1, 2$, the minimal models are unique and they are given by [4] as

$$g=1:\quad k=1,\ P_1={}^t(3, 1),\quad h(u, v)=v^3+\lambda_1 u+R_1'(u, v),\quad \dim R_1'=4$$
$$g=2:\quad k=1,\ P_1={}^t(5, 3),\quad h(u, v)=v^5+\lambda_1 u^3+R_1'(u, v),\quad \dim R_1'=7.$$

M. Miyanishi recently obtained the classification for $g\leqq 4$ from a different viewpoint ([7]).

§2. Minimal numerical G. A. M-solutions.

The classification of the case $g\leqq 3$ is done in [4]. We consider first the numerical G. A. M-solutions of (P-g) and (B) for a given g, $3\leqq g\leqq 7$.

CASE (1) $k=1$. Then by (P-g), we have $(a_1-1)(c_1-1)=2g$. Using this equality and the property $a_1>c_1>1$, $b_1=a_1-c_1$ and $\gcd(a_1, b_1)=1$, we get the following numerical G. A. M-solutions:

For $g=3$, $P_1={}^t(a_1, b_1)={}^t(7, 5)$ or $P_1={}^t(4, 1)$. For $g=4$, $P_1={}^t(9, 7)$ or $P_1={}^t(5, 2)$. For $g=5$, $P_1={}^t(11, 9)$. For $g=6$, $P_1={}^t(13, 11)$ or $P_1={}^t(7, 4)$ or $P_1={}^t(5, 1)$. For $g=7$, $P_1={}^t(15, 13)$ or $P_1={}^t(8, 5)$.

We can easily see that each case corresponds to geometric G. A. M-solutions.

For example, for $g=3$, $k=1$ and $P_1={}^t(7, 5)$, h_1 can be written as $h_1(u, v)=h_0^7+\xi_1 u^5+R_1'(u, v)$ where $R_1'(u, v)$ is a linear combination of monomials in $\mathcal{N}_1(\mathcal{W}; 7)'$, where $h_0(u, v)=v+tu$, $t\in \boldsymbol{C}$, $u^\alpha h_0^\beta \in \mathcal{N}_1(\mathcal{W}; 7)'$ if and only if $7\alpha+5\beta>35$ and $\alpha+\beta\leqq 7$ and $\beta<6$. Therefore R' is a linear combination of 10 monomials

$$\mathcal{N}_1(P_1; 7)' = \{u^2h_0^5, u^3h_0^4, u^3h_0^3, u^4h_0^3, u^4h_0^2, u^5h_0^2, u^5h_0, u^6h_0, u^6, u^7\}, \ \dim R_1'=10$$

The corresponding affine curve is defined by $f(x, y)=y_0^7+\xi_1 x^2+R_1''(x, y)$ where $y_0=y+t$ and $R_1''(x, y)$ is a generic linear combination of 10 monomials $\{y_0^5, y_0^4, xy_0^3, y_0^3, xy_0^2, y_0^2, xy_0, y_0, x, 1\}$. Of course we have to choose the coefficients so that C has no other singularity than $\rho=[1; 0; 0]$. The dimension of such polynomials is $1+11=12$. The first 1 is the dimension of h_0 and can be ignored by a linear change of coordinates $(u, v)\mapsto(u, v+tv)$ which keeps G. A. M-solutions stable. It is easy to observe that $(a_1, c_1)=(2g+1, 2)$ is always numerical G. A. M-solution for any g.

By a similar argument, the above numerical G. A. M-solutions correspond to the following geometric G. A. M-solutions:

$g=3$, $P_1={}^t(4, 1)$, $h_1(u, v)=h_0^4+\xi_1 u+R_1'(u, v)$, $\dim R_1'=8$
$g=4$, $P_1={}^t(9, 7)$, $h_1(u, v)=h_0^9+\xi_1 u^7+R_1'(u, v)$, $\dim R_1'=13$
$g=4$, $P_1={}^t(5, 2)$, $h_1(u, v)=h_0^5+\xi_1 u^2+R_1'(u, v)$, $\dim R_1'=10$
$g=5$, $P_1={}^t(11, 9)$, $h_1(u, v)=h_0^{11}+\xi_1 u^9+R_1'(u, v)$, $\dim R_1'=16$
$g=6$, $P_1={}^t(13, 11)$, $h_1(u, v)=h_0^{13}+\xi_1 u^{11}+R_1'(u, v)$, $\dim R_1'=19$
$g=6$, $P_1={}^t(7, 4)$, $h_1(u, v)=h_0^7+\xi_1 u^4+R_1'(u, v)$, $\dim R_1'=14$
$g=6$, $P_1={}^t(5, 1)$, $h_1(u, v)=h_0^5+\xi_1 u+R_1'(u, v)$, $\dim R_1'=13$
$g=7$, $P_1={}^t(15, 13)$, $h_1(u, v)=h_0^{15}+\xi_1 u^{13}+R_1'(u, v)$, $\dim R_1'=22$
$g=7$, $P_1={}^t(8, 5)$, $h_1(u, v)=h_0^8+\xi_1 u^5+R_1'(u, v)$, $\dim R_1'=16$.

We omit the description of $\mathcal{N}_1(P_1; a_1)'$ as it is easy to be computed.

(2) We consider the case $k>1$.

First, taking $(1-A_{2,k})\times(\mathrm{B})+A_{2,k}\times(\mathrm{P}\text{-}g)$, we get the following inequality ((★), § 8, [4]).

$$A_{2,k}\leqq \frac{2g-1}{(a_1-1)(c_1-1)-1}\leqq 2g-1 \tag{2.1}$$

This is useful for the computation of minimal numerical G. A. M-solutions. For example, we consider the case $g=3$ and $k\geqq 2$. By (2.1), we have the possibility: $A_{2,k}=2, 3, 4, 5$. This implies that $k=2$ or $k=3$, $a_2=a_3=2$.

(a) If $k=2$, $A_{2,k}=a_2$. If $a_2=3, 4, 5$, $(a_1-1)(c_1-1)-1=1$ by (2.1). If $a_2=2$, either $(a_1-1)(c_1-1)-1=1$ or 2 by (2.1). We consider the case $(a_1-1)(c_1-1)-1=1$ first. Hence, $a_1=3$, $c_1=2$, $b_1=1$. b_2 is determined by (P-g) so that $b_2=9, 17, 71/3, 30$ according to $a_2=2, 3, 4, 5$ respectively. The last two cases are impossible as b_2 is an integer with $\gcd(a_2, b_2)=1$. Assume that $a_2=2$ and $(a_1-1)(c_1-1)-1=2$. Then $a_1=4$, $c_1=2$, $b_1=2$. This contradicts $\gcd(a_1, b_1)=1$.

(b) Assume that $k=3$, $a_2=a_3=2$. Then by (2.1), $a_1=3$, $b_1=1$ and by (P-g) and (B) we need to have

$$-71+6b_2+b_3=0, \quad 96-8b_2-2b_3 \geqq 0.$$

This has no positive integral solution. Thus the minimal numerical G. A. M-solutions for $g=3$, $k\geqq 2$ are $\mathcal{W}(3, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 9)\}$, $\mathcal{W}(3, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 17)\}$.

By a similar computation, the minimal numerical G. A. M-solutions for $4\leqq g\leqq 7$, $k\geqq 2$ are given as follows. In the notation $\mathcal{W}(a, b; c)$, $g=a$ and $k=b$.

CASE (2) $k=2$. We know that $A_{2,k}=a_2\leqq 2g-1\leqq 13$ for $g\leqq 7$. Using (P-g) and (B), we get the following numerical G. A. M-solutions.

$\mathcal{W}(4, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 7)\}$, $\mathcal{W}(4, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 16)\}$,
$\mathcal{W}(4, 2; 3)=\{P_1={}^t(3, 1), P_2={}^t(4, 23)\}$, $\mathcal{W}(5, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 5)\}$,
$\mathcal{W}(5, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(5, 29)\}$, $\mathcal{W}(6, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 3)\}$,
$\mathcal{W}(6, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 14)\}$, $\mathcal{W}(6, 2; 3)=\{P_1={}^t(3, 1), P_2={}^t(6, 35)\}$,
$\mathcal{W}(7, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 1)\}$, $\mathcal{W}(7, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 13)\}$,
$\mathcal{W}(7, 2; 3)=\{P_1={}^t(3, 1), P_2={}^t(4, 21)\}$, $\mathcal{W}(7, 2; 4)=\{P_1={}^t(3, 1), P_2={}^t(5, 28)\}$,
$\mathcal{W}(7, 2; 5)=\{P_1={}^t(3, 1), P_2={}^t(7, 41)\}$, $\mathcal{W}(4, 2; 4)=\{P_1={}^t(5, 3), P_2={}^t(2, 19)\}$,
$\mathcal{W}(5, 2; 3)=\{P_1={}^t(5, 3), P_2={}^t(2, 17)\}$, $\mathcal{W}(6, 2; 4)=\{P_1={}^t(5, 3), P_2={}^t(2, 15)\}$,
$\mathcal{W}(6, 2; 5)=\{P_1={}^t(5, 3), P_2={}^t(3, 29)\}$, $\mathcal{W}(7, 2; 6)=\{P_1={}^t(5, 3), P_2={}^t(2, 13)\}$,
$\mathcal{W}(7, 2; 7)=\{P_1={}^t(5, 3), P_2={}^t(3, 28)\}$, $\mathcal{W}(6, 2; 6)=\{P_1={}^t(7, 5), P_2={}^t(2, 27)\}$,
$\mathcal{W}(7, 2; 8)=\{P_1={}^t(7, 5), P_2={}^t(2, 25)\}$, $\mathcal{W}(6, 2; 7)=\{P_1={}^t(4, 1), P_2={}^t(2, 23)\}$,
$\mathcal{W}(7, 2; 9)=\{P_1={}^t(4, 1), P_2={}^t(2, 21)\}$.

CASE (3) $k\geqq 3$. First we show that $k\geqq 4$ is impossible. In fact, by (2.1) if $k\geqq 4$, $A_{2,k}\geqq 2^3=8$ and therefore either $k=4$, and $A_{2,k}=8$ or $A_{2,k}=12$. In both cases, $(a_1-1)(c_1-1)-1=1$, so $a_1=3$, $b_1=1$. Thus (i) $(a_2, a_3, a_4)=(2, 2, 2)$, or (ii) $(a_2, a_3, a_4)=(3, 2, 2)$ or (iii) $(a_2, a_3, a_4)=(2, 3, 2)$ or (iv) $(a_2, a_3, a_4)=(2, 2, 3)$. For example, we consider the case (i). Solving (P-g) in b_4, b_4 is a linear function $\varphi(b_2, b_3)$ of b_2 and b_3. Then substituting $b_4=\varphi(b_2, b_3)$ in (B), the positivity of b_4 and (B) says that $b_2=11, 12$ is the only possibility. As $\gcd(a_2, b_2)=1$, $b_2=11$. Substituting this in φ and (B) again, we see that there exists no positive integer b_3 for $4\leqq g\leqq 7$. The other cases (ii)$\sim$(iv) are also impossible by a similar argument.

Therefore $k=3$. By (2.1), $A_{2,k}\leqq 13$. Thus $A_{2,k}=4, 6, 8, 9, 10, 12$ and $(a_1-1)(c_1-1)-1=1, 2, 3$. This gives the possible values of a_1, b_1, a_2, a_3. In each case, we first solve (P-g) in b_3 to express b_3 as a linear function $\varphi(b_2)$ and the positivity of $\varphi(b_2)$ and (B) determine the minimal numerical G. A. M-solutions as follows.

$$\mathcal{W}(4, 3; 1)=\{P_1={}^t(3, 1),\ P_2={}^t(2, 11),\ P_3={}^t(2, 3)\},$$
$$\mathcal{W}(5, 3; 1)=\{P_1={}^t(3, 1),\ P_2={}^t(2, 11),\ P_3={}^t(2, 1)\},$$
$$\mathcal{W}(6, 3; 1)=\{P_1={}^t(3, 1),\ P_2={}^t(2, 9),\ P_3={}^t(2, 11)\},$$
$$\mathcal{W}(7, 3; 1)=\{P_1={}^t(3, 1),\ P_2={}^t(2, 9),\ P_3={}^t(2, 9)\},$$
$$\mathcal{W}(6, 3; 2)=\{P_1={}^t(3, 1),\ P_2={}^t(2, 11),\ P_3={}^t(3, 5)\},$$
$$\mathcal{W}(7, 3; 2)=\{P_1={}^t(3, 1),\ P_2={}^t(2, 11),\ P_3={}^t(3, 4)\},$$
$$\mathcal{W}(6, 3; 3)=\{P_1={}^t(3, 1),\ P_2={}^t(3, 17),\ P_3={}^t(2, 5)\},$$
$$\mathcal{W}(7, 3; 3)=\{P_1={}^t(3, 1),\ P_2={}^t(3, 17),\ P_3={}^t(2, 3)\}.$$

§ 3. Geometric existence and proof of Main Theorem.

To prove the Main Theorem (1.9), we have to check the existence (or non-existence) of geometric G. A. M-solutions corresponding to minimal numerical G. A. M-solutions obtained in § 2. For this purpose, we use the criterion given by Theorem (1.8). The case $k=1$ is already seen in § 2.

CASE I. $k=2$. We compute the characteristic Tschirnhausen approximate polynomials. By definition,

$$\bar{h}_0=v,\quad \bar{h}_1(u, v)=v^{a_1}+\xi_1u^{b_1},\quad \bar{h}_2=h_1^{a_2}+v^{\nu_0}u^{\mu}$$

where ν_0, μ are the integral solution of $m_1(0, \nu_0, \mu)=\nu_0m_1(h_0)+\mu m_1(u)=a_2m_1(h_1)+b_2$ with $\nu_0<a_1$. As $m_1(h_0)=b_1$, $m_1(h_1)=a_1b_1$, $m_1(u)=a_1$, this reduces to

$$\nu_0b_1+\mu a_1=a_2a_1b_1+b_2,\quad \nu_0<a_1. \tag{3.1}$$

The moduli set $\mathfrak{N}_2(\mathcal{W}; A_{1,2})'$ is the set of monomials $M=h_1^{\nu_1}h_0^{\nu_0}u^{\mu}$ which satisfy

$$a_2(\nu_1a_1b_1+\nu_0b_1+\mu a_1)+\nu_1b_2>a_2^2a_1b_1+a_2b_2, \tag{3.2}$$
$$\nu_1<a_2-1,\ \nu_0<a_1,\ a_1\nu_1+\nu_0+\mu\leqq a_1a_2.$$

In the following, $h_0(u, v)=v+t_{0,1}u$, $t_{0,1}\in\boldsymbol{C}$. (We may assume that $h_0=v$ by a linear change of coordinates.)

CASE II-1. $P_1={}^t(3, 1)$.

1. Consider the case $\mathcal{W}(3, 2; 1)=\{P_1={}^t(3, 1),\ P_2={}^t(2, 9)\}$. Then (3.1) gives the equality: $\nu_0+3\mu=15$ and $\nu_0<3$. Thus we get $(\nu_0, \mu)=(0, 5)$. Thus $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2u^5$ and $\deg\bar{h}_2=6$. Thus $\mathcal{W}(3, 2; 1)$ has a geometric G. A. M-solution $h_2(u, v)$ which is written as

$$h_0(u, v) = v + t_{0,1}u,$$
$$h_1(u, v) = h_0(u, v)^3 + \xi_1 u + R_1', \quad \mathcal{N}_1(\mathcal{W}; 3)' = \{u^2, u^3, h_0 u, h_0 u^2\}$$
$$h_2(u, v) = h_1(u, v)^2 + \xi_2 u^5 + R_2', \quad \mathcal{N}_2(\mathcal{W}; 6)' = \{u^6, h_0 u^5\}.$$

This corresponds to (3) of $g=3$ in Theorem (1.8). The corresponding polynomial which defines the smooth affine curve of genus 3 is

$$f_1(x, y) = (y+t_{0,1})^3 + \xi_1 x^2 + t_{1,1}x + t_{1,2} + t_{1,3}(y+t_{0,1})x + t_{1,4}(y+t_{0,1})$$
$$f_2(x, y) = f_1(x, y)^2 + \xi_2 x + t_{2,1} + t_{2,2}(y+t_{0,1})$$

and $\{f_2(x, y)=0\} \subset \boldsymbol{C}^2$ is smooth as long as $t_{2,1}$ is generic after giving other arbitrary coefficients. The dimension of such polynomials $h_2(u, v)$ or $f_2(x, y)$ is $1+5+3=9$. We omit this correspondence $h(u, v) \mapsto f(x, y)$ hereafter. Until No. 17, $P_1={}^t(3, 1)$ and $\bar{h}_1$, h_1 are the same as above.

2. Let $\mathcal{W}=\mathcal{W}(3, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 17)\}$. Then we have $\bar{h}_2(u, v)=\bar{h}_1(u, v)^3+\xi_2 v^2 u^8$. We see that $\deg \bar{h}_2(u, v)=10>9$. Thus there is no geometric G. A. M-solution in this case.

3. Let $\mathcal{W}=\mathcal{W}(4, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 7)\}$. Then $\bar{h}_2(u, v)=\bar{h}_1^2+\xi_2 v u^4$ and $\deg \bar{h}_2(u, v)=6$. Thus this case has a geometrical G. A. M-solution. A general G. A. M-solution $h_2(u, v)$ is written as $h_2(u, v)=h_1(u, v)^2+\xi_2 h_0(u, v)u^4+R_2'$ where R_2' is a generic linear combination of $\mathcal{N}_2(\mathcal{W}; 6)'=\{u^5, u^6, h_0 u^5, h_0^2 u^4\}$. This corresponds to (3) of $g=4$.

4. Let $\mathcal{W}=\mathcal{W}(4, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 16)\}$. Then $\bar{h}_2(u, v)=\bar{h}_1(u, v)^3+\xi_2 v u^8$ and $\deg \bar{h}_2(u, v)=9$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^3+\xi_2 h_0 u^8+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 9)'=\{u^9, h_1 u^6\}$. This corresponds to (4) of $g=4$.

5. Let $\mathcal{W}=\mathcal{W}(4, 2; 3)=\{P_1={}^t(3, 1), P_2={}^t(4, 23)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^4+\xi_2 v^2 u^{11}$. As $\deg \bar{h}_2(u, v)=13>12$, this case has no geometric G. A. M-solution.

6. Let $\mathcal{W}=\mathcal{W}(5, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 5)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v^2 u^3$ and $\deg \bar{h}_2(u, v)=5$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^2+\xi_2 h_0^2 u^3+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 6)'=\{u^4, u^5, u^6, h_0 u^4, h_0 u^5, h_0^2 u^4\}$. This corresponds to (2) of $g=5$.

7. Let $\mathcal{W}=\mathcal{W}(5, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(5, 29)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^5+\xi_2 v^2 u^{14}$. As $\deg \bar{h}_2(u, v)=16>15$, this case has no geometric G. A. M-solution.

8. Let $\mathcal{W}=\mathcal{W}(6, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 3)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 u^3$ and $\deg \bar{h}_2(u, v)=6$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^2+\xi_2 h_0^2 u^3+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 6)'=\{u^4, u^5, u^6, h_0 u^3, h_0 u^4, h_0 u^5, h_0^2 u^3, h_0^2 u^4\}$. This corresponds to (4) of $g=6$.

9. Let $\mathcal{W}=\mathcal{W}(6, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 14)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^3+\xi_2 v^2 u^7$ and $\deg \bar{h}_2(u, v)=9$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^3+\xi_2 h_0^2 u^7+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 9)'=\{u^8, u^9, h_0^2 u^7, h_1 u^6, h_1 h_0 u^5\}$. This corresponds to (6) of $g=6$.

10. Let $\mathcal{W}=\mathcal{W}(6, 2; 3)=\{P_1={}^t(3, 1), P_2={}^t(6, 35)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^6+\xi_2 v^2 u^{17}$. As $\deg \bar{h}_2(u, v)=19>18$, this case has no geometric G. A. M-solution.

11. Let $\mathcal{W}=\mathcal{W}(7, 2; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 1)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v u^2$ and $\deg \bar{h}_2(u, v)=3$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^2+\xi_2 h_0 u^2+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 6)'=\{u^3, u^4, u^5, u^6, h_0u^3, h_0u^4, h_0u^5, h_0^2u^2, h_0^2u^3, h_0^2u^4\}$. This corresponds to (3) of $g=7$.

12. Let $\mathcal{W}=\mathcal{W}(7, 2; 2)=\{P_1={}^t(3, 1), P_2={}^t(3, 13)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^3+\xi_2 v u^7$ and $\deg \bar{h}_2(u, v)=9$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^3+\xi_2 h_0 u^7+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 9)'=\{u^8, u^9, h_0u^8, h_0^2u^7, h_1u^5, h_1u^6, h_1h_0u^5\}$. This corresponds to (4) of $g=7$.

13. Let $\mathcal{W}=\mathcal{W}(7, 2; 3)=\{P_1={}^t(3, 1), P_2={}^t(4, 21)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^4+\xi_2 u^{11}$ and $\deg \bar{h}_2(u, v)=12$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^4+\xi_2 u^{11}+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 12)'=\{u^{12}, h_0u^{11}, h_1u^9, h_1h_0u^8, h_1^2u^6\}$ and this corresponds to (5) of $g=7$.

14. Let $\mathcal{W}=\mathcal{W}(7, 2; 4)=\{P_1={}^t(3, 1), P_2={}^t(5, 28)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^5+\xi_2 v u^{14}$ and $\deg h_2(u, v)=15$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^5+\xi_2 h_0 u^{14}+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 15)'=\{u^{15}, h_1u^{12}, h_1^2u^9, h_1^3u^6\}$. This correspnds to (6) of $g=7$.

15. Let $\mathcal{W}=\mathcal{W}(7, 2; 5)=\{P_1={}^t(3, 1), P_2={}^t(7, 41)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^7+\xi_2 v^2 u^{20}$ and as $\deg h_2(u, v)=22>21$, this case has no geometric G. A. M-solution.

16. (a) Let $\mathcal{W}=\{P_1={}^t(3, 1), P_2={}^t(2, 11)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v^2 u^5$. As $\deg h_2(u, v)=7>6$, this case has no geometric G. A. M-solution. This proves also the non-existence of geometric G. A. M-solution for $\mathcal{W}(4, 3; 1)$, $\mathcal{W}(5, 3; 1)$, $\mathcal{W}(6, 3; 2)$, $\mathcal{W}(7, 3; 2)$.

CASE I-2. $P_1={}^t(5, 3)$. In No. 17∼22, $\bar{h}_1(u, v)$, $h_1(u, v)$ are as in No. 17.

17. Let $\mathcal{W}=\mathcal{W}(4, 2; 4)=\{P_1={}^t(5, 3), P_2={}^t(2, 19)\}$. $\bar{h}_1(u, v)=v^5+\xi_1 u^3$, $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v^3 u^8$ and $\deg h_2(u, v)=11>10$. This there is no geometric G. A. M-solution in this case.

18. Let $\mathcal{W}=\mathcal{W}(5, 2; 3)=\{P_1={}^t(5, 3), P_2={}^t(2, 17)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v^4 u^7$ and $\deg h_2(u, v)=11>10$. Thus there is no geometric G. A. M-solution in this case.

19. Let $\mathcal{W}=\mathcal{W}(6, 2; 4)=\{P_1={}^t(5, 3), P_2={}^t(2, 15)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 u^9$ and $\deg h_2(u, v)=10$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^2+\xi_2 u^9+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 10)'=\{u^{10}, h_0u^9, h_0^2u^8\}$ and this corresponds to (5) of $g=6$.

20. Let $\mathcal{W}=\mathcal{W}(6, 2; 5)=\{P_1={}^t(5, 3), P_2={}^t(3, 29)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^3+\xi_2 v^3 u^{13}$ and $\deg h_2(u, v)=16>115$. Thus there is no geometric G. A. M-solution in this case.

21. Let $\mathcal{W}=\mathcal{W}(7, 2; 6)=\{P_1={}^t(5, 3), P_2={}^t(2, 13)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v u^8$ and $\deg h_2(u, v)=10$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^2+\xi_2 h_0 u^8+R_2'$ and $\mathcal{N}_2(\mathcal{W}; 10)'=\{u^9, u^{10}, h_0u^9, h_0^2u^8, h_0^3u^7\}$. This corresponds to (7)

of $g=7$.

22. Let $\mathcal{W}=\mathcal{W}(7, 2; 7)=\{P_1={}^t(5, 3), P_2={}^t(3, 28)\}$. $\bar{h}_2(u, v)=\bar{h}_1(u, v)^3+\xi_2 v u^{14}$ and $\deg h_2(u, v)=15$. A generic geometric G. A. M-solution is $h_2(u, v)=h_1(u, v)^3+\xi_2 h_0 u^{14}+R_2'$ and $\mathcal{M}_2(\mathcal{W}; 15)'=\{u^{15}, h_1 u^{10}\}$, corresponding to (8) of $g=7$.

CASE I-3. $P_1={}^t(7, 5)$.

23. Let $\mathcal{W}=\mathcal{W}(6, 2; 6)=\{P_1={}^t(7, 5), P_2={}^t(2, 27)\}$. $\bar{h}_1(u, v)=v^7+\xi_1 u^5$ and $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v^4 u^{11}$ and $\deg \bar{h}_2(u, v)=15>14$. Thus there is no geometric G. A. M-solution in this case.

24. Let $\mathcal{W}=\mathcal{W}(7, 2; 8)=\{P_1={}^t(7, 5), P_2={}^t(2, 25)\}$. $\bar{h}_1$ is as above and $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v^5 u^{10}$ and $\deg \bar{h}_2(u, v)=15>14$. Thus there is no geometric G. A. M-solution in this case.

CASE I-4. $P_1={}^t(4, 1)$.

25. Let $\mathcal{W}=\mathcal{W}(6, 2; 7)=\{P_1={}^t(4, 1), P_2={}^t(2, 23)\}$. $\bar{h}_1(u, v)=v^4+\xi_1 u$ and $\bar{h}_2(u, v)=\bar{h}_1(u, v)^2+\xi_2 v^3 u^7$. As $\deg \bar{h}_2(u, v)=10>8$, there is no geometric G. A. M-solution in this case.

26. Let $\mathcal{W}=\mathcal{W}(7, 2; 9)=\{P_1={}^t(4, 1), P_2={}^t(2, 21)\}$. $\bar{h}_1(u, v)$ is as above and $h_2(u, v)=h_1(u, v)^2+\xi_2 h_0 u^7+t u^8$ and $\mathcal{M}_2(\mathcal{W}; 8)'=\{u^8\}$, corresponding to (9) of $g=7$.

CASE II. $k=3$. Now we consider the existence of geometric G. A. M-solutions for $k=3$. We have shown in No. 16 that $\mathcal{W}(4, 3; 1)$, $\mathcal{W}(5, 3; 1)$, $\mathcal{W}(6, 3; 2)$ and $\mathcal{W}(7, 3; 2)$ have no geometric G. A. M-solutions. By No. 2, $\mathcal{W}(6, 3; 3)$ and $\mathcal{W}(7, 3; 3)$ do not have geometric G. A. M-solutions. So we have to consider only the existence of $\mathcal{W}(6, 3; 1)$ and $\mathcal{W}(7, 3; 1)$. By Theorem (1.6), the third characteristic Tschirnhausen polynomial $\bar{h}_3(u, v)$ is determined as $\bar{h}_3(u, v)=\bar{h}_2(u, v)^{a_3}+\xi_3 \bar{h}_1(u, v)^{\nu_1} v^{\nu_0} u^{\mu}$ where

$$\nu_1 m_2(h_1)+\nu_0 m_2(h_0)+\mu m_2(u)=a_3 m_2(h_2)+b_3, \quad \nu_1<a_2, \ \nu_0<a_1 \tag{3.3}$$

$$m_2(u)=a_1 a_2, \ m_2(h_0)=a_2 b_1, \ m_2(h_1)=a_2 a_1 b_1+b_2, \ m_2(h_2)=a_1 b_1 a_2^2+a_2 b_2 \tag{3.4}$$

When the characteristic Tschirnhausen polynomial satisfies $\deg \bar{h}_3(u, v)=a_1 a_2 a_3$, the moduli space $\mathcal{M}_3(\mathcal{W})'$ is defined by $\mathcal{M}_3(\mathcal{W}; A_3)'=\{h_2^{\nu_2} h_1^{\nu_1} h_0^{\nu_0} u^{\mu}; \nu_2<a_3-1, \ \nu_1<a_2, \ \nu_0<a_1, \ (\star)\}$ where

$$(\star) \quad a_3(\nu_2 m_2(h_2)+\nu_1 m_2(h_1)+\nu_0 m_2(h_0)+\mu m_2(u))+b_3\nu_2>a_3^2 m_2(h_2)+a_3 b_3$$

27. Let $\mathcal{W}=\mathcal{W}(6, 3; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 9), P_3={}^t(2, 11)\}$. Then $\bar{h}_1(u, v)=v^3-\xi_1 u$ and $\bar{h}_2(u, v)=\bar{h}_1^2+\xi_2 u^5$ as we have seen in No. 1. (3.3) reduces to $15\nu_1+2\nu_0+6\mu=71$ and thus we have

$$\bar{h}_3=\bar{h}_2^2+\xi_3 \bar{h}_1(u, v) v u^9, \quad \deg \bar{h}_3(u, v)=13>12.$$

Thus there is no geometric G. A. M-solution in this case.

28. Let $\mathcal{W}=\mathcal{W}(7, 3; 1)=\{P_1={}^t(3, 1), P_2={}^t(2, 9), P_3={}^t(2, 9)\}$. Then $\bar{h}_1$, $\bar{h}_2$ are as in No. 27 and (3.3) reduces to: $15\nu_1+2\nu_0+6\mu=69$.

$$\bar{h}_3(u, v)=\bar{h}_2(u, v)^2+\xi_3\bar{h}_1(u, v)u^9, \quad \deg\bar{h}_3(u, v)=12.$$

Thus $\mathcal{N}_3(\mathcal{W})=\{h_1^{\nu_1}h_0^{\nu_0}u^{\mu}; 15\nu_1+2\nu_0+6\mu\geqq 69, \nu_1<2, \nu_0<3\}$. Therefore $\mathcal{N}_3(\mathcal{W}; 12)'=\{u^{12}\}$ and a generic geometric G. A. M-solution is given by $h_3(u, v)=h_2(u, v)^2+\xi_3 h_1(u, v)u^9+tu^{12}$ where $h_1(u, v)$ and $h_2(u, v)$ are as in No. 1. This completes the proof of the Main Theorem (1.9).

APPENDIX. *Let $\mathcal{W}=\{P_1, \cdots, P_k\}$, $P_i={}^t(a_i, b_i)$, be a given weight sequence and assume that $\mathcal{P}_k(\mathcal{W}; A_{1,k})\neq\emptyset$. Then $u^{A_{1,k}}\in\mathcal{N}_k(\mathcal{W}; A_{1,k})$.*

PROOF. Let $h(u, v)\in\mathcal{P}_k(\mathcal{W}; A_{1,k})$ and let $h_i(u, v)$ be the respective Tschirnhausen approximate polynomials as before. Let $\mathcal{M}_j(\mathcal{W})=h_{j-2}^{\nu_{j-2}^{(j)}}\cdots h_0^{\nu_0^{(j)}}u^{\mu^{(j)}}$ be the characteristic monomials. By the assumption, we have

$$\nu_{j-2}^{(j)}m_{j-1}(h_{j-2})+\cdots+\nu_0^{(j)}m_{j-1}(h_0)+\mu^{(j)}m_{j-1}(u)=a_j m_{j-1}(h_{j-1})+b_j \tag{3.5}$$

$$\deg_{(u,v)}\mathcal{M}_j(\mathcal{W})=\nu_{j-2}^{(j)}A_{1,j-2}+\cdots+\nu_0^{(j)}A_{1,0}+\mu^{(j)}\leqq A_{1,j} \tag{3.6}$$

where $A_{1,0}=1$. We will show that

$$m_{j-1}(h_i)<A_{1,i}A_{1,j-1}, \quad j\geqq i+2. \tag{3.7}$$

Assuming this for a moment, we consider the monomial $M:=u^{\nu_{j-2}^{(j)}A_{1,j-2}+\cdots+\nu_0^{(j)}A_{1,0}+\mu^{(j)}}$. Then $\deg_{(u,v)}M=\sum_{l=0}^{j-2}\nu_l^{(j)}A_{1,l}+\mu^{(j)}=\deg_{(u,v)}\mathcal{M}_j(\mathcal{W})\leqq A_{1,j}$ and by (3.7) we have

$$\begin{aligned}
&(\nu_{j-2}^{(j)}A_{1,j-2}+\cdots+\nu_0^{(j)}A_{1,0}+\mu^{(j)})m_{j-1}(u)\\
&=(\nu_{j-2}^{(j)}A_{1,j-2}+\cdots+\nu_0^{(j)}A_{1,0}+\mu^{(j)})A_{1,j-1}\\
&\geqq\sum_{i=0}^{j-2}\nu_i^{(j)}m_{j-1}(h_i)+\mu^{(j)}A_{1,j-1}\\
&=m_{j-1}(\mathcal{M}(\mathcal{W}))=a_j m_{j-1}(h_{j-1})+b_j.
\end{aligned}$$

This implies that $M\in\mathcal{N}_j(\mathcal{W}; A_{1,j})$. Therefore $u^{A_{1,j}}\in\mathcal{N}_j(\mathcal{W}; A_{1,j})$ by (3.6).

PROOF OF (3.7). We now show (3.7) by the induction on i. As $m_1(h_0)=b_1<a_1$ and $m_{j-1}(h_0)=A_{2,j-1}b_1<A_{1,j-1}$, the assertion is true for $i=0$. Assume (3.7) for $i<\alpha$. Then for any $j\geqq\alpha+2$,

$$\begin{aligned}
m_{j-1}(h_\alpha)&=A_{\alpha+2,j-1}(a_{\alpha+1}m_\alpha(h_\alpha)+b_{\alpha+1}) \qquad \text{by (c)}\\
&=A_{\alpha+2,j-1}\Big(\sum_{l=0}^{\alpha-1}\nu_l^{(\alpha+1)}m_\alpha(h_l)+\mu^{(\alpha+1)}m_\alpha(u)\Big) \qquad \text{by (3.5)}
\end{aligned}$$

$$\leqq A_{\alpha+2,j-1}\Big(\sum_{l=0}^{\alpha-1}\nu_l^{(\alpha+1)}A_{1,l}A_{1,\alpha}+\mu^{(\alpha+1)}A_{1,\alpha}\Big) \quad \text{by induction's assumption}$$

$$\leqq A_{\alpha+2,j-1}A_{1,\alpha}A_{1,\alpha+1} = A_{1,j-1}A_{1,\alpha}.$$

Here $A_{\alpha+2,\alpha-1}=1$ by definition. This completes the proof.

Appendix by Yuji NAKAZAWA

In this appendix, we give the list of the classification for the cases $8\leqq g\leqq 16$ and the examples of $f(x, y)$. We omit the proof as the method is essentially same as them of $3\leqq g\leqq 7$. (The classification for $g\geqq 17$ can be computed in the similar way.)

We consider a smooth affine curve $C^a=\{f(x, y)=0\}\subset \boldsymbol{C}^2$ of degree n with one place at infinity, say at $\rho=(1;0;0)$ and let g be the genus of the smooth compactification of C^a. By the assumption, $f(x, y)$ is written as

(1.1) $$f(x, y)=(y^{a_1}+\xi_1 x^{c_1})^{A_2}+(\text{lower terms}),\quad \xi_1\in \boldsymbol{C}^*,\ c_1<a_1,\ n=a_1A_2$$

where a_1, c_1, A_2 are integers and $\gcd(a_1, c_1)=1$. We say $C^a=\{f(x, y)=0\}$ is *minimal* if $c_1\geqq 2$. If $c_1=1$, we can take the change of affine coordinates: $x'=y^{a_1}+\xi_1 x$, $y'=y$ so that the degree of $\deg f'(x', y'):=f(\xi_1^{-1}(x'-y'^{a_1}), y')$ is strictly less than n.

The homogeneous polynomial $F(X, Y, Z):=f(X/Z, Y/Z)Z^n$ defines projective curve C in $\boldsymbol{P}^2$ and $F(X, Y, Z)$ is written as

$$F(X, Y, Z)=(Y^{a_1}+\xi_1 X^{c_1}Z^{b_1})^{A_2}+(\text{lower terms}),\quad b_1=a_1-c_1,\ n=a_1A_2$$

In the affine space $U_0:=\boldsymbol{P}^2-\{X=0\}$ with the affine coordinates $u=Z/X$, $v=Y/X$, $C\cap U_0$ is defined by $\{(u, v)\in \boldsymbol{C}^2;\ h(u, v)=0\}$ where

(1.2) $$\begin{aligned} h(u, v)&:=F(1, v, u)=u^n f(u^{-1}, vu^{-1})\\ f(x, y)&=F(x, y, 1)=x^n h(x^{-1}, yx^{-1})\end{aligned}$$

and ρ corresponds to the origin in this coordinates U_0. Note that

(1.1)′ $$h(u, v)=(v^{a_1}+\xi_1 u^{b_1})^{A_2}+(\text{higher terms}).$$

Observe that $h(u, v)$ is a monic polynomial of degree n in v and the degree of $h(u, v)$ is also n. We call $h(u, v)$ *the polynomial at infinity* of $f(x, y)$. The determination of $f(x, y)$ is thus equivalent to that of $h(u, v)$.

THEOREM. *Let $C^a=\{f(x, y)=0\}$ be a minimal smooth affine curve with one place at infinity, say $\rho=[1;0;0]$, of genus $g=8, \cdots, 16$. Then the corresponding weight sequence to the polynomial at infinity $h(u, v)$ of $f(x, y)$ is one of the following.*

We can calculate the polynomial $h(u, v)$ for the case $g\geqq 9$ as $g=8$. In the list, $\xi_i\in \boldsymbol{C}^*$ and $h_0(u, v)=v+tu$ with $t\in \boldsymbol{C}$. The coefficients in R_1', R_2', R_3', R_4' are generically chosen so that the corresponding affine curve is smooth.

$g=8$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(17, 15)$. $h(u, v)=h_1(u, v)=h_0^{17}+\xi_1u^{15}+R_1'(u, v)$, $\dim R_1'=25$ *and* $\mathcal{N}_1(\mathcal{W}; 17)'=\{h_0^a u^b; a+b\leqq 17, 17b+15a>255, a<16\}$. *An example of $f(x, y)$ is given by* $f(x, y)=y^{17}+\xi_1x^2+c$. (*The coefficient c is generically chosen so that the affine curve* $C^a=\{f(x, y)=0\}$ *is smooth.*)

(2) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 11)$. $h_1(u, v)=h_0(u, v)^5+\xi_1u^3+R_1'(u, v)$, $h_2(u, v)=h_1(u, v)^2+\xi_2u^7v^2+R_2'(u, v)$, $\dim R_1'=7$, $\mathcal{N}_1(\mathcal{W}; 5)'=\{u^4, u^5, h_0u^3, h_0u^4, h_0^2u^2, h_0^2u^3, h_0^3u^2\}$, $\dim R_2'=7$ and $\mathcal{N}_2(\mathcal{W}; 10)'=\{u^9, u^{10}, h_0u^8, h_0u^9, h_0^2u^8, h_0^3u^7, h_0^4u^6\}$. *An example is given by* $f(x, y)=(y^5+\xi_1x^2)^2+\xi_2xy^2+c$.

$g=9$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(19, 17)$. *An example is* $f(x, y)=y^{19}+\xi_1x^2+c$.

(2) $\rho_\infty(C)=1$, $P_1={}^t(10, 7)$. *An example is* $f(x, y)=y^{10}+\xi_1x^3+c$.

(3) $\rho_\infty(C)=1$, $P_1={}^t(7, 3)$. *An example is* $f(x, y)=y^7+\xi_1x^4+c$.

(4) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 11)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^3+\xi_2xy^2+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(5, 27)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^5+\xi_2x+c$.

(6) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 9)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^2+\xi_2xy^3+c$.

(7) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(3, 26)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^3+\xi_2y^2+c$.

(8) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 17)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^2+\xi_2xy+c$.

(9) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 21)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^2+\xi_2x+c$.

(10) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(2, 27)$. *An example is* $f(x, y)=(y^5+\xi_1x^3)^2+\xi_2y+c$.

(11) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(2, 5)$, $h_1(u, v)=h_0(u, v)^3+\xi_1u+R_1'$, $\dim R_1'=4$, $\mathcal{N}_1(\mathcal{W}; 3)'=\{u^2, u^3, h_0u, h_0u^2\}$, $h_2(u, v)=h_1(u, v)^2+\xi_2u^5+R_2'$, $\dim R_2'=2$, $\mathcal{N}_2(\mathcal{W}; 6)'=\{u^6, h_0u^5\}$, $h(u, v)=h_3(u, v)=h_2(u, v)^2+\xi_3h_1(u, v)h_0u^8+R_3'(u, v)$, $\dim R_3'=4$ *and* $\mathcal{N}_3(\mathcal{W}; 12)=\{u^{11}, u^{12}, h_0u^{11}, h_1(u, v)u^9\}$. *An example of $f(x, y)$ is given by* $f(x, y)=(f_1(x, y)^2+\xi_2x)^2+\xi_3f_1(x, y)y+c$, $f_1(x, y)=y^3+\xi_1x^2$.

$g=10$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(21, 19)$. *An example is* $f(x, y)=y^{21}+\xi_1x^2+c$.

(2) $\rho_\infty(C)=1$, $P_1={}^t(11, 8)$. *An example is* $f(x, y)=y^{11}+\xi_1x^3+c$.

(3) $\rho_\infty(C)=1$, $P_1={}^t(6, 1)$. *An example is* $f(x, y)=y^6+\xi_1x^5+c$.

(4) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 10)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^3+\xi_2x^2y+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(4, 19)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^4+\xi_2xy+c$.

(6) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(7, 40)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^7+\xi_2y+c$.

(7) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 7)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^2+\xi_2xy^4+c$.

(8) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(3, 25)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^3+\xi_2x+c$.

(9) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 15)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^2+\xi_2y^3+c$.

(10) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 19)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^2+\xi_2xy+c$.

(11) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(3, 40)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^3+\xi_2y+c$.

(12) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(2, 25)$. *An example is* $f(x, y)=(y^5+\xi_1x^3)^2+\xi_2x+c$.

(13) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(2, 15)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy)^2+\xi_3f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(14) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(2, 3)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2x)^2+\xi_3f_1(x, y)x+c$, $f_1(x, y)=y^3+\xi_1x^2$.

$g=11$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(23, 21)$. *An example is* $f(x, y)=y^{23}+\xi_1x^2+c$.

(2) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(5, 26)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^5+\xi_2y^2+c$.

(3) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 5)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^2+\xi_2x^3+c$.

(4) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 13)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^2+\xi_2x^2y+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 17)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^2+\xi_2xy^2+c$.

(6) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(2, 1)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2x)^2+\xi_3f_1(x, y)y^2+c$, $f_1(x, y)=y^3+\xi_1x^2$.

$g=12$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(7, 2)$. *An example is* $f(x, y)=y^7+\xi_1x^5+c$.

(2) $\rho_\infty(C)=1$, $P_1={}^t(9, 5)$. *An example is* $f(x, y)=y^9+\xi_1x^4+c$.

(3) $\rho_\infty(C)=1$, $P_1={}^t(13, 10)$. *An example is* $f(x, y)=y^{13}+\xi_1x^3+c$.

(4) $\rho_\infty(C)=1$, $P_1={}^t(25, 23)$. *An example is* $f(x, y)=y^{25}+\xi_1x^2+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 8)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^3+\xi_2x^2y^2+c$.

(6) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 3)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^2+\xi_2x^3y+c$.

(7) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(3, 23)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^3+\xi_2xy+c$.

(8) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(5, 48)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^5+\xi_2y+c$.

(9) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 11)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^2+\xi_2xy^3+c$.

(10) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(3, 32)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^3+\xi_2x+c$.

(11) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 15)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^2+\xi_2xy^3+c$.

(12) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(3, 38)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^3+\xi_2y^2+c$.

(13) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(2, 21)$. *An example is* $f(x, y)=(y^5+\xi_1x^3)^2+\xi_2y^3+c$.

(14) $\rho_\infty(C)=2$, $P_1={}^t(9, 7)$, $P_2={}^t(2, 27)$. *An example is* $f(x, y)=(y^9+\xi_1x^2)^2+\xi_2x+c$.

(15) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(2, 11)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy)^2+\xi_3f_1(x, y)y+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(16) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 16)$, $P_3={}^t(2, 3)$. *An example is given by* $f(x, y)=(f_1(x, y)^3+\xi_2y)^2+\xi_3x+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(17) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(3, 14)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2x)^3+\xi_3y+c$, $f_1(x, y)=y^3+\xi_1x^2$.

$g=13$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(14, 11)$. *An example is* $f(x, y)=y^{14}+\xi_1x^3+c$.

(2) $\rho_\infty(C)=1$, $P_1={}^t(27, 25)$. *An example is* $f(x, y)=y^{27}+\xi_1x^2+c$.

(3) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 7)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^3+\xi_2x^3y+c$.

(4) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(4, 17)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^4+\xi_2xy^2+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(7, 39)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^7+\xi_2x+c$.

(6) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(9, 52)$. *An example is* $f(x, y)=(y^3+\xi_1x^2)^9+\xi_2y+c$.

(7) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 1)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^2+\xi_2x^3y^2+c$.

(8) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(3, 22)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^3+\xi_2y^4+c$.

(9) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 9)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^2+\xi_2x^3y+c$.

(10) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 13)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^2+\xi_2xy^4+c$.

(11) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(2, 19)$. *An example is* $f(x, y)=(y^5+\xi_1x^3)^2+\xi_2xy^2+c$.

(12) $\rho_\infty(C)=2$, $P_1={}^t(9, 7)$, $P_2={}^t(2, 25)$. *An example is* $f(x, y)=(y^9+\xi_1x^2)^2+\xi_2xy+c$.

(13) $\rho_\infty(C)=2$, $P_1={}^t(9, 7)$, $P_2={}^t(3, 52)$. *An example is* $f(x, y)=(y^9+\xi_1x^2)^3+\xi_2y+c$.

(14) $\rho_\infty(C)=2$, $P_1={}^t(7, 4)$, $P_2={}^t(2, 39)$. *An example is* $f(x, y)=(y^7+\xi_1x^3)^2+\xi_2y+c$.

(15) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 5)$, $P_3={}^t(2, 21)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy^2)^2+\xi_3f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(16) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(2, 9)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy)^2+\xi_3f_1(x, y)x+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(17) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 16)$, $P_3={}^t(2, 1)$. *An example is given by* $f(x, y)=(f_1(x, y)^3+\xi_2y)^2+\xi_3f_1(x, y)x+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(18) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 16)$, $P_3={}^t(3, 16)$. *An example is given by* $f(x, y)=(f_1(x, y)^3+\xi_2y)^3+\xi_3f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1x^2$.

$g=14$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(8, 3)$. *An example is* $f(x, y)=y^8+\xi_1x^5+c$.

(2) $\rho_\infty(C)=1$, $P_1={}^t(29, 27)$. *An example is* $f(x, y)=y^{29}+\xi_1x^2+c$.

(3) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(4, 35)$. *An example is* $f(x, y)=(y^5+\xi_1x^2)^4+\xi_2x+c$.

(4) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 7)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^2+\xi_2x^2y^3+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 11)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^2+\xi_2xy^5+c$.

(6) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(2, 17)$. *An example is* $f(x, y)=(y^5+\xi_1x^3)^2+\xi_2x^2y+c$.

(7) $\rho_\infty(C)=2$, $P_1={}^t(9, 7)$, $P_2={}^t(2, 23)$. *An example is* $f(x, y)=(y^9+\xi_1x^2)^2+\xi_2xy^2+c$.

(8) $\rho_\infty(C)=2$, $P_1={}^t(5, 1)$, $P_2={}^t(2, 35)$. *An example is* $f(x, y)=(y^5+\xi_1x^4)^2+\xi_2x+c$.

(9) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(2, 7)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy)^2+\xi_3f_1(x, y)y^2+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(10) $\rho_\infty(C)=3$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 15)$, $P_3={}^t(2, 15)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2 x)^2+\xi_3 f_1(x, y)+c$, $f_1(x, y)=y^5+\xi_1 x^2$.

$g=15$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(7, 1)$. *An example is* $f(x, y)=y^7+\xi_1 x^6+c$.

(2) $\rho_\infty(C)=1$, $P_1={}^t(11, 7)$. *An example is* $f(x, y)=y^{11}+\xi_1 x^4+c$.

(3) $\rho_\infty(C)=1$, $P_1={}^t(16, 13)$. *An example is* $f(x, y)=y^{16}+\xi_1 x^3+c$.

(4) $\rho_\infty(C)=1$, $P_1={}^t(31, 29)$. *An example is* $f(x, y)=y^{31}+\xi_1 x^2+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 5)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^3+\xi_2 x^3 y^2+c$.

(6) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(5, 24)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^5+\xi_2 x^2+c$.

(7) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(8, 45)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^8+\xi_2 x+c$.

(8) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(3, 20)$. *An example is* $f(x, y)=(y^5+\xi_1 x^2)^3+\xi_2 x^2+c$.

(9) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 5)$. *An example is* $f(x, y)=(y^4+\xi_1 x^3)^2+\xi_2 x^4 y+c$.

(10) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(3, 29)$. *An example is* $f(x, y)=(y^4+\xi_1 x^3)^3+\xi_2 xy+c$.

(11) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(4, 45)$. *An example is* $f(x, y)=(y^4+\xi_1 x^3)^4+\xi_2 y+c$.

(12) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 9)$. *An example is* $f(x, y)=(y^7+\xi_1 x^2)^2+\xi_2 xy^6+c$.

(13) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(3, 35)$. *An example is* $f(x, y)=(y^7+\xi_1 x^2)^3+\xi_2 x+c$.

(14) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(2, 15)$. *An example is* $f(x, y)=(y^5+\xi_1 x^3)^2+\xi_2 x^3+c$.

(15) $\rho_\infty(C)=2$, $P_1={}^t(9, 7)$, $P_2={}^t(2, 21)$. *An example is* $f(x, y)=(y^9+\xi_1 x^2)^2+\xi_2 xy^3+c$.

(16) $\rho_\infty(C)=2$, $P_1={}^t(9, 7)$, $P_2={}^t(3, 50)$. *An example is* $f(x, y)=(y^9+\xi_1 x^2)^3+\xi_2 y^2+c$.

(17) $\rho_\infty(C)=2$, $P_1={}^t(11, 9)$, $P_2={}^t(2, 33)$. *An example is* $f(x, y)=(y^{11}+\xi_1 x^2)^2+\xi_2 x+c$.

(18) $\rho_\infty(C)=2$, $P_1={}^t(7, 4)$, $P_2={}^t(2, 35)$. *An example is* $f(x, y)=(y^7+\xi_1 x^3)^2+\xi_2 x+c$.

(19) $\rho_\infty(C)=2$, $P_1={}^t(8, 5)$, $P_2={}^t(2, 45)$. *An example is* $f(x, y)=(y^8+\xi_1 x^3)^2+\xi_2 y+c$.

(20) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 5)$, $P_3={}^t(2, 17)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2 xy^2)^2+\xi_3 f_1(x, y)y+c$, $f_1(x, y)=y^3+\xi_1 x^2$.

(21) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(2, 5)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2 xy)^2+\xi_3 f_1(x, y)xy+c$, $f_1(x, y)=y^3+\xi_1 x^2$.

(22) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(3, 26)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2 xy)^3+\xi_3 y+c$, $f_1(x, y)=y^3+\xi_1 x^2$.

(23) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(3, 11)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2 x)^3+\xi_3 f_1(x, y)y+c$, $f_1(x, y)=y^3+\xi_1 x^2$.

(24) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(4, 21)$, $P_3={}^t(2, 21)$. *An example is given by* $f(x, y)=(f_1(x, y)^4+\xi_2 x)^2+\xi_3 f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1 x^2$.

(25) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(4, 21)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2 x)^4+\xi_3 f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1 x^2$.

(26) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 16)$, $P_3={}^t(3, 14)$. *An example is given by* $f(x, y)=(f_1(x, y)^3+\xi_2 y)^3+\xi_3 f_1(x, y)^2+c$, $f_1(x, y)=y^3+\xi_1 x^2$.

(27) $\rho_\infty(C)=3$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 21)$, $P_3={}^t(2, 9)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2 y)^2+\xi_3 f_1(x, y)+c$, $f_1(x, y)=y^4+\xi_1 x^3$.

(28) $\rho_\infty(C)=4$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(2, 9)$, $P_4={}^t(2, 9)$, h_1 *and* h_2 *are as* (11) *of* $g=9$, $h_3(u, v)=h_2(u, v)^2+\xi_3 h_1(u, v)u^9+R_3'(u, v)$, $\dim R_3'=1$ *and* $\mathcal{N}_3(\mathcal{W}; 12)'=\{u^{12}\}$, $h_4(u, v)=h_3(u, v)^2+\xi_4 h_2(u, v)u^{18}+R_4'(u, v)$, $\dim R_4'=1$ *and* $\mathcal{N}_4(\mathcal{W}; 24)'=\{u^{24}\}$. *An example is given by* $f(x, y)=f_3(x, y)^2+\xi_4 f_2(x, y)+c$, $f_1=y^3+\xi_1 x^2$, $f_2=f_1^2+\xi_2 x$, $f_3=f_2^2+\xi_3 f_1$.

$g=16$:

(1) $\rho_\infty(C)=1$, $P_1={}^t(9, 4)$. *An example is* $f(x, y)=y^9+\xi_1 x^5+c$.

(2) $\rho_\infty(C)=1$, $P_1={}^t(17, 14)$. *An example is* $f(x, y)=y^{17}+\xi_1 x^3+c$.

(3) $\rho_\infty(C)=1$, $P_1={}^t(33, 31)$. *An example is* $f(x, y)=y^{33}+\xi_1 x^2+c$.

(4) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 4)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^3+\xi_2 x^4 y+c$.

(5) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(4, 15)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^4+\xi_2 x^3+c$.

(6) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(6, 31)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^6+\xi_2 xy+c$.

(7) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(7, 38)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^7+\xi_2 y^2+c$.

(8) $\rho_\infty(C)=2$, $P_1={}^t(3, 1)$, $P_2={}^t(11, 64)$. *An example is* $f(x, y)=(y^3+\xi_1 x^2)^{11}+\xi_2 y+c$.

(9) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(3, 19)$. *An example is* $f(x, y)=(y^5+\xi_1 x^2)^3+\xi_2 xy^3+c$.

(10) $\rho_\infty(C)=2$, $P_1={}^t(5, 3)$, $P_2={}^t(5, 46)$. *An example is* $f(x, y)=(y^5+\xi_1 x^2)^5+\xi_2 y^2+c$.

(11) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(2, 3)$. *An example is* $f(x, y)=(y^4+\xi_1 x^3)^2+\xi_2 x^3 y^3+c$.

(12) $\rho_\infty(C)=2$, $P_1={}^t(4, 1)$, $P_2={}^t(3, 28)$. *An example is* $f(x, y)=(y^4+\xi_1x^3)^3+\xi_2x^2+c$.

(13) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(2, 7)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^2+\xi_2x^3+c$.

(14) $\rho_\infty(C)=2$, $P_1={}^t(7, 5)$, $P_2={}^t(3, 34)$. *An example is* $f(x, y)=(y^7+\xi_1x^2)^3+\xi_2y^4+c$.

(15) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(2, 13)$. *An example is* $f(x, y)=(y^5+\xi_1x^3)^2+\xi_2xy^4+c$.

(16) $\rho_\infty(C)=2$, $P_1={}^t(5, 2)$, $P_2={}^t(3, 40)$. *An example is* $f(x, y)=(y^5+\xi_1x^3)^3+\xi_2x+c$.

(17) $\rho_\infty(C)=2$, $P_1={}^t(9, 7)$, $P_2={}^t(2, 19)$. *An example is* $f(x, y)=(y^9+\xi_2x^2)^2+\xi_2xy^4+c$.

(18) $\rho_\infty(C)=2$, $P_1={}^t(11, 9)$, $P_2={}^t(2, 31)$. *An example is* $f(x, y)=(y^{11}+\xi_1x^2)^2+\xi_2xy+c$.

(19) $\rho_\infty(C)=2$, $P_1={}^t(11, 9)$, $P_2={}^t(3, 64)$. *An example is* $f(x, y)=(y^{11}+\xi_1x^2)^3+\xi_2y+c$.

(20) $\rho_\infty(C)=2$, $P_1={}^t(5, 1)$, $P_2={}^t(2, 31)$. *An example is* $f(x, y)=(y^5+\xi_1x^4)^2+\xi_2xy+c$.

(21) $\rho_\infty(C)=2$, $P_1={}^t(7, 4)$, $P_2={}^t(2, 33)$. *An example is* $f(x, y)=(y^7+\xi_1x^3)^2+\xi_2y^3+c$.

(22) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 3)$, $P_3={}^t(2, 27)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2x^3)^2+\xi_3f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(23) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 5)$, $P_3={}^t(2, 15)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy^2)^2+\xi_3f_1(x, y)x+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(24) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(2, 3)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy)^2+\xi_3f_1(x, y)x^2+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(25) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 13)$, $P_3={}^t(2, 25)$. *An example is given by* $f(x, y)=(f_1(x, y)^3+\xi_2xy)^2+\xi_3f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(26) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(3, 14)$, $P_3={}^t(2, 15)$. *An example is given by* $f(x, y)=(f_1(x, y)^3+\xi_2y^2)^2+\xi_3x+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(27) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 7)$, $P_3={}^t(3, 25)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2xy)^3+\xi_3f_1(x, y)+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(28) $\rho_\infty(C)=3$, $P_1={}^t(3, 1)$, $P_2={}^t(2, 9)$, $P_3={}^t(3, 10)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2x)^3+\xi_3y^2+c$, $f_1(x, y)=y^3+\xi_1x^2$.

(29) $\rho_\infty(C)=3$, $P_1={}^t(5, 3)$, $P_2={}^t(2, 15)$, $P_3={}^t(2, 11)$. *An example is given by* $f(x, y)=(f_1(x, y)^2+\xi_2x)^2+\xi_3f_1(x, y)y+c$, $f_1(x, y)=y^5+\xi_1x^2$.

References

[1] S. S. Abhyankar and T. Moh, Newton-Puiseux expansion and generalized Tschirnhausen transformation I, J. Reine Angew. Math., **260** (1973), 47–83.

[2] S.S. Abhyankar and T. Moh, Newton-Puiseux expansion and generalized Tschirnhausen transformation II, J. Reine Angew. Math., **261** (1973), 29-54.
[3] S.S. Abhyankar and T. Moh, Embeddings of line in the plane, J. Reine Angew. Math., **276** (1975), 148-166.
[4] N. A'Campo and M. Oka, Geometry of plane curves via Tschirnhausen resolution tower, Osaka J. Math., **33** (1996), 1003-1033.
[5] P. Jaworski, Normal forms and bases of local rings of irreducible germs of functions of two variables, J. Soviet Math., **50** (1990), 1350-1364.
[6] J. Milnor, Singular Points of Complex Hypersurface, Annals Math. Studies, vol. 61, Princeton Univ. Press, Princeton, 1968.
[7] M. Miyanishi, Minimization of the embeddings of the curves into the affine plane, J. Math. Kyoto Univ., **36** (1996), 311-329.
[8] T.T. Moh, On analytic irreducibility at ∞ of a pencil of curves, Proc. Amer. Soc., **44** (1974), 22-24.
[9] W.D. Neumann, Complex algebraic curves via their links at infinity, Invent. Math., **98** (1989), 445-489.
[10] T. Oda, Convex Bodies and Algebraic Geometry, Springer, Berlin-Heidelberg-New York, 1987.
[11] M. Oka, Polynomial normal form of a plane curve with a given weight sequence, Chinese Quart. J. Math., **10**, no. 4 (1995), 53-61.
[12] M. Suzuki, Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\boldsymbol{C}^2$, Journal of Math. Soc. Japan, **26** (1974), 241-257.
[13] E.W.v. Tschirnhausen, Acta eruditorum, Leiptiz, 1683.

Yuji NAKAZAWA
Department of Mathematics
Tokyo Institute of Technology
Oh-Okayama, Meguro-ku
Tokyo 152, Japan

Mutsuo OKA
Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1
Hachioji-shi, Tokyo 192-03, Japan

E-mail address: Yuji Nakazawa: nakazawa@math.titech.ac.jp
Mutsuo Oka: oka@math.metro-u.ac.jp