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# Powers of ideals in Cohen-Macaulay rings

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## 1. Introduction.

Let $I$ be an ideal in a Noetherian local ring $A$ with maximal ideal $\mathfrak{m}$ and assume that the field $A/\mathfrak{m}$ is infinite. For each integer $n \geqq 1$, let $I^{(n)} = \{a \in A \mid sa \in I^n$ for some $s \in A \setminus \bigcup_{p \in \mathrm{Min}_A A/I} p\}$ and call it the $n$-th symbolic power of $I$. In this paper we are going to investigate the conditions under which $I^{(n)} = I^n$ for all $n$. As is well-known, when $I$ is a prime ideal and the local ring $A_I$ is regular, $I^{(n)} = I^n$ for all $n \geqq 1$ if and only if the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geqq 0} I^n/I^{n+1}$ is an integral domain. Recalling this fact, in [**Ho**] Hochster gave a certain algorithm to check whether $\mathrm{G}(I)$ is an integral domain. Thereafter his paper has led numerous works and researches on this subject, cf. [**CN**], [**Hu1**], [**Hu2**], [**Hu4**], [**HH1**], [**HU**], [**RV**], [**SV**], [**T**]; among them we are especially interested in [**HH1**], where Huckaba and Huneke gave a criterion for the equality $I^{(n)} = I^n$ for all $n \geqq 1$ in terms of the local analytic spreads of $I$ in the case where the analytic spread $\lambda(I)$ of $I$ itself is relatively small. In the present paper we shall inherit the study of Huckaba and Huneke to develop their argument for the ideals of higher analytic deviation. But before going into the detail, we would like to fix some basic definitions.

We put $\lambda(I) = \dim A/\mathfrak{m} \otimes_A \mathrm{G}(I)$ and call it the analytic spread of $I$ (cf. [**NR**]). Then we have Burch's inequalities $\mathrm{ht}_A I \leqq \lambda(I) \leqq \dim A - \inf_{n \geqq 1}\{\mathrm{depth}\, A/I^n\}$ (cf. [**Bu**]). An ideal $J$ of $A$ is said to be a reduction of $I$, if $J \subseteqq I$ and $I^{n+1} = JI^n$ for some $n \geqq 0$. For each reduction $J$ of $I$ we put $\mathrm{r}_J(I) = \min\{n \geqq 0 \mid I^{n+1} = JI^n\}$ and call it the reduction number of $I$ with respect to $J$. A reduction $J$ of $I$ is said to be minimal, if it is minimal among the reductions of $I$. As is well-known, this is equivalent to saying that $J$ is generated by $\lambda(I)$ elements ([**NR**]).

If $I^{(n)} = I^n$ for all $n \geqq 1$, we have $\mathrm{Ass}_A A/I^n = \mathrm{Min}_A A/I$ for all $n \geqq 1$, so that $\mathrm{depth}\, A_Q/I^n A_Q > 0$ for any $Q \in \mathrm{V}(I) \setminus \mathrm{Min}_A A/I$; hence, because $\lambda(I_Q) \leqq \mathrm{ht}_A Q - \inf_{n > 0} \mathrm{depth}\, A_Q/I^n A_Q$ by Burch's inequality, we have $\lambda(I_Q) < \mathrm{ht}_A Q$ for any $Q \in \mathrm{V}(I) \setminus \mathrm{Min}_A A/I$. In their paper [**HH1**] Huckaba and Huneke proved that this condition $\lambda(I_Q) < \mathrm{ht}_A Q$ for all $Q \in \mathrm{V}(I) \setminus \mathrm{Min}_A A/I$ characterizes the equality $I^{(n)} = I^n$ $(n \geqq 1)$ for a certain class of ideals $I$ having $\lambda(I) - \mathrm{ht}_A I \leqq 2$. Following

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[**HH1**], we define $\mathrm{ad}(I)=\lambda(I)-\mathrm{ht}_A I$ and call it the analytic deviation of $I$. With this notation the main result of our paper can be stated as follows, which is a natural generalization of Huckaba and Huneke's results to the case of $\mathrm{ad}(I)$ arbitrary.

THEOREM (1.1). *Let $I$ be an unmixed ideal in a $d$-dimensional Cohen-Macaulay local ring $A$ with infinite residue field. Let $s=\mathrm{ht}_A I$ and assume that for all $Q\in \mathrm{V}(I)$ with $\mathrm{ht}_A Q<\max\{s+1, \lambda(I)\}$, the ideal $I_Q$ of $A_Q$ is generated by $\mathrm{ht}_A Q$ elements. Let $\alpha\in \boldsymbol{Z}$ with $\alpha>\mathrm{ad}(I)$ and assume that $\mathrm{depth}(A/I^n)_Q\geqq \min\{\alpha-n, \mathrm{ht}_A Q-s-n\}$ for all $Q\in \mathrm{V}(I)$ and $1\leqq n\leqq \mathrm{ad}(I)$. Then the following conditions are equivalent.*

(1) $I^{(n)}=I^n$ *for all* $n\geqq 1$.

(2) $\lambda(I_Q)<\dim A_Q$ *for any* $Q\in \mathrm{V}(I)\setminus \mathrm{Min}_A A/I$ *with* $\mathrm{ht}_A Q\leqq \lambda(I)$.

(3) $\lambda(I_Q)<\dim A_Q$ *for any* $Q\in \mathrm{V}(I)\setminus \mathrm{Min}_A A/I$.

*When this is the case, we have $\mathrm{r}_J(I)\leqq \mathrm{ad}(I)$ for any special reduction $J$ of $I$ (see* (2.1) *for the definition of special reductions) and* $\mathrm{depth}\ A/I^n\geqq \min\{\alpha-\mathrm{ad}(I), d-\lambda(I)\}$ *for all* $n\geqq 1$.

Our proof of Theorem (1.1) is based on the calculation of $\mathrm{depth}(A/J^m I^{\mathrm{ad}(I)})_Q$ for $m\geqq 0$ and $Q\in \mathrm{V}(I)$, where $J=(a_1, a_2, \cdots, a_{\lambda(I)})$ is a special reduction of $I$. The notion of special reduction was introduced by Aberbach and Huneke [**AH**, Definition 5.1] and there they guaranteed its existence, for example, in the case where $I_Q$ is a complete intersection for all $Q\in \mathrm{V}(I)$ with $\mathrm{ht}_A Q<\lambda(I)$ ([**AH**, Section 6]). In [**U**] Ulrich proved that $I$ has a special reduction if and only if $I$ satisfies the condition $G_{\lambda(I)}$ in the sense of Artin and Nagata [**AN**], i.e., $I_Q$ is generated by $\mathrm{ht}_A Q$ elements for all $Q\in \mathrm{V}(I)$ with $\mathrm{ht}_A Q<\lambda(I)$, which we assume in our Theorem (1.1). In Section 2 of our paper we will give a brief summary on special reductions. In Section 3 we shall compute, modifying the generators $a_1, a_2, \cdots, a_{\lambda(I)}$ of a given special reduction $J$ of $I$, the depth of $(A/(a_1, \cdots, a_i)^m I^n)_Q$ for $Q\in \mathrm{V}(I)$ in terms of $m$, $n$ and $i$, so that we will have the information necessary for the proof of Theorem (1.1), that is the case $i=\lambda(I)$ and $n=\mathrm{ad}(I)$.

When $A$ is a Gorenstein ring and $A/I$ is Cohen-Macaulay, we can improve Theorem (1.1) and have the following.

THEOREM (1.2). *Assume that $A$ is a $d$-dimensional Gorenstein ring with infinite residue class field and let $I$ be an ideal in $A$ of height $s$. Assume that $A/I$ is a Cohen-Macaulay ring and that for all $Q\in \mathrm{V}(I)$ with $\mathrm{ht}_A Q<\max\{s+1, \lambda(I)\}$, $I_Q$ is generated by $\mathrm{ht}_A Q$ elements. Suppose $\alpha\geqq \mathrm{ad}(I)$ and assume that* $\mathrm{depth}(A/I^n)_Q\geqq \min\{\alpha-n, \mathrm{ht}_A Q-s-n\}$ *for all $Q\in \mathrm{V}(I)$ and $1\leqq n\leqq \mathrm{ad}(I)$. Then the conditions* (1), (2) *and* (3) *stated in Theorem* (1.1) *are equivalent to each other. And when this is the case, we have $\mathrm{r}_J(I)\leqq \max\{0, \mathrm{ad}(I)-1\}$ for any special* *reduction* $J$ *of* $I$ *and* depth $A/I^n \geqq \min\{\alpha - \mathrm{ad}(I)+1, d-\lambda(I)\}$ *for all* $n \geqq 1$.

If $\mathrm{ad}(I)=2$, the above theorem covers [**HH1**, Theorem 3.5].

The method we adopt in this paper is quite useful when we discuss the Cohen-Macaulay and the Gorenstein property of the associated graded ring

$$\mathrm{G}(I) = \bigoplus_{n \geqq 0} I^n/I^{n+1}.$$

For example, after the modification of a system of generators $a_1, a_2, \cdots, a_{\lambda(I)}$ of a given special reduction $J$ stated above, we can prove that $a_1, a_2, \cdots, a_s$ is an $A$-regular sequence and $(a_1, a_2, \cdots, a_s)A \cap I^n = (a_1, a_2, \cdots, a_s)I^{n-1}$ for all $n \geqq 1$, which implies that $a_1^*, a_2^*, \cdots, a_s^*$ is an $\mathrm{G}(I)$-regular sequence ($a_i^*$ denotes the initial form of $a_i$ in $\mathrm{G}(I)$). Moreover the results derived from the proof of the theorems above guarantees that depth $\mathrm{G}(I)=d$ if depth $A/I^n$ is big enough for $1 \leqq n \leqq \mathrm{ad}(I)$. But we will not refer to it any more in this paper since the subsequent paper [**GNN**] is devoted to a precise investigation on this subject.

Throughout this paper let $(A, \mathfrak{m})$ denote a Noetherian local ring with infinite residue class field. For an ideal $I$ of $A$ we denote by $\mathrm{V}(I)$ the set of prime ideals in $A$ containing $I$. Let $\mathrm{Min}_A A/I$ be the set of minimal elements in $\mathrm{V}(I)$. We put $\mathrm{Assh}_A A/I = \{Q \in \mathrm{Min}_A A/I \mid \dim A/I = \dim A/Q\}$. The number of a minimal system of generators for an $A$-module $M$ shall be denoted by $\mu_A(M)$ and for a prime ideal $Q$ in $A$, we write $\mu_Q(M) = \mu_{A_Q}(M_Q)$.

## 2. Special reductions.

Let $(A, \mathfrak{m})$ be a Noetherian local ring having infinite residue field and let $I$ be an ideal of $A$. We put $s = \mathrm{ht}_A I$ and $l = \lambda(I)$. In this section we recall the definition of special reduction of $I$ given by Aberbach and Huneke and prove its existence in the case where $I$ satisfies the condition $G_l$ in the sense of Artin and Nagata [**AN**].

DEFINITION (2.1) (cf. [**AH**, Definition 5.1]). We say that $J$ is a special reduction of $I$ if $J$ is a minimal reduction of $I$ and if there exists a system of generators $a_1, a_2, \cdots, a_l$ of $J$ such that $I_Q = (a_1, a_2, \cdots, a_{\mathrm{ht}_A Q})A_Q$ for all $Q \in \mathrm{V}(I)$ with $\mathrm{ht}_A Q < l$ (in the case where $\mathrm{ht}_A Q = 0$, this equality reads that $I_Q = (0)$). In particular, if $s = l$, then any minimal reduction is a special reduction.

Let $K$ be an ideal contained in $I$ and let $Q \in \mathrm{V}(I)$. Then the equality $K_Q = I_Q$ holds if and only if $K : I \not\subseteq Q$. Hence a minimal reduction $J$ of $I$ is special if and only if we can choose a system of generators $a_1, a_2, \cdots, a_l$ of $J$ so that $\mathrm{ht}_A(I + (a_1, \cdots, a_i)A : I) > i$ for all $s \leqq i < l$. Thus our definition of special reduction is the same as that in [**AH**]. The following result is due to Ulrich [**U**].

PROPOSITION (2.2). *Let $J$ be a reduction of $I$. Then the following conditions are equivalent.*

(1) *There exists a special reduction of $I$ contained in $J$.*

(2) $J_Q=I_Q$ *and* $\mu_Q(I)\leqq \mathrm{ht}_A Q$ *for all* $Q\in \mathrm{V}(I)$ *with* $\mathrm{ht}_A Q<l$.

PROOF. The implication (1)$\Rightarrow$(2) is obvious by the definition of special reduction, so we have to prove the converse. For that it is enough to show in the case where $J=I$ since a special reduction of $J$ is a special reduction of $I$ as well if $J_Q=I_Q$ for all $Q\in\mathrm{V}(I)$ with $\mathrm{ht}_A Q<l$. In the following as assume $\mu_Q(I)\leqq\mathrm{ht}_A Q$ for all $Q\in\mathrm{V}(I)$ with $\mathrm{ht}_A Q<l$ and inductively choose elements $a_1, a_2, \cdots, a_l$ in $I$ so that the conditions;

(i) $a_1^*, a_2^*, \cdots, a_i^*$ is a s.s.o.p. for $S$, where $S=A/\mathfrak{m}\otimes_A \mathrm{G}(I)=\bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$ and $a_j^*$ denotes the image of $a_j$ in $I/\mathfrak{m}I=S_1$,

(ii) $\mu_Q(I/(a_1, \cdots, a_i))\leqq\max\{0, \mathrm{ht}_A Q-i\}$ for any $Q\in\mathrm{V}(I)$ with $\mathrm{ht}_A Q<l$

are satisfied for all $1\leqq i\leqq l$. Then it is easy to see that $(a_1, a_2, \cdots, a_l)A$ is a special reduction of $I$.

Now suppose $1\leqq i\leqq l$ and we have already taken the elements $a_1, a_2, \cdots, a_{i-1}$ satisfying the required conditions, namely, we assume that (i)$'$ $a_1^*, \cdots, a_{i-1}^*$ is a s.s.o.p. for $S$ and (ii)$'$ $\mu_p(I/(a_1, \cdots, a_{i-1}))\leqq\max\{0, \mathrm{ht}_A p-i+1\}$ for any $p\in\mathrm{V}(I)$ with $\mathrm{ht}_A p<l$ (notice that if $i=1$, then (1)$'$ insists nothing and (ii)$'$ is just the condition (2)). Let $\mathcal{F}$ be the set of $Q\in\mathrm{V}(I)$ such that $i\leqq\mathrm{ht}_A Q<l$ and $\mu_Q(I/(a_1, \cdots, a_{i-1}))=\mathrm{ht}_A Q-i+1$.

CLAIM. *$\mathcal{F}$ is a finite set.*

PROOF OF CLAIM. Let $Q\in\mathcal{F}$. We put $\mathrm{ht}_A Q=k$. Then as $\mu_Q(I/(a_1, \cdots, a_{i-1}))=k-i+1$ we have $Q\supseteqq\mathcal{B}_k$, where $\mathcal{B}_k=\mathrm{ann}_A\wedge^{k-i+1}I/(a_1, \cdots, a_{i-1})$. On the other hand, if $p\in\mathrm{V}(I)$ and if $\mathrm{ht}_A p<k$, by the condition (ii)$'$ we see $\mu_p(I/(a_1, \cdots, a_{i-1}))\leqq\max\{0, \mathrm{ht}_A p-i+1\}<k-i+1$ and so $p\not\supseteqq\mathcal{B}_k$. This means $\mathrm{ht}_A(I+\mathcal{B}_k)\geqq k$. Thus we have $Q\in\mathrm{Min}_A A/(I+\mathcal{B}_k)$. Hence $\mathcal{F}\subseteqq\bigcup_{i\leq k<l}\mathrm{Min}_A A/(I+\mathcal{B}_k)$, which implies that $\mathcal{F}$ is finite as is required.

For $Q\in\mathcal{F}$, we denote by $V(Q)$ the subspace $((a_1, \cdots, a_{i-1})A_Q+QI_Q)\cap I+\mathfrak{m}I/\mathfrak{m}I$ of the vector space $S_1=I/\mathfrak{m}I$. Notice that $V(Q)\neq S_1$. Actually, if $V(Q)=S_1$, then $((a_1, \cdots, a_{i-1})A_Q+QI_Q)\cap I=I$ by Nakayama's lemma and so $(a_1, \cdots, a_{i-1})A_Q+QI_Q=I_Q$, which implies $(a_1, \cdots, a_{i-1})A_Q=I_Q$. But this cannot happen since $\mu_Q(I/(a_1, \cdots, a_{i-1}))=\mathrm{ht}_A Q-i+1\geqq 1$ as $Q\in\mathcal{F}$. Let $P\in\mathrm{Assh}_S S/(a_1^*, \cdots, a_{i-1}^*)S$ and put $W(P)=P\cap S_1$. Then $W(P)$ is also a subspace of $S_1$ and $W(P)\neq S_1$. Therefore there exists an element $a_i\in I$ whose image in $S_1$ is not included in any $W(P)$ ($P\in\mathrm{Assh}_S S/(a_1^*, \cdots, a_{i-1}^*)S$) nor $V(Q)$ ($Q\in\mathcal{F}$), since we are assuming that $A/\mathfrak{m}$ is infinite. Then obviously $a_1^*, \cdots, a_i^*$ is a s.s.o.p. for $S$. So, in the following we prove that the condition (ii) is satisfied. If $\mathrm{ht}_A Q<i$,

then $\mathrm{ht}_A Q-i+1\leqq 0$ and so we have $\mu_Q(I/(a_1, \cdots, a_{i-1}))=0$ by the condition (ii)′, which implies $\mu_Q(I/(a_1, \cdots, a_i))=0=\max\{0, \mathrm{ht}_A Q-i\}$. Let us consider the case where $\mathrm{ht}_A Q\geqq i$. If $Q\in\mathcal{F}$, then $\mu_Q(I/(a_1, \cdots, a_{i-1}))=\mathrm{ht}_A Q-i+1$ and $a_i^*\notin V(Q)$, which means that $a_i$ is a part of a minimal system of generators of $I_Q/(a_1, \cdots, a_{i-1})A_Q$. Hence we have $\mu_Q(I/(a_1, \cdots, a_i))=\mathrm{ht}_A Q-i=\max\{0, \mathrm{ht}_A Q-i\}$. Even if $Q\notin\mathcal{F}$ we have $\mu_Q(I/(a_1, \cdots, a_i))\leqq\mathrm{ht}_A Q-i$ since $\mu_Q(I/(a_1, \cdots, a_i))\leqq \mu_Q(I/(a_1, \cdots, a_{i-1}))<\mathrm{ht}_A Q-i+1$. This completes the proof of Proposition (2.2).

REMARK (2.3). In the proof of (2.2) we can take $a_{s+1}$ so that, for any $Q\in\mathrm{Min}_A A/I$ with $I_Q\neq(0)$, it is a part of a minimal system of generators of $I_Q$. In fact, because $\mathrm{Min}_A A/I$ is finite, we can choose $a_{s+1}$ so that its image in $S_1$ is not in the proper subspace $QI_Q\cap I+\mathfrak{m}I/\mathfrak{m}I$ for all $Q\in\mathrm{Min}_A A/I$ with $I_Q\neq(0)$.

COROLLARY (2.4). *Let $J$ be a special reduction of $I$. Then, for any $Q\in\mathrm{V}(I)$, there exists a special reduction of $I_Q$ contained in $J_Q$.*

PROOF. We put $l'=\lambda(I_Q)$ and take any $p\in\mathrm{V}(I)$ such that $p\subseteqq Q$ and $\mathrm{ht}_A p<l'$. Then as $l'\leqq l$ we have $J_p=I_p$ and $\mu_p(I)\leqq\mathrm{ht}_A p$, which means $(J_Q)_{pA_Q}=(I_Q)_{pA_Q}$ and $\mu_{pA_Q}(I_Q)\leqq\mathrm{ht}_{A_Q} pA_Q$. Hence by (2.2) we see that $J_Q$ contains a special reduction of $I_Q$.

COROLLARY (2.5) (cf. [**AH**, Proposition 6.4]). *If $I_Q$ is generated by a regular sequence of length $s$ for any $Q\in\mathrm{V}(I)$ such that $\mathrm{ht}_A Q<l$, then any minimal reduction of $I$ is a special reduction.*

## 3. The depth of $(A/J^m I^{\mathrm{ad}(I)})_Q$ for $Q\in\mathrm{V}(I)$.

In this section let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and $I$ an ideal in $A$ having a special reduction $J$. We put $s=\mathrm{ht}_A I$, $l=\lambda(I)$ and assume that $a_1, a_2, \cdots, a_l$ is a system of generators of $J$ such that

(3.1) $\qquad I_Q=(a_1, a_2, \cdots, a_{\mathrm{ht}_A Q})A_Q$ for all $Q\in\mathrm{V}(I)$ with $\mathrm{ht}_A Q<l$.

For $1\leqq i\leqq l$ we write $J_i=(a_1, a_2, \cdots, a_i)A$. In particular $J_0=(0)$.

We begin with modifying $a_1, a_2, \cdots, a_l$ so that they enjoy the property in the following lemma and (3.1) is still satisfied after the modification.

LEMMA (3.2). *We may assume that, for any $1\leqq i\leqq l$, $a_i\notin Q$ if $Q\in(\mathrm{Ass}\, A\cup(\bigcup_{m\geqq 1}\mathrm{Ass}_A A/J_{i-1}{}^m))\setminus\mathrm{V}(I)$.*

PROOF. Let $a_i'\in J^2$ for $i=1, 2, \cdots, l$. Notice that when we replace $a_i$ by $a_i+a_i'$, the elements $a_1, \cdots, a_l$ again forms a minimal system of generators of $J$ and moreover they still enjoy the property of (3.1). Actually, if $Q\in\mathrm{V}(I)$, then $J^2\subseteqq QI$ and so, in $I_Q$, we have $a_i\equiv a_i+a_i' \bmod QI_Q$ for $i=1, 2, \cdots, l$, which means $I_Q=(a_1+a_1', \cdots, a_{\mathrm{ht}_A Q}+a'_{\mathrm{ht}_A Q})A_Q$ when $\mathrm{ht}_A Q<l$. Therefore we will inductively choose adequate elements $a_i'$ in $J^2$ for $1\leqq i\leqq l$ so that after the replacing the required conditions are satisfied.

Now suppose $1\leqq i\leqq n$ and assume that we have already modified $a_1, \cdots, a_{i-1}$ (if $i=1$, this insists nothing). We put $\mathcal{F}=(\mathrm{Ass}\, A\cup(\bigcup_{m\geqq 1}\mathrm{Ass}_A A/J_{i-1}{}^m))\setminus \mathrm{V}(I)$. By [**Br**] we see that $\mathcal{F}$ is finite. If $a_i\notin Q$ for any $Q\in\mathcal{F}$, we do not change $a_i$. So let us consider the case where $a_i\in Q$ for some $Q\in\mathcal{F}$. Let $\{Q_1, Q_2, \cdots, Q_q\}$ be the set derived from $\mathcal{F}$ by deleting the smaller elements when there exist relations of inclusions. We may assume $a_i\in Q_1\cap\cdots\cap Q_p$ $(1\leqq p\leqq q)$ and $a_i\notin Q_{p+1}\cup\cdots\cup Q_q$. Because all of $J^2, Q_{p+1}, \cdots, Q_q$ is not contained in any of $Q_1, \cdots, Q_p$, there exists $a_i'\in J^2\cap Q_{p+1}\cap\cdots\cap Q_q$ such that $a_i'\notin Q_1\cup\cdots\cup Q_p$. Then we easily see that $a_i+a_i'\notin Q_1\cup\cdots\cup Q_q$. Thus replacing $a_i$ by $a_i+a_i'$ we have $a_i\notin Q$ for any $Q\in\mathcal{F}$. Repeating this procedure until $i=l$ we get the required assertion.

In the rest of this section we assume that $a_1, a_2, \cdots, a_l$ is a system of generators of $J$ having the property of (3.1) and (3.2).

LEMMA (3.3). *$a_1, a_2, \cdots, a_s$ is an $A$-regular sequence.*

PROOF. Suppose $1\leqq i\leqq s$ and assume that $a_1, \cdots, a_{i-1}$ is an $A$-regular sequence. Let $Q\in\mathrm{Ass}_A A/J_{i-1}$. Then we have $\mathrm{ht}_A Q=i-1<s$ as $A/J_{i-1}$ is Cohen-Macaulay, and so $Q\not\supseteq I$. Hence $a_i\notin Q$ by (3.2). Therefore $a_i$ is a non-zero-divisor on $A/J_{i-1}$. Thus we can prove that $a_1, \cdots, a_i$ is an $A$-regular sequence for $1\leqq i\leqq s$ by induction on $i$.

LEMMA (3.4). *$((0): a_i)\cap I=(0)$ for any $1\leqq i\leqq l$.*

PROOF. It is enough to show that $((0): a_i)A_Q\cap I_Q=(0)$ for any $Q\in\mathrm{Ass}\, A$. If $I\subseteq Q$, then $I_Q=(0)$ as $\mathrm{ht}_A Q=0$ (see Definition (2.1)). And if $I\not\subseteq Q$, then $a_i\notin Q$ by (3.2) and so $((0): a_i)A_Q=(0)$ as $a_i$ is a unit in $A_Q$. Thus in any case we get the required assertion.

Our purpose of this section is to prove the following lemma, which is the most important result in this paper from the technical point of view.

LEMMA (3.5). *Let $N$ be an integer such that $0\leqq N\leqq \mathrm{ad}(I)$. Assume that, for a fixed integer $\alpha\geqq N$, $\mathrm{depth}(A/I^n)_Q\geqq\min\{\alpha-n, \mathrm{ht}_A Q-s-n\}$ if $Q\in\mathrm{V}(I)$ and $1\leqq n\leqq N$. Then we have*

(3.6) $$\mathrm{depth}(A/J_i{}^m I^n)_Q\geqq\min\{\alpha-n, \mathrm{ht}_A Q-s-n\},$$

*where $Q\in\mathrm{V}(I)$, $m\geqq 0$, $0\leqq n\leqq N$ and $0\leqq i\leqq n+s$.*

PROOF. We prove Lemma (3.5) by "triple" induction on $m$, $n$ and $i$. We begin with induction on $m$. But if $m=0$, the required inequality (3.6) is just the hypothesis since $J_i^0 I^n = I^n$ for all $i$ and $n$. So we fix $m \geqq 0$ and assuming that (3.6) holds for this $m$ we will prove

$$\text{(3.7)} \qquad \operatorname{depth}(A/J_i^{m+1}I^n)_Q \geqq \min\{\alpha - n,\ \operatorname{ht}_A Q - s - n\}$$

for any $Q \in \mathrm{V}(I)$, $0 \leqq n \leqq N$ and $0 \leqq i \leqq n+s$ by induction on $n$. For that, in the case where $n=0$, it is enough to compute $\operatorname{depth}(A/J_i^{m+1})_Q$ for any $Q \in \mathrm{V}(I)$ and $0 \leqq i \leqq s$. However if $0 \leqq i \leqq s$, $A/J_i^{m+1}$ is a $d-i$ dimensional Cohen-Macaulay ring since $a_1, \cdots, a_i$ is an $A$-regular sequence by (3.3), and so we get, for any $Q \in \mathrm{V}(I)$, $\operatorname{depth}(A/J_i^{m+1})_Q = \operatorname{ht}_A Q - i \geqq \min\{\alpha, \operatorname{ht}_A Q - s\}$, which is the inequality derived from (3.7) substituting $n=0$. Now we fix $0 \leqq n < N$ and assuming that (3.7) holds for this $n$ we will prove

$$\text{(3.8)} \qquad \operatorname{depth}(A/J_i^{m+1}I^{n+1})_Q \geqq \min\{\alpha - n - 1,\ \operatorname{ht}_A Q - s - n - 1\}$$

for any $Q \in \mathrm{V}(I)$ and $0 \leqq i \leqq n+s+1$ by induction on $i$. But again this is obvious if $i=0$ as $J_0^{m+1}I^{n+1} = (0)$ and as $\operatorname{depth} A_Q = \operatorname{ht}_A Q$. Therefore in the following we consider $\operatorname{depth}(A/J_{i+1}^{m+1}I^{n+1})_Q$ assuming the inequality (3.8) for a fixed integer $0 \leqq i < n+s+1$.

Here we need the following

CLAIM 1. $(J_i^{m+1} : a_{i+1}) \cap J_{i+1}^m I^{n+1} = J_i^{m+1} I^n$.

Suppose this is true. Then we can determine the Kernel of the natural surjection $\varphi : A/J_i^{m+1}I^{n+1} \to A/J_{i+1}^{m+1}I^{n+1}$ as follows: First notice $J_{i+1}^{m+1} = J_i^{m+1} + a_{i+1}J_{i+1}^m$ and so

$$\begin{aligned} \operatorname{Ker}\varphi &\cong a_{i+1}J_{i+1}^m I^{n+1}/J_i^{m+1}I^{n+1} \cap a_{i+1}J_{i+1}^m I^{n+1} \\ &= a_{i+1}J_{i+1}^m I^{n+1}/a_{i+1}J_i^{m+1}I^n \end{aligned}$$

since

$$J_i^{m+1}I^{n+1} \cap a_{i+1}J_{i+1}^m I^{n+1} \subseteqq a_{i+1}((J_i^{m+1} : a_{i+1}) \cap J_{i+1}^m I^{n+1}) = a_{i+1}J_i^{m+1}I^n$$

by Claim 1. Next let $x \in J_{i+1}^m I^{n+1}$ be an element in the kernel of the surjection

$$J_{i+1}^m I^{n+1} \xrightarrow{a_{i+1}} a_{i+1}J_{i+1}^m I^{n+1}/a_{i+1}J_i^{m+1}I^n.$$

Then there exists $y \in J_i^{m+1}I^n$ such that $a_{i+1}x = a_{i+1}y$. This means $x=y$ since $x - y \in ((0) : a_{i+1}) \cap J_{i+1}^m I^{n+1} \subseteqq ((0) : a_{i+1}) \cap I = (0)$ by (3.4), and so $x \in J_i^{m+1}I^n$. Hence we have an isomorphism $\operatorname{Ker}\varphi \cong J_{i+1}^m I^{n+1}/J_i^{m+1}I^n$. Thus we get an exact sequence

$$\text{(3.9)} \qquad 0 \longrightarrow J_{i+1}^m I^{n+1}/J_i^{m+1}I^n \longrightarrow A/J_i^{m+1}I^{n+1} \xrightarrow{\varphi} A/J_{i+1}^{m+1}I^{n+1} \longrightarrow 0,$$

which plays a key role in our proof together with the natural exact sequence

(3.10) $\quad 0 \longrightarrow J_{i+1}{}^m I^{n+1}/J_i{}^{m+1}I^n \longrightarrow A/J_i{}^{m+1}I^n \longrightarrow A/J_{i+1}{}^m I^{n+1} \longrightarrow 0.$

Let us recall our hypothesis of induction on $m$ and $n$ respectively insisting

$$\operatorname{depth}(A/J_{i+1}{}^m I^{n+1})_Q \geqq \min\{\alpha-n-1,\ \operatorname{ht}_A Q-s-n-1\}$$

and

$$\operatorname{depth}(A/J_i{}^{m+1}I^n)_Q \geqq \min\{\alpha-n,\ \operatorname{ht}_A Q-s-n\}.$$

Then applying Depth Lemma (cf. [**HH**, Remark 1]) to the exact sequence derived from (3.10) by localization at $Q$, we get

$$\operatorname{depth}_{A_Q}(J_{i+1}{}^m I^{n+1}/J_i{}^{m+1}I^n)_Q \geqq \min\{\alpha-n,\ \operatorname{ht}_A Q-s-n\}.$$

This fact and the hypothesis of induction on $i$ that

$$\operatorname{depth}(A/J_i{}^{m+1}I^{n+1})_Q \geqq \min\{\alpha-n-1,\ \operatorname{ht}_A Q-s-n-1\}$$

imply the required inequality

$$\operatorname{depth}(A/J_{i+1}{}^{m+1}I^{n+1})_Q \geqq \min\{\alpha-n-1,\ \operatorname{ht}_A Q-s-n-1\}$$

by Depth Lemma applied to the exact sequence derived from (3.9) localizing at $Q$. In order to prove Claim 1 we need some preparations. In the following arguments $m$, $n$ and $i$ denotes the integers fixed above.

CLAIM 2. *Let* $0\leqq j\leqq n+s$. *Then*

$$(J_j{}^{m+1}:a_{j+1})\cap J_{j+1}{}^m I^{j-s+1}=J_j{}^{m+1}I^{j-s}.$$

PROOF OF CLAIM 2. If $j<s$, then as $a_{j+1}$ is a non-zero-divisor over $A/J_j{}^{m+1}$ (cf. Proof of Corollary (3.3)) and as $I^{j-s}=I^{j-s+1}=A$ we have

$$\begin{aligned}(J_j{}^{m+1}:a_{j+1})\cap J_{j+1}{}^m I^{j-s+1} &= J_j{}^{m+1}\cap J_{j+1}{}^m\\ &= J_j{}^{m+1}\\ &= J_j{}^{m+1}I^{j-s}.\end{aligned}$$

So let us consider the case where $j\geqq s$. We take any $Q\in\operatorname{Ass}_A A/J_j{}^{m+1}I^{j-s}$. It is enough to show $((J_j)_Q^{m+1}:_{A_Q}a_{j+1})\cap(J_{j+1})_Q^m I_Q^{j-s+1}=(J_j)_Q^{m+1}I_Q^{j-s}$ since $(J_j{}^{m+1}:a_{j+1})\cap J_{j+1}{}^m I^{j-s+1}\supseteqq J_j{}^{m+1}I^{j-s}$. If $I\nsubseteq Q$, then $Q\in\operatorname{Ass}_A A/J_j{}^{m+1}$ and so $a_{j+1}\notin Q$ by (3.2), which means that $a_{j+1}$ is a unit in $A_Q$. Hence we have

$$\begin{aligned}((J_j)_Q^{m+1}:_{A_Q}a_{j+1})\cap(J_{j+1})_Q^m I_Q^{j-s+1} &= (J_j)_Q^{m+1}\cap(J_{j+1})_Q^m\\ &= (J_j)_Q^{m+1}\\ &= (J_j)_Q^{m+1}I_Q^{j-s}.\end{aligned}$$

In the case where $I\subseteqq Q$, by the hypothesis of induction on $n$ (notice $0\leqq j-s\leqq n$) we get

$$0=\operatorname{depth}(A/J_j{}^{m+1}I^{j-s})_Q \geqq \min\{\alpha-j+s,\ \operatorname{ht}_A Q-j\},$$

which implies $\operatorname{ht}_A Q-j\leqq 0$ as $\alpha-j+s\geqq\alpha-(n+s)+s=\alpha-n\geqq N-n>0$. Then as $\operatorname{ht}_A Q\leqq j\leqq n+s<\operatorname{ad}(I)+s=l$ we see $(J_j)_Q=(J_{j+1})_Q=I_Q$ by (3.1) and so we have

$$\begin{aligned}((J_j)_Q^{m+1}:_{A_Q} a_{j+1})\cap(J_{j+1})_Q^m I_Q^{j-s+1} &= (I_Q^{m+1}:_{A_Q} a_{j+1})\cap I_Q^{m+j-s+1}\\ &= I_Q^{m+j-s+1}\\ &= (J_j)_Q^{m+1} I_Q^{j-s},\end{aligned}$$

which completes the proof of Claim 2.

CLAIM 3. *Let $Q\in \mathrm{V}(I)$ such that $\operatorname{ht}_A Q\leqq n+s$ and let $j\leqq n+s$. Then, for any $q\geqq j-s$, we have*

$$((J_j)_Q^{m+1}:_{A_Q} a_{j+1})\cap(J_{j+1})_Q^m I_Q^{q+1}=(J_j)_Q^{m+1} I_Q^q.$$

PROOF OF CLAIM 3. We take any $Q\in\mathrm{V}(I)$ such that $\operatorname{ht}_A Q\leqq n+s$ and fix it. We will prove the equality above by descending induction on $j$. Let $j=n+s$. Then $(J_j)_Q=(J_{j+1})_Q=I_Q$. Hence, for any $q\geqq j-s=n\geqq 0$, we have

$$\begin{aligned}((J_j)_Q^{m+1}:_{A_Q} a_{j+1})\cap(J_{j+1})_Q^m I_Q^{q+1} &= (I_Q^{m+1}:_{A_Q} a_{j+1})\cap I_Q^{m+q+1}\\ &= I_Q^{m+q+1}\\ &=(J_j)_Q^{m+1} I_Q^q.\end{aligned}$$

Now we suppose $j<n+s$ and assume $((J_{j+1})_Q^{m+1}:_{A_Q} a_{j+2})\cap(J_{j+2})_Q^m I_Q^{q+1}=(J_{j+1})_Q^{m+1} I_Q^q$ for any $q\geqq j+1-s$. We will show the required equality for $q\geqq j-s$ by induction on $q$. But in Claim 2 we have already seen that it holds if $q=j-s$. So we suppose $q\geqq j-s+1$ and assume $((J_j)_Q^{m+1}:_{A_Q} a_{j+1})\cap(J_{j+1})_Q^m I_Q^q=(J_j)_Q^{m+1} I_Q^{q-1}$. Then we have

$$\begin{aligned}&((J_j)_Q^{m+1}:_{A_Q} a_{j+1})\cap(J_{j+1})_Q^m I_Q^{q+1}\\ &= ((J_j)_Q^{m+1}:_{A_Q} a_{j+1})\cap(J_{j+1})_Q^m I_Q^q\cap(J_{j+1})_Q^m I_Q^{q+1}\end{aligned}$$

by the inductive hypothesis on $q$

$$\begin{aligned}&= (J_j)_Q^{m+1} I_Q^{q-1}\cap(J_{j+1})_Q^m I_Q^{q+1}\\ &\subseteqq (J_j)_Q^{m+1}\cap(J_{j+1})_Q^{m+1}\cap(J_{j+2})_Q^m I_Q^{q+1}\\ &\subseteqq (J_j)_Q^{m+1}\cap((J_{j+1})_Q^{m+1}:_{A_Q} a_{j+2})\cap(J_{j+2})_Q^m I_Q^{q+1}\end{aligned}$$

by the inductive hypothesis on $j$

$$\begin{aligned}&= (J_j)_Q^{m+1}\cap(J_{j+1})_Q^{m+1} I_Q^q\\ &= (J_j)_Q^{m+1}\cap((J_j)_Q^{m+1}+a_{j+1}(J_{j+1})_Q^m) I_Q^q\\ &= (J_j)_Q^{m+1} I_Q^q+a_{j+1}(((J_j)_Q^{m+1}:_{A_Q} a_{j+1})\cap(J_{j+1})_Q^m I_Q^q)\end{aligned}$$

by the inductive hypothesis on $q$

$$= (J_j)_Q^{m+1} I_Q^q + a_{j+1}(J_j)_Q^{m+1} I_Q^{q-1}$$
$$= (J_j)_Q^{m+1} I_Q^q .$$

Hence we get the required equality and the proof of Claim 3 is completed.

PROOF OF CLAIM 1. Let us take any $Q \in \mathrm{Ass}_A A/J_i{}^{m+1} I^n$. It is enough to show $((J_i)_Q^{m+1} :_{A_Q} a_{i+1}) \cap (J_{i+1})_Q^m I_Q^{n+1} = (J_i)_Q^{m+1} I_Q^n$ since the inclusion $(J_i{}^{m+1} : a_{i+1}) \cap J_{i+1}{}^m I^{n+1} \supseteqq J_i{}^{m+1} I^n$ is obvious. If $I \not\subseteq Q$, then $Q \in \mathrm{Ass}_A A/J_i{}^{m+1}$ and so $a_{i+1} \notin Q$ by (3.2), which means $a_{i+1}$ is a unit in $A_Q$. Hence we have

$$\begin{aligned} ((J_i)_Q^{m+1} :_{A_Q} a_{i+1}) \cap (J_{i+1})_Q^m I_Q^{n+1} &= (J_i)_Q^{m+1} \cap (J_{i+1})_Q^m \\ &= (J_i)_Q^{m+1} \\ &= (J_i)_Q^{m+1} I_Q^n . \end{aligned}$$

In the case where $I \subseteqq Q$, by the hypothesis of induction on $n$ we see

$$0 = \mathrm{depth}(A/J_i{}^{m+1} I^n)_Q \geqq \min\{\alpha - n,\ \mathrm{ht}_A Q - s - n\} .$$

This implies $\mathrm{ht}_A Q - s - n \leqq 0$ as $\alpha - n \geqq N - n > 0$, and so $\mathrm{ht}_A Q \leqq n + s$. Then we have already seen in Claim 3 the required equality. Thus we have seen Claim 1 and Proof of Lemma (3.5) is completed.

Let $N = \mathrm{ad}(I)$ and $\alpha = \mathrm{ad}(I) + 1$. Then substituting $m = 1$, $n = \mathrm{ad}(I)$ and $i = l$ in (3.6) we have

$$\mathrm{depth}(A/JI^{\mathrm{ad}(I)})_Q \geqq \min\{1,\ \mathrm{ht}_A Q - l\}$$

for any $Q \in \mathrm{V}(I)$, and so $\mathrm{ht}_A Q \leqq l$ if $Q \in \mathrm{Ass}_A A/JI^{\mathrm{ad}(I)}$, from which we can prove $\mathrm{r}_J(I) \leqq \mathrm{ad}(I)$ under a suitable condition (see Section 4). Thus if we get certain information on $A/JI^N$ for an integer $N$, then this $N$ may bound $\mathrm{r}_J(I)$, so we would like to make it as small as possible. For example, if $A$ is Gorenstein and if $A/I$ is Cohen-Macaulay, we can get to know about $A/JI^{\mathrm{ad}(I)-1}$ as is described in the rest of this section. For that we assume that *$a_{s+1}$ is a part of a minimal system of generators for $I_Q$ for all $Q \in \mathrm{Min}_A A/I$ with $I_Q \neq (0)$* (cf. Remark (2.3)). We aim to prove the following

LEMMA (3.11). *Let $A$ be Gorenstein and $A/I$ Cohen-Macaulay. Let $N$ be an integer such that $0 \leqq N \leqq \mathrm{ad}(I)$. Assume that, for a fixed integer $\alpha \geqq N$,* $\mathrm{depth}(A/I^n)_Q \geqq \min\{\alpha - n,\ \mathrm{ht}_A Q - s - n\}$ *if $Q \in \mathrm{V}(I)$ and $1 \leqq n \leqq N$. Then we have*

$$\mathrm{depth}(A/J_i{}^m I^{i-s-1})_Q \geqq \min\{\alpha - i + s + 1,\ \mathrm{ht}_A Q - i\}, \tag{3.12}$$

*where $m \geqq 0$, $s + 1 \leqq i \leqq s + N$ and $Q \in \mathrm{V}(I)$.*

Of course it is enough to consider only the case where $N>0$, so we assume it in the rest of this section.

LEMMA (3.13). *Let $A$ be a Gorenstein ring and $A/I$ a Cohen-Macaulay ring. We put $K=J_s$. Then, for any $m\geqq 0$, the following assertions hold.*

(1) *$A/K^{m+1}:I$ is a $d-s$ dimensional Cohen-Macaulay ring.*

(2) *$K^{m+1}:a_{s+1}=K^{m+1}:I$.*

PROOF. (1) If $s=0$, then $K=(0)$ and so by [**PS**, Proposition 1.3] this assertion is true. Hence we have to prove in the case where $s>0$. Let us consider the exact sequence

$$0 \longrightarrow K^m/K^m(K:I) \longrightarrow A/K^m(K:I) \longrightarrow A/K^m \longrightarrow 0.$$

Because $a_1, \cdots, a_s$ is an $A$-regular sequence by (3.3), $\operatorname{depth}_A A/K^m=d-s$ and moreover $K^m/K^{m+1}$ is $A/K$-free, from which we see that $K^m/K^m(K:I)$ is $A/(K:I)$-free as $K^m/K^m(K:I)\cong K^m/K^{m+1}\otimes_A A/K:I$, and so $\operatorname{depth}_A K^m/K^m(K:I)=d-s$ since $A/K:I$ is a $d-s$ dimensional Cohen-Macaulay ring by [**PS**, Proposition 1.3]. Therefore by the exact sequence above we get $\operatorname{depth} A/K^m(K:I)=d-s$, which means $A/K^m(K:I)$ is a $d-s$ dimensional Cohen-Macaulay ring since $\dim A/K^m(K:I)\leqq\dim A/K^{m+1}=d-s$. In the following we prove $K^{m+1}:I=K^m(K:I)$. We take any $Q\in\operatorname{Ass}_A A/K^m(K:I)=\operatorname{Assh}_A A/K^m(K:I)$. It is enough to show $K_Q^{m+1}:_{A_Q}I_Q=K_Q^m(K_Q:_{A_Q}I_Q)$ since the inclusion $K^{m+1}:I\supseteqq K^m(K:I)$ is obvious. If $I\not\subseteq Q$, then we have

$$\begin{aligned} K_Q^{m+1}:_{A_Q}I_Q &= K_Q^{m+1}:_{A_Q}A_Q \\ &= K_Q^{m+1} \\ &= K_Q^m(K_Q:_{A_Q}A_Q) \\ &= K_Q^m(K_Q:I_Q). \end{aligned}$$

In the case where $I\subseteqq Q$, we see $I_Q=K_Q$ as $\operatorname{ht}_A Q=s$, and so

$$\begin{aligned} K_Q^{m+1}:_{A_Q}I_Q &= K_Q^{m+1}:_{A_Q}K_Q \\ &= K_Q^m \quad (\text{as } s>0) \\ &= K_Q^m(K_Q:_{A_Q}K_Q) \\ &= K_Q^m(K_Q:_{A_Q}I_Q). \end{aligned}$$

Thus we get the assertion (1).

(2) The inclusion $K^{m+1}:a_{s+1}\supseteqq K^{m+1}:I$ is obvious, so take any $Q\in\operatorname{Ass}_A A/K^{m+1}:I$ and will show $K_Q^{m+1}:_{A_Q}a_{s+1}=K_Q^{m+1}:_{A_Q}I_Q$. If $I\not\subseteq Q$, then $Q\in\operatorname{Ass}_A A/K^{m+1}$ and so $a_{s+1}\notin Q$ by (3.2), which means $a_{s+1}$ is a unit in $A_Q$. Hence we have

$$K_Q^{m+1} :_{A_Q} a_{s+1} = K_Q^{m+1} :_{A_Q} A_Q = K_Q^{m+1} :_{A_Q} I_Q .$$

If $I \subseteqq Q$, then $I_Q = K_Q$ since $\mathrm{ht}_A Q = s$ by (1). Moreover if $I_Q \neq (0)$ (in the case where $I_Q = (0)$, the required equality is trivial), by our assumption $a_{s+1}$ is a part of a minimal system of generators of $K_Q$ which forms an $A_Q$-regular sequence. Therefore

$$\begin{aligned} K_Q^{m+1} :_{A_Q} a_{s+1} &= K_Q^m \\ &= K_Q^{m+1} :_{A_Q} K_Q \\ &= K_Q^{m+1} :_{A_Q} I_Q \end{aligned}$$

and this completes the proof.

PROOF OF LEMMA (3.11). We prove by double induction on $m$ and $i$. We begin with induction on $m$. But if $m=0$, the required inequality (3.12) easily follows from the hypothesis as $J_i^0 I^{i-s-1} = I^{i-s-1}$ for all $i$. So we fix $m \geqq 0$ and assuming that (3.12) holds for this $m$ we will prove

(3.14) $$\mathrm{depth}(A/J_i^{m+1} I^{i-s-1})_Q \geqq \min\{\alpha - i + s + 1, \mathrm{ht}_A Q - i\}$$

for any $s+1 \leqq i \leqq s+N$ and $Q \in \mathrm{V}(I)$ by induction on $i$. Let us begin with the case where $i = s+1$. We put $K = J_s$ and consider the natural surjection $\varphi : A/K^{m+1} \to A/J_{s+1}^{m+1}$. Then, as $J_{s+1}^{m+1} = K^{m+1} + a_{s+1} J_{s+1}^m$,

$$\begin{aligned} \mathrm{Ker}\,\varphi &\cong a_{s+1} J_{s+1}^m / K^{m+1} \cap a_{s+1} J_{s+1}^m \\ &= a_{s+1} J_{s+1}^m / a_{s+1}(K^{m+1} : I) \end{aligned}$$

since $K^{m+1} \cap a_{s+1} J_{s+1}^m \subseteqq a_{s+1}(K^{m+1} : a_{s+1}) = a_{s+1}(K^{m+1} : I) = a_{s+1} K^m (K : I) \subseteqq K^{m+1} \cap a_{s+1} J_{s+1}^m$ by (3.13). Let $x \in J_{s+1}^m$ be an element in the kernel of the surjection

$$J_{s+1}^m \xrightarrow{a_{s+1}} a_{s+1} J_{s+1}^m / a_{s+1}(K^{m+1} : I) .$$

Then there exists $y \in K^{m+1} : I$ such that $a_{s+1} x = a_{s+1} y$. This means $x - y \in (0) : a_{s+1}$, so $(x-y) I \in ((0) : a_{s+1}) \cap I = (0)$ by (3.4). Hence $x - y \in (0) : I \subseteqq K^{m+1} : I$ and so $x \in K^{m+1} : I$. Thus $\mathrm{Ker}\,\varphi \cong J_{s+1}^m / K^{m+1} : I$ and we have the exact sequences

(3.15) $$0 \longrightarrow J_{s+1}^m / K^{m+1} : I \longrightarrow A/K^{m+1} \xrightarrow{\varphi} A/J_{s+1}^{m+1} \longrightarrow 0$$

and

(3.16) $$0 \longrightarrow J_{s+1}^m / K^{m+1} : I \longrightarrow A/K^{m+1} : I \longrightarrow A/J_{s+1}^m \longrightarrow 0 .$$

Let $Q \in \mathrm{V}(I)$. By the hypothesis of induction on $m$, we see $\mathrm{depth}(A/J_{s+1}^m)_Q \geqq \min\{\alpha, \mathrm{ht}_A Q - s - 1\}$, and by Lemma (3.13) we have $\mathrm{depth}(A/K^{m+1} : I)_Q = \mathrm{ht}_A Q - s$. Hence applying Depth Lemma to the exact sequence derived from (3.16) by localization at $Q$ we get $\mathrm{depth}_{A_Q}(J_{s+1}^m / K^{m+1} : I)_Q \geqq \min\{\alpha + 1, \mathrm{ht}_A Q - s\}$. Then again by Depth Lemma applied to (3.15) after localization at $Q$ we see

$$\operatorname{depth}(A/J_{s+1}{}^{m+1})_Q \geqq \min\{\alpha,\ \operatorname{ht}_A Q-s-1\}$$

since $\operatorname{depth}(A/K^{m+1})_Q=\operatorname{ht}_A Q-s$. Thus we have seen that (3.14) holds for $i=s+1$. So, in the following we fix $s+1\leqq i<s+N$ and assuming that (3.14) holds for this $i$, we will compute $\operatorname{depth}(A/J_{i+1}{}^{m+1}I^{i-s})_Q$.

CLAIM. $(J_i{}^{m+1}: a_{i+1})\cap J_{i+1}{}^m I^{i-s}=J_i{}^{m+1}I^{i-s-1}$.

PROOF OF CLAIM. The inclusion $(J_i{}^{m+1}: a_{i+1})\cap J_{i+1}{}^m I^{i-s}\supseteqq J_i{}^{m+1}I^{i-s-1}$ is obvious, so we take any $Q\in\operatorname{Ass}_A A/J_i{}^{m+1}I^{i-s-1}$ and will prove $((J_i)_Q^{m+1}:_{A_Q} a_{i+1})\cap(J_{i+1})_Q^{m+1}I_Q^{i-s-1}=(J_i)_Q^{m+1}I_Q^{i-s-1}$. If $I\nsubseteq Q$, then $Q\in\operatorname{Ass}_A A/J_i{}^{m+1}$ and so $a_{i+1}\notin Q$, which implies the required equality as we have seen many times in this paper. Suppose $I\subseteqq Q$. Then by the hypothesis of induction on $i$, we see

$$0=\operatorname{depth}(A/J_i{}^{m+1}I^{i-s-1})_Q\geqq\min\{\alpha-i+s+1,\ \operatorname{ht}_A Q-i\},$$

which means $\operatorname{ht}_A Q\leqq i$ as $\alpha-i+s+1\geqq N-i+s+1=((s+N)-i)+1>0$ and so $I_Q=(J_i)_Q=(J_{i+1})_Q$. Hence we have

$$\begin{aligned}((J_i)_Q^{m+1}:_{A_Q} a_{i+1})\cap(J_{i+1})_Q^m I_Q^{i-s} &= (I_Q^{m+1}:_{A_Q} a_{i+1})\cap I_Q^{m+i-s}\\ &= I_Q^{m+i-s}\quad(\text{as } i-s\geqq 1)\\ &= (J_i)_Q^{m+1}I_Q^{i-s-1}.\end{aligned}$$

Thus we have proved Claim.

By Claim, similarly as (3.9), we get an exact sequence

$$(3.17)\qquad 0\longrightarrow J_{i+1}{}^m I^{i-s}/J_i{}^{m+1}I^{i-s-1}\longrightarrow A/J_i{}^{m+1}I^{i-s}\longrightarrow A/J_{i+1}{}^{m+1}I^{i-s}\longrightarrow 0$$

since

$$\begin{aligned}J_{i+1}{}^{m+1}I^{i-s}/J_i{}^{m+1}I^{i-s} &\cong a_{i+1}J_{i+1}{}^m I^{i-s}/J_i{}^{m+1}I^{i-s}\cap a_{i+1}J_{i+1}{}^m I^{i-s}\\ &= a_{i+1}J_{i+1}{}^m I^{i-s}/a_{i+1}((J_i{}^{m+1}: a_{i+1})\cap J_{i+1}{}^m I^{i-s})\\ &\cong J_{i+1}{}^m I^{i-s}/J_i{}^{m+1}I^{i-s-1}.\end{aligned}$$

Together with (3.17) we consider the natural exact sequence

$$(3.18)\qquad 0\longrightarrow J_{i+1}{}^m I^{i-s}/J_i{}^{m+1}I^{i-s-1}\longrightarrow A/J_i{}^{m+1}I^{i-s-1}\longrightarrow A/J_{i+1}{}^m I^{i-s}\longrightarrow 0.$$

Let us recall the inductive hypothesis on $m$ and $i$ respectively insisting, for any $Q\in V(I)$,

$$\operatorname{depth}(A/J_{i+1}{}^m I^{i-s})_Q\geqq\min\{\alpha-i+s,\ \operatorname{ht}_A Q-i-1\}$$

and

$$\operatorname{depth}(A/J_i{}^{m+1}I^{i-s-1})_Q\geqq\min\{\alpha-i+s+1,\ \operatorname{ht}_A Q-i\}.$$

These inequality imply

$$\operatorname{depth}_{A_Q}(J_{i+1}{}^m I^{i-s}/J_i{}^{m+1}I^{i-s-1})_Q\geqq\min\{\alpha-i+s+1,\ \operatorname{ht}_A Q-i\}$$

by Depth Lemma applied to (3.18). Then again applying Depth Lemma to (3.17) we get

$$\operatorname{depth}(A/J_{i+1}{}^{m+1}I^{i-s})_Q \geqq \min\{\alpha-i+s,\ \operatorname{ht}_A Q-i-1\}$$

since, by Lemma (3.5),

$$\operatorname{depth}(A/J_i{}^{m+1}I^{i-s})_Q \geqq \min\{\alpha-i+s,\ \operatorname{ht}_A Q-i\},$$

and we have completed the proof of Lemma (3.11).

## 4. Proof of Theorem (1.1) and Theorem (1.2).

In this section we prove the Theorems (1.1) and (1.2). However these proofs are quite similar. So we precisely describe only the proof of Theorem (1.1) and Proof of Theorem (1.2) shall be given briefly, being indicated the different points. Throughout this section we put $s=\operatorname{ht}_A I$ and $l=\lambda(I)$.

PROOF OF THEOREM (1.1). As is stated in Introduction, (1) implies (3) by Burch's inequality and the implication (3)$\Rightarrow$(2) is obvious. So we assume the condition (2) and prove (1) together with the last assertions by induction on $\operatorname{ad}(I)$. If $\operatorname{ad}(I)=0$, then $I$ must be a complete intersection (cf. [**CN**, Theorem]). Actually, in this case a minimal reduction $J$ of $I$ is a complete intersection. Then, for any $Q\in\operatorname{Ass}_A A/J$, we have $Q\in\mathrm{V}(I)$ and $\operatorname{ht}_A Q=s$, which implies $\mu_Q(I)=s$ by the assumption, and so $I_Q=J_Q$ since a complete intersection ideal has no proper reduction. Thus we see $I=J$, whence $\mathrm{r}_J(I)=0$ and $I$ is a complete intersection. This means, for all $n\geqq 1$, $I^{(n)}=I^n$ as $A/I^n$ is Cohen-Macaulay. Now suppose $\operatorname{ad}(I)>0$ and assume that Theorem (1.1) is true for ideals with smaller analytic deviation than $I$. By (2.2) there exists a special reduction $J$ of $I$ and we can choose a minimal system of generators of $J$ so that (3.1) and (3.2) are satisfied. Let $Q\in\operatorname{Ass}_A A/JI^{\operatorname{ad}(I)}$. Then by Lemma (3.5) we have

$$0=\operatorname{depth}(A/JI^{\operatorname{ad}(I)})_Q \geqq \min\{\alpha-\operatorname{ad}(I),\ \operatorname{ht}_A Q-l\},$$

which means $\operatorname{ht}_A Q\leqq l$ as $\alpha>\operatorname{ad}(I)$. Suppose $Q\notin\operatorname{Min}_A A/I$. Then $\lambda(I_Q)<\dim A_Q=\operatorname{ht}_A Q$ by the condition (2). Hence $\operatorname{ad}(I_Q)=\lambda(I_Q)-\operatorname{ht}_{A_Q} I_Q<\operatorname{ht}_A Q-\operatorname{ht}_{A_Q} I_Q\leqq l-s=\operatorname{ad}(I)$. Thus we get $\operatorname{ad}(I_Q)<\operatorname{ad}(I)$, which holds even if $Q\in\operatorname{Min}_A A/I$. We would like to apply the inductive hypothesis to $I_Q$. So, we have to verify that $I_Q$ satisfies the assumptions of Theorem (1.1). In fact, of course, $\alpha>\operatorname{ad}(I_Q)$ and if $pA_Q\in\mathrm{V}(I_Q)$ ($p\in\operatorname{Spec} A$), then, for all $1\leqq n\leqq\operatorname{ad}(I_Q)$, we have $\operatorname{depth}(A_Q/I_Q^n)_{pA_Q}=\operatorname{depth}(A/I^n)_p\geqq\min\{\alpha-n,\ \operatorname{ht}_A p-s-n\}=\min\{\alpha-n,\ \operatorname{ht}_{A_Q} pA_Q-\operatorname{ht}_{A_Q} I_Q-n\}$ (here we used the assumption that $I$ is unmixed, which guarantee $s=\operatorname{ht}_{A_Q} I_Q$ in our situation). Moreover it is quite easy to see that $I_Q$ has the property $G_{\lambda(I_Q)}$ in the sence of Artin and Nagata and it satisfies the condition (2). Therefore by the inductive hypothesis, for any special reduction $J'$ of $I_Q$, we have $\mathrm{r}_{J'}(I_Q)\leqq \mathrm{ad}(I_Q)$, i.e., $I_Q^{\mathrm{ad}(I_Q)+1}=J'I_Q^{\mathrm{ad}(I_Q)}$. Recall that by Corollary (2.4) there exists a special reduction of $I_Q$ contained in $J_Q$. Hence we have $I_Q^{\mathrm{ad}(I_Q)+1}=J_QI_Q^{\mathrm{ad}(I_Q)}$, and so $I_Q^{\mathrm{ad}(I)+1}=J_QI_Q^{\mathrm{ad}(I)}$. This implies $I^{\mathrm{ad}(I)+1}=JI^{\mathrm{ad}(I)}$ since we took $Q\in \mathrm{Ass}_A A/JI^{\mathrm{ad}(I)}$ arbitrary. Thus we see $\mathrm{r}_J(I)\leqq \mathrm{ad}(I)$. Moreover we have

$$\text{(4.1)} \qquad \mathrm{depth}(A/I^n)_Q \geqq \min\{\alpha-\mathrm{ad}(I),\ \mathrm{ht}_A Q-l\}$$

for all $n\geqq 1$. In fact, if $n\leqq \mathrm{ad}(I)$, it is trivial by the assumption, and if $n>\mathrm{ad}(I)$, we get the inequality above by Lemma (3.5) since $I^n=J^{n-\mathrm{ad}(I)}I^{\mathrm{ad}(I)}$. Let $n\geqq 1$ and $Q\in \mathrm{Ass}_A A/I^n$. By (4.1) we have $\mathrm{ht}_A Q\leqq l$. Then again by the inductive hypothesis we see $(I_Q)^{(n)}=I_Q^n$, and so $(I^{(n)})_Q=I_Q^n$ since $(I_Q)^{(n)}=(I^{(n)})_Q$ by the definition of symbolic powers. Therefore $I^{(n)}=I^n$. We get the last assertion of this theorem from (4.1) substituting $Q=\mathfrak{m}$ and the proof is completed.

Proof of Theorem (1.2). Similarly as Theorem (1.1) the content of this theorem is the implication (2)$\Rightarrow$(1) and the last assertions. Assume the condition (2). We prove by induction on $\mathrm{ad}(I)$. If $\mathrm{ad}(I)=0$, then $I$ is a complete intersection, and so we easily get the required assertions. Suppose $\mathrm{ad}(I)>0$ and Theorem (1.2) is true for ideals with analytic deviation less than $\mathrm{ad}(I)$. By (2.2) there exists a special reduction $J$ of $I$. Let $Q\in \mathrm{Ass}_A A/JI^{\mathrm{ad}(I)-1}$. Then by Lemma (3.11) we see $\mathrm{ht}_A Q\leqq l$, whence $\lambda(I_Q)<\dim A_Q$ unless $Q\in \mathrm{Min}_A A/I$ by the condition (2). Thus $\mathrm{ad}(I_Q)<\mathrm{ad}(I)$. Hence by the inductive hypothesis we have $\mathrm{r}_{J'}(I_Q)\leqq \mathrm{ad}(I_Q)-1$ for any special reduction $J'$ of $I_Q$. This means $I_Q^{\mathrm{ad}(I)}=J_QI_Q^{\mathrm{ad}(I)-1}$. Therefore we get $I^{\mathrm{ad}(I)}=JI^{\mathrm{ad}(I)-1}$, i.e., $\mathrm{r}_J(I)\leqq \mathrm{ad}(I)-1$. Then by Lemma (3.11) we see

$$\text{(4.2)} \qquad \mathrm{depth}(A/I^n)_Q \geqq \min\{\alpha-\mathrm{ad}(I)+1,\ \mathrm{ht}_A Q-l\}$$

for all $n\geqq 1$ and $Q\in \mathrm{V}(I)$. Let $n\geqq 1$ and $Q\in \mathrm{Ass}_A A/I^n$. By (4.2) we have $\mathrm{ht}_A Q\leqq l$. Then again by the inductive hypothesis the equality $(I^{(n)})_Q=(I_Q)^{(n)}=I_Q^n$ holds. Therefore $I^{(n)}=I^n$. We get the last assertion of this theorem from (4.2) in the case where $Q=\mathfrak{m}$ and the proof is completed.

Example (4.3). We consider an example given by Vasconcelos [**V**, (3.3) Examples]. Let $B=k[\{X_{ij}\}_{1\leq i\leq 3,1\leq j\leq 4}]_{(\{X_{ij}\})}$ ($k$ is an infinite field) and let $K$ be the ideal in $B$ generated by the maximal minors of the generic matrix $X=(X_{ij})_{1\leq i\leq 3,1\leq j\leq 4}$. We denote by $f_i$ the determinant of the matrix derived from $X$ deleting $i$th column. By [**Hu3**, Proposition 1.1] $f_1, f_2, f_3, f_4$ is a $d$-sequence. Hence $\lambda(K)=4$ and so $\mathrm{ad}(K)=2$. As is well known, if $p\in \mathrm{V}(K)$ and $\mathrm{ht}_A p\leqq 6$, we have $\mu_p(K)=2$. Therefore by [**HH1**, Theorem 3.5] and [**HH2**, Theorem 3.1] we see that the associated graded ring $\mathrm{G}(K)=\bigoplus_{n\geq 0}K^n/K^{n+1}$ is a Cohen-Macaulay integral domain, and so it must be Gorenstein by [**Ho**, Proposition].

Let $A=B[T_1, T_2, T_3, T_4]$ ($T_i$'s are new indeterminate) and let $\varphi: A \to \mathrm{G}(K)$ be the homomorphism of $B$-algebras such that $\varphi(T_i)=f_i^*$ for $1 \leqq i \leqq 4$, where $f_i^*$ denotes the initial form of $f_i$ in $\mathrm{G}(K)$. Let $I$ be the kernel of $\varphi$. Then $\mathrm{ht}_A I=4$ and $A/I$ is Gorenstein. Because $f_1, f_2, f_3, f_4$ is a $d$-sequence, the Rees algebra $\mathrm{R}(K)=\bigoplus_{n\geq 1} K^n$ is isomorphic to the symmetric algebra $\mathrm{S}(K)$ by [**Hu5**, Theorem 3.1], and so we have $I=(f_1, f_2, f_3, f_4, g_1, g_2, g_3)A$, where $g_i=\sum_{j=1}^4 X_{ij}T_j$ for $1 \leqq i \leqq 3$. As is noted in [**V**, (3.3) Example], $I$ is generated by a $d$-sequence, which means $\lambda(I_M)=7$, where $M$ is the graded maximal ideal of $A$. So, $\mathrm{ad}(I_M)=3$. Let $Q \in \mathrm{V}(I)$ such that $\mathrm{ht}_A Q \leqq 6$. We put $p=Q \cap B$. Of course $\mathrm{ht}_A p \leqq 6$. Then, as is mentioned above, $\mu_p(K)=2$ and so we may assume $K_p=(f_3, f_4)A_p$. Thus we get the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{J} \arrow[r] & B_p[T_3, T_4] \arrow[r, "\psi"] \arrow[d] & \mathrm{G}(K_p) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I_p \arrow[r] & A_p \arrow[r, "\varphi"] \arrow[d] & \mathrm{G}(K_p) \arrow[r] & 0 \\
 & & (T_1, T_2)A_p \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $\psi(T_i)=f_i^*$ for $i=3, 4$ and $\mathcal{J}=\mathrm{Ker}\,\psi$. Because $f_3, f_4$ is a regular sequence on $B_p$, we have $\mathcal{J}=(f_3, f_4)B_p[T_3, T_4]$. Consequently $I_p$ is generated by 4 elements, and so $\mu_Q(I)=4$. Moreover $A/I^2$ is Cohen-Macaulay by [**V**, (3.3) Example]. Thus $I_M \subsetneqq A_M$ satisfies the standard assumptions of Theorem (1.2) in the case where $\alpha=13$. Hence we have $I^{(n)}=I^n$ for all $n \geqq 1$ if $\lambda(I_Q) \leqq 6$ for any $Q \in \mathrm{V}(I)$ such that $\mathrm{ht}_A Q=7$. Let $Q$ be such a prime ideal. Again we put $p=Q \cap B$. Because $\mathrm{ht}_A p \leqq 7$, we may assume $x_{11} \notin p$. Then, considering the entries of $X$ in $B_p$, we get a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X'_{22} & X'_{23} & X'_{24} \\ 0 & X'_{32} & X'_{33} & X'_{34} \end{pmatrix}$$

by elementary transformation, which means $K_p=(f_2, f_3, f_4)B_p$ and we get the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathfrak{T} \arrow[r] & B_p[T_2, T_3, T_4] \arrow[r, "\rho"] \arrow[d] & \mathrm{G}(K_p) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I_p \arrow[r] & A_p \arrow[r, "\varphi"] \arrow[d] & \mathrm{G}(K_p) \arrow[r] & 0 \\
 & & T_1A_p \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $\rho(T_i)=f_i^*$ for $i=3, 4$ and $\mathfrak{T}=\mathrm{Ker}\,\rho$. Notice that

$$\mathfrak{T} = (f_2, f_3, f_4, h_2, h_3)B_p[T_2, T_3, T_4],$$

where $h_i=\sum_{j=2}^{4} X'_{ij}T_j$ for $j=2, 3$. Therefore $I_p$ is generated by 6 elements, and so we get $\mu_Q(I)=6$ as is required.

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