J. Math. Soc. Japan
Vol. 47, No. 1, 1995

A theorem of Hardy-Littlewood for harmonic functions satisfying Hölder's condition

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(Received May 31, 1993)

1. Introduction.

Our aim in this paper is to give an extension of a result of Hardy-Littlewood [**2**, Theorems 40 and 41] for holomorphic functions on the unit disc.

Let $B(x, r)$ denote the open ball centered at x with radius r. We denote by $\boldsymbol{B}$ the unit ball $B(0, 1)$ of R^n, and by $d(x)$ the distance of x from the boundary $\partial\boldsymbol{B}$, that is, $d(x)=1-|x|$.

An easy modification of the proof of [**1**, Theorem 5.1] deduces the following results (see also [**3**, Theorem 15.8]).

THEOREM A. *Let u be a harmonic function on $\boldsymbol{B}$ and $0<\alpha\leqq 1$. Then u satisfies*

$$|\nabla u(x)| \leqq Md(x)^{\alpha-1} \qquad \text{for any } x\in\boldsymbol{B}$$

if and only if

$$|u(x)-u(y)| \leqq M|x-y|^{\alpha} \qquad \text{for any } x\in\boldsymbol{B} \text{ and } y\in\boldsymbol{B}, \tag{1}$$

where ∇ denotes the gradient.

If u satisfies (1), then we say that u satisfies Hölder's condition of exponent α in $\boldsymbol{B}$.

In this paper let M denote various constants, whose value may change from one occurrence to the next.

THEOREM B. *Let u be a harmonic function on $\boldsymbol{B}$. Then u satisfies*

$$|\nabla u(x)| \leqq Md(x)^{-1} \qquad \text{for any } x\in\boldsymbol{B}$$

if and only if $u\in BMO(\boldsymbol{B})$, that is,

$$\frac{1}{|B|}\int_B \left| u(y)-\frac{1}{|B|}\int_B u(z)dz \right| dy \leqq M$$

for any open ball $B=B(x, r)\subseteqq\boldsymbol{B}$.

For a nonnegative integer k, denote by $\nabla_k u$ the gradient iterated k times of u, that is,

$$|\nabla_k u(x)| = \Big(\sum_{|\mu|=k} \frac{k!}{\mu!} |D^\mu u(x)|^2\Big)^{1/2}.$$

Finally we have the following result, whose proof seems to be derived nowhere in a complete form.

THEOREM C. *Let u be a harmonic function on $\boldsymbol{B}$ and $0<\alpha\leqq 2$. Then u satisfies*

$$|\nabla_2 u(x)| \leqq Md(x)^{\alpha-2} \qquad \textit{for any } x\in\boldsymbol{B}$$

if and only if

$$|u(x+y)+u(x-y)-2u(x)| \leqq M|y|^\alpha \tag{2}$$

whenever $x\in\boldsymbol{B}$ and y with $x\pm y\in\boldsymbol{B}$.

If $0<\alpha<1$, then (2) is equivalent to (1). We give generalizations of Theorems A, B and C; we thus establish a complete proof of Theorem C. For this purpose, consider a positive nondecreasing function k on $(0, \infty)$ satisfying the doubling condition:

$$k(2t) \leqq Mk(t) \qquad \text{for any } t>0. \tag{3}$$

Define

$$h(r) = rk(1/r)$$

for $r>0$ and $h(0)=0$. Note that h also satisfies the doubling condition.

THEOREM 1. *Suppose $t^\beta k(1/t)$ is nondecreasing on $(0, \infty)$ for some $0<\beta<1$. Let u be a harmonic function on $\boldsymbol{B}$. Then u satisfies*

$$|\nabla u(x)| \leqq Mk(d(x)^{-1}) \qquad \textit{for any } x\in\boldsymbol{B} \tag{4}$$

if and only if

$$|u(x)-u(y)| \leqq Mh(|x-y|) \qquad \textit{for any } x \textit{ and } y \textit{ in } \boldsymbol{B}. \tag{5}$$

Clearly, Theorem 1 gives a generalization of Theorem A.

REMARK. Following Smith and Stegenga [4], we define the quasi-hyperbolic metric with respect to k by

$$K_{k,\boldsymbol{B}}(x, y) = \inf_\gamma \int_\gamma k(d(z)^{-1})ds,$$

where the infimum is taken over all rectifiable arcs γ in $\boldsymbol{B}$ joining x and y with the arc length s. If u satisfies (4), then

$$|u(x)-u(y)| \leqq MK_{k,\boldsymbol{B}}(x, y).$$

We shall prove later that $K_{k,\boldsymbol{B}}(x, y) \leqq Mh(|x-y|)$ for any $x \in \boldsymbol{B}$ and $y \in \boldsymbol{B}$.

THEOREM 2. *Suppose $h(t)=tk(1/t)$ is nondecreasing on $(0, \infty)$. Let u be a harmonic function on $\boldsymbol{B}$. Then u satisfies* (4) *if and only if*

$$\frac{1}{|B|}\int_B |u(y)-u(x)|\,dy \leqq Mh(r) \tag{6}$$

for any open ball $B=B(x, r) \subseteqq \boldsymbol{B}$.

If we take $k(t)=t$, then (6) implies that $u \in BMO(\boldsymbol{B})$, so that Theorem 2 gives an extension of Theorem B.

Letting k be as above, we define

$$h^*(r) = r^2 k(1/r)$$

for $r>0$ and $h^*(0)=0$ for the sake of convenience.

THEOREM 3. *Suppose $t^{\beta}k(1/t)$ is nondecreasing on $(0, \infty)$ for some $0<\beta<2$. Let u be a harmonic function on $\boldsymbol{B}$. Then u satisfies*

$$|\nabla_2 u(x)| \leqq Mk(d(x)^{-1}) \qquad \text{for any } x \in \boldsymbol{B} \tag{7}$$

if and only if

$$|u(x+y)+u(x-y)-2u(x)| \leqq Mh^*(|y|) \tag{8}$$

whenever $x \in \boldsymbol{B}$ any y with $x \pm y \in \boldsymbol{B}$.

If we take $k(t)=t^{2-\alpha}$ with $0<\alpha \leqq 2$, then Theorem 3 implies Theorem C.

THEOREM 4. *Suppose $h^*(t)=t^2k(1/t)$ is nondecreasing on $(0, \infty)$. Let u be a harmonic function on $\boldsymbol{B}$. Then u satisfies* (7) *if and only if*

$$\frac{1}{|B|}\int_B |u(x+y)+u(x-y)-2u(x)|\,dy \leqq Mh^*(r) \tag{9}$$

for any open ball $B=B(0, r)$ with $0<r<d(x)$.

2. Proof of Theorem 1.

For a proof of Theorem 1, we prepare some lemmas.

First we start with a mean-value inequality for harmonic functions.

LEMMA 1 (cf. Stein [**5**, Appendix C.3]). *If u is a harmonic function on $B(x, r)$, then*

$$|\nabla_k u(x)| \leqq M_k r^{-n-k} \int_{B(x,r)} |u(y)| dy$$

for any nonnegative integer k, where M_k is a positive constant independent of x and r.

Define

$$\tilde{h}(r) = \int_0^r k(1/s) ds$$

for $r>0$ and $\tilde{h}(0)=0$. Note that $\tilde{h}$ satisfies the doubling condition on $(0, \infty)$ and

$$\tilde{h}(r) \geqq k(1/r) \int_0^r ds = rk(1/r). \tag{10}$$

LEMMA 2. *If u is a continuously differentiable function on $\boldsymbol{B}$ satisfying* (4), *then*

$$|u(x+y)-u(x)| \leqq M\tilde{h}(|y|) \tag{11}$$

whenever $x \in \boldsymbol{B}$ and y is of the form rx, $0<r<d(x)/|x|$.

PROOF. Since $d(x+ty)=d(x)-t|y| \geqq (1-t)|y|$, we have by (4),

$$\begin{aligned}
|u(x+y)-u(x)| &\leqq \int_0^1 \left| \frac{d}{dt} u(x+ty) \right| dt \\
&\leqq \int_0^1 |(\nabla u)(x+ty)| \, |y| dt \\
&\leqq M|y| \int_0^1 k(d(x+ty)^{-1}) dt \\
&\leqq M|y| \int_0^1 k(1/(1-t)|y|) dt \\
&\leqq M \int_0^{|y|} k(1/s) ds \\
&= M\tilde{h}(|y|).
\end{aligned}$$

Hence Lemma 2 is proved.

COROLLARY. *Suppose $\tilde{h}(1)<\infty$. If u is a continuously differentiable function on $\boldsymbol{B}$ satisfying* (4), *then u is bounded on $\boldsymbol{B}$.*

In fact, we have by (11)

$$\begin{aligned}
|u(x)| &\leqq |u(x/2)| + M\tilde{h}(|x|/2) \\
&\leqq \sup_{B(0,1/2)} |u| + M\tilde{h}(1/2)
\end{aligned}$$

for $x \in \boldsymbol{B}$, so that u is bounded on $\boldsymbol{B}$.

LEMMA 3. *If u is a continuously differentiable function on $\boldsymbol{B}$ satisfying* (4), *then* (11) *holds for $x \in \boldsymbol{B}$ any y with $x+y \in \boldsymbol{B}$.*

PROOF. Case 1: $|y| \geqq 1/4$. This case follows readily from Corollary to Lemma 2.

Case 2: $|x| \leqq 1/2$ and $|y| \leqq 1/4$. Applying the mean value theorem, we find t_0, $0<t_0<1$, such that

$$|u(x+y)-u(x)| \leqq |y||\nabla u(x+t_0 y)|.$$

Since $|\nabla u|$ is bounded on $B(0, 3/4)$, we obtain (11).

Case 3: $|x| \geqq 1/2$ and $|y| \leqq 1/4$. This is the most difficult case. To conquer the present case, take $x^*=(|x|-|y|)x/|x|$ and $y^*=(|x|-|y|)y/|x|$. We write

$$\begin{aligned} |u(x+y)-u(x)| &\leqq |u(x+y)-u(x^*+y^*)|+|u(x^*+y^*)-u(x^*)|+|u(x^*)-u(x)| \\ &= A+B+C\,. \end{aligned}$$

Since $|x^*-x|=|y|$, we apply Lemma 2 to prove

$$C \leqq M\tilde{h}(|y|).$$

Noting that $|(x+y)-(x^*+y^*)| \leqq |x+y||y|/|x| \leqq 2|y|$, we have by Lemma 2 again

$$A \leqq M\tilde{h}(2|y|) \leqq M\tilde{h}(|y|).$$

Applying the mean value theorem, we find t_0 such that $0<t_0<1$ and

$$B \leqq |y^*||\nabla u(x^*+t_0 y^*)| \leqq M|y^*|k(d(x^*+t_0 y^*)^{-1}).$$

Note that for $0<t<1$,

$$\begin{aligned} d(x^*+ty^*) &\geqq |(x+ty)-(x^*+ty^*)| \\ &= |x+ty||y|/|x| \\ &\geqq (|x|-|y|)|y|/|x| \\ &\geqq |y|/2\,. \end{aligned} \tag{12}$$

Since $|y^*| \leqq |y|$, we finally establish

$$B \leqq M|y|k(2/|y|) \leqq M\tilde{h}(|y|),$$

with the aid of (10). Thus the proof is complete.

LEMMA 4. *If u is a harmonic function on $\boldsymbol{B}$ satisfying* (5), *then*

$$|\nabla u(x)| \leqq Md(x)^{-1}h(d(x)) \qquad \textit{for any } x \in \boldsymbol{B}\,. \tag{13}$$

PROOF. For fixed $x \in \boldsymbol{B}$, consider the function v:

$$v(z) \equiv u(z)-u(x),$$

which is harmonic in $\boldsymbol{B}$. Applying Lemma 1, we have for $r=d(x)$,

$$\begin{aligned}|\nabla u(x)| = |\nabla v(x)| &\leqq Mr^{-n-1}\int_{B(x,r)}|v(z)|\,dz \\ &\leqq Mr^{-n-1}h(r)\int_{B(x,r)}dz \\ &\leqq Mr^{-1}h(r).\end{aligned}$$

Therefore (13) follows.

By Lemmas 3 and 4, Theorem 1 is proved if we note that

$$\begin{aligned}\tilde{h}(r) &= \int_0^r k(1/s)\,ds \\ &= \int_0^r \{s^{\beta}k(1/s)\}\,s^{-\beta}\,ds \\ &\leqq r^{\beta}k(1/r)\int_0^r s^{-\beta}\,ds \\ &= (1-\beta)^{-1}rk(1/r) \\ &= (1-\beta)^{-1}h(r).\end{aligned}$$

3. Proof of Theorem 2.

PROOF OF THEOREM 2. First we show the only if part. Since $d(x+ty)\geqq d(x)-t|y|$, we have

$$\begin{aligned}|u(x+y)-u(x)| &\leqq M|y|\int_0^1 k\Big(\frac{1}{d(x)-t|y|}\Big)dt \\ &\leqq M\int_0^{|y|} k\Big(\frac{1}{d(x)-t}\Big)dt.\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(z)-u(x)|\,dz &\leqq \frac{M}{|B(0,r)|}\int_{B(0,r)}\Big\{\int_0^{|y|} k\Big(\frac{1}{d(x)-t}\Big)dt\Big\}dy \\ &\leqq \frac{M}{r^n}\int_0^r\int_{\Sigma_{n-1}}\Big\{\int_0^{\rho} k\Big(\frac{1}{d(x)-t}\Big)dt\Big\}\rho^{n-1}\,d\sigma\, d\rho \\ &\leqq \frac{M}{r^n}\int_0^r\Big\{\int_t^r \rho^{n-1}d\rho\Big\}k\Big(\frac{1}{r-t}\Big)dt \\ &\leqq \frac{M}{r}\int_0^r (r-t)k\Big(\frac{1}{r-t}\Big)dt\end{aligned}$$

$$\leqq \frac{M}{r} r k(1/r) \int_0^r dt$$
$$= Mh(r)\,,$$

so that (6) holds.

Next we show the if part. For fixed $x \in \boldsymbol{B}$, consider the function v:

$$v(z) \equiv u(z) - u(x)\,.$$

Since v is harmonic in $B = B(x, r)$ with $r = d(x)$, by Lemma 1 and (6), we have

$$\begin{aligned} |\nabla u(x)| &= |\nabla v(x)| \leqq M r^{-n-1} \int_B |v(z)|\, dz \\ &\leqq M r^{-1} h(r) \\ &= Mk(d(x)^{-1})\,. \end{aligned}$$

Hence (4) follows, and the proof is complete.

4. Proof of Theorem 3.

For a proof of Theorem 3, we need some lemmas.

Define

$$\bar{h}(r) = \int_0^r s k(1/s) ds$$

for $r > 0$ and $\bar{h}(0) = 0$. Note that $\bar{h}$ satisfies the doubling condition on $(0, \infty)$ and

$$\bar{h}(r) \geqq k(1/r) \int_0^r s ds = 2^{-1} r^2 k(1/r)\,.$$

LEMMA 5. *Let u be a continuously twice differentiable function on $\boldsymbol{B}$ satisfying* (7). *If $x \in \boldsymbol{B}$ and $y = rx$ with $0 < r < d(x)/|x|$, then*

$$|u(x+y) + u(x-y) - 2u(x)| \leqq M\bar{h}(|y|)\,. \tag{14}$$

PROOF. By the mean value theorem we have

$$|u(x+y) + u(x-y) - 2u(x)| \leqq |y|^2 \int_0^1 (1-t) |\nabla_2 u(x+ty) + \nabla_2 u(x-ty)|\, dt\,. \tag{15}$$

Since $d(x \pm ty) = d(x) \mp t|y| \geqq |y| - t|y|$, we have by (7),

$$\begin{aligned} |u(x+y) + u(x-y) - 2u(x)| &\leqq M \int_0^{|y|} (|y| - s) k((|y| - s)^{-1}) ds \\ &= M\bar{h}(|y|)\,. \end{aligned}$$

COROLLARY. *Suppose $\bar{h}(1) < \infty$. If u is a continuously twice differentiable function on $\boldsymbol{B}$ satisfying* (7), *then u is bounded on $\boldsymbol{B}$.*

In fact, we obtain from (14),

$$|u(x)| \leqq 2|u(3x/4)|+|u(x/2)|+M\bar{h}(|x|/4)$$
$$\leqq 3 \sup_{B(0,3/4)} |u|+M\bar{h}(1/4),$$

which shows that u is bounded on $\boldsymbol{B}$.

LEMMA 6. *If u is a continuously twice differentiable function on $\boldsymbol{B}$ satisfying* (7), *then* (14) *holds for $x \in \boldsymbol{B}$ and y with $x+y \in \boldsymbol{B}$.*

PROOF. Case 1: $|y| \geqq 1/4$. This case follows readily from the boundedness of u, which was shown in Corollary to Lemma 5.

Case 2: $|x| \leqq 1/2$ and $|y| \leqq 1/4$. This case follows from (15) and the fact that $|\nabla_2 u|$ is bounded on $B(0, 3/4)$.

Case 3: $|x| \geqq 1/2$ and $|y| \leqq 1/4$. Take $x^*=(|x|-|y|)x/|x|$, $x^{**}=(|x|-2|y|)\cdot x/|x|$, $y^*=(|x|-|y|)y/|x|$ and $y^{**}=(|x|-2|y|)y/|x|$. Write

$$\begin{aligned}|u(x+y)+u(x-y)-2u(x)| &\leqq 2|u(x)+u(x^{**})-2u(x^*)| \\ &\quad +|u(x+y)+u(x^{**}+y^{**})-2u(x^*+y^*)| \\ &\quad +|u(x-y)+u(x^{**}-y^{**})-2u(x^*-y^*)| \\ &\quad +2|u(x^*+y^*)+u(x^*-y^*)-2u(x^*)| \\ &\quad +|u(x^{**}+y^{**})+u(x^{**}-y^{**})-2u(x^{**})| \\ &= 2A+B+C+2D+E\,.\end{aligned}$$

Let $w=x-x^*=x^*-x^{**}$. Then, obviously, $|w|=|y|$. Lemma 5 yields

$$A = |u(x^*+w)+u(x^*-w)-2u(x^*)| \leqq M\bar{h}(|y|).$$

Next let $w_1=(x+y)-(x^*+y^*)=(x^*+y^*)-(x^{**}+y^{**})$ and $w_2=(x-y)-(x^*-y^*)=(x^*-y^*)-(x^{**}-y^{**})$. Then, as above, we have

$$B \leqq M\bar{h}(|w_1|)$$

and

$$C \leqq M\bar{h}(|w_2|).$$

Since $|w_1| \leqq 2|y|$ and $|w_2| \leqq 2|y|$,

$$B \leqq M\bar{h}(|y|), \qquad C \leqq M\bar{h}(|y|).$$

Noting that $d(x^* \pm ty^*) \geqq |x \pm ty|-|x^* \pm ty^*| \geqq (|x|-|y|)|y|/|x| \geqq |y|/2$ for $0<t<1$, we have by (15) and (7),

$$\begin{aligned}D &\leqq M|y^*|^2\int_0^1 (1-t)\{k(d(x^*+ty^*)^{-1})+k(d(x^*-ty^*)^{-1})\}\,dt \\ &\leqq M|y|^2 k(2/|y|) \\ &\leqq M\bar{h}(|y|).\end{aligned}$$

Similarly

$$E \leqq M\bar{h}(|y|),$$

and hence the proof is complete.

LEMMA 7. *If u is a harmonic function on $\boldsymbol{B}$ satisfying* (8) *for $x \in \boldsymbol{B}$ and y with $x \pm y \in \boldsymbol{B}$, then*

$$|\nabla_2 u(x)| \leqq M d(x)^{-2} h^*(d(x)) \qquad \text{whenever } x \in \boldsymbol{B}.$$

PROOF. For fixed $x \in \boldsymbol{B}$, we put

$$v(y) \equiv u(x+y)+u(x-y)-2u(x)$$

and $r=d(x)$. Since the function v is harmonic in $B(0, r)$, applying Lemma 1, we have

$$\begin{aligned} |\nabla_2 u(x)| = 2^{-1}|\nabla_2 v(0)| &\leqq M r^{-n-2}\int_{B(0,r)} |v(y)|\, dy \\ &\leqq M r^{-n-2} h^*(r)\int_{B(0,r)} dy \\ &\leqq M r^{-2} h^*(r). \end{aligned}$$

Therefore, the required conclusion follows.

Now Theorem 3 is proved by Lemmas 6 and 7 if one notes that

$$\begin{aligned} \bar{h}(r) &= \int_0^r \left\{t^{\beta} k\left(\frac{1}{t}\right)\right\} t^{1-\beta}\, dt \\ &\leqq r^{\beta} k(1/r)\int_0^r t^{1-\beta}\, dt \\ &= (2-\beta)^{-1} r^2 k(1/r) \\ &= (2-\beta)^{-1} h^*(r). \end{aligned}$$

5. Proof of Theorem 4.

PROOF OF THEOREM 4. First we show the only if part. Since $d(x \pm ty) \geqq d(x)-t|y|$, we have by (15),

$$\begin{aligned} |u(x+y)+u(x-y)-2u(x)| &\leqq M|y|^2\int_0^1 (1-t) k\left(\frac{1}{d(x)-t|y|}\right) dt \\ &\leqq M\int_0^{|y|} (|y|-s) k\left(\frac{1}{d(x)-s}\right) ds\,. \end{aligned}$$

Hence we have for $B=B(0, r)$ with $0<r<d(x)$,

$$\begin{aligned}\frac{1}{|B|}\int_B |u(x+y)+u(x-y)-2u(x)|\,dy &\leqq \frac{M}{|B|}\int_B\left\{\int_0^{|y|}(|y|-s)k\left(\frac{1}{d(x)-s}\right)ds\right\}dy \\ &\leqq \frac{M}{r^n}\int_0^r\left\{\int_0^\rho(\rho-s)k\left(\frac{1}{r-s}\right)ds\right\}\rho^{n-1}d\rho \\ &= \frac{M}{r^n}\int_0^r\left\{\int_s^r(\rho-s)\rho^{n-1}d\rho\right\}k\left(\frac{1}{r-s}\right)ds \\ &\leqq \frac{M}{r}\int_0^r(r-s)^2k\left(\frac{1}{r-s}\right)ds \\ &\leqq \frac{M}{r}r^2k(1/r)\int_0^r ds \\ &= Mh^*(r).\end{aligned}$$

Hence, in view of Lemma 7, the proof is complete.

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