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Littlewood-Paley-Stein inequality for a symmetric diffusion

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1. Introduction.

The probabilistic approach to the Littlewood-Paley-Stein inequality was begun by Meyer [**18**]. Recently Bakry and Emery introduced the concept of Γ_2. They used it to discuss the hypercontractivity. Further Bakry [**4**] established the Littlewood-Paley-Stein inequality for a diffusion process under the condition that Γ_2 is non-negative and subsequently, in [**6**] he obtained it for a diffusion process on a complete Riemannian manifold under conditions for Ricci curvature and the Hessian of the density function, which assures equivalently that Γ_2 is bounded from below. The main purpose of this paper is to extend his result to the case that Γ_2 is bounded from below under the general setting. Moreover we discuss the sections of Hermitian bundles. We begin with introducing Γ_2.

Let M be a complete separable metric space and m be a Borel measure on M. Suppose we are given an m-symmetric diffusion process $(X_t, P_x)_{x\in M}$ on M and let e^{tL} be the corresponding symmetric semigroup on $L^2(M; m)$ with the generator L. We assume that the diffusion $(X_t, P_x)_{x\in M}$ is *conservative* and that there exists a dense subspace $\mathcal{A}$ in $L^2(M; m)$ such that

(i) $\mathcal{A}$ is an algebra,

(ii) $\mathcal{A}\subseteqq\bigcap_{1\leqq p<\infty}L^p(M; m)\cap\mathrm{Dom}(L)$,

(iii) $\mathcal{A}$ is stable under the operation of L.

Then we can define a sesquilinear map Γ: $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ by

$$\Gamma(f, g)=\frac{1}{2}\{L(f\bar{g})-(Lf)\bar{g}-f(L\bar{g})\}$$

where $^-$ denotes the complex conjugate. Then Γ_2 is defined by

$$\Gamma_2(f, g)=\frac{1}{2}\{L\Gamma(f, g)-\Gamma(Lf, g)-\Gamma(f, Lg)\}.$$

We simply denote $\Gamma_2(f, f)$ and $\Gamma(f, f)$ by $\Gamma_2(f)$ and $\Gamma(f)$, respectively.

More generally, we consider a trivial vector bundle $E=M\times\boldsymbol{C}^n$ and denote the set of all sections whose components belong to $\mathcal{A}$ by $\mathcal{A}(\boldsymbol{C}^n)$. Then L can be easily extended to the space of sections of E. Similarly, Γ can be extended

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to $\mathcal{A}(\boldsymbol{C}^n)$ in natural way. We consider an operator of the form $L-U$ where $U(x)$ is $n\times n$ Hermitian matrices which we call a potential. We assume that U is locally bounded and further there exists $\beta\geqq 0$ such that

$$U(x)\geqq -\beta I_n \qquad \text{for} \quad x\in M$$

where I_n is the identity matrix. For this operator $L-U$, we define $\vec{\Gamma}_2$ by

$$\vec{\Gamma}_2(u, v)=\frac{1}{2}\{L\Gamma(u, v)-\Gamma((L-U)u, v)-\Gamma(u, (L-U)v)\} \qquad \text{for} \quad u, v\in\mathcal{A}(\boldsymbol{C}^n).$$

The semigroup on $L^2(M; m)\otimes\boldsymbol{C}^n$ generated by $L-U$ is not a contraction semigroup in general and we consider the following generator $\vec{L}$;

$$\vec{L}=L-U(x)-\alpha I_n$$

where α is a positive constant. Taking α to be large enough, the semigroup generated by $\vec{L}$ is contraction and we can define Littlewood-Paley G-functions associated with $\vec{L}$ (for precise definition, see section 2).

We suppose that $\vec{\Gamma}_2$ is *bounded from below*, i.e., there exist constants a, $b\geqq 0$ such that

$$\vec{\Gamma}_2(u)\geqq -a\Gamma(u)-b|u|^2.$$

We assume that the above inequality holds for not only $u\in\mathcal{A}(\boldsymbol{C}^n)$ but also $\vec{P}_t u$, $u\in\mathcal{A}(\boldsymbol{C}^n)$ where $\{\vec{P}_t\}$ is the semigroup generated by $\vec{L}$. Here we implicitly suppose that $\vec{\Gamma}_2$ is well-defined for $\vec{P}_t u$, $u\in\mathcal{A}(\boldsymbol{C}^n)$.

Our main results below will be to establish the Littlewood-Paley-Stein inequality for such G-functions. Moreover we also discuss the case that a fiber space is a Hilbert space. We discuss two examples. First one is considered on an abstract Wiener space. In this case, we have to consider a vector bundle whose fiber is a Hilbert space. Second one is a Laplacian acting on a vector bundle over a complete Riemannian manifold. As applications we will discuss, in another papers, the problem related to the Riesz transformation and Sobolev spaces on an abstract Wiener space ([**23**]) and on a complete Riemannian manifold ([**30**]).

The organization of this paper is as follows. In section 2, we give estimates of $\Gamma(\vec{P}_t)$ and $\Gamma(\vec{Q}_t)$, $\vec{P}_t$, $\vec{Q}_t$ being a semigroup and a Cauchy semigroup generated by $\vec{L}$, respectively. In these estimates, the assumption that $\vec{\Gamma}_2$ is bounded from below is crucial. In section 3, we introduce the Littlewood-Paley G-functions and H-functions and discuss the relation among them. In section 4 we give estimates of G-functions and H-functions and thereby obtain a proof of Littlewood-Paley-Stein inequalities. Here we follow a probabilistic proof of Meyer [**18**] and Bakry [**6**], in which inequalities for submartingales play an important role. We give examples in section 5.

2. Symmetric diffusion.

Let M be a complete separable metric space and m be a σ-finite Borel measure on M. By $L^2(M; m)$, we denote the complex L^2-space. Let $(X_t, P_x)_{x\in M}$ be an m-symmetric diffusion process on M. We assume that the diffusion is *conservative*. Then the corresponding contraction semigroup $\{P_t\}$ on $L^2(M; m)$ is given by;

$$P_t f(x) = E_x[f(X_t)], \qquad f \in L^2(M; m) \tag{2.1}$$

where E_x stands for the expectation with respect to the probability measure P_x. Let L be the generator of $\{P_t\}$. We assume that there exists a dense subspace $\mathcal{A}$ in $L^2(M; m)$ satisfying (i), (ii) and (iii) in section 1.

As in section 1 we define sesquilinear maps Γ and Γ_2 as follows. For $f, g\in\mathcal{A}$,

$$\Gamma(f, g) = \frac{1}{2}\{L(f\bar{g})-(Lf)\bar{g}-f(L\bar{g})\}, \tag{2.2}$$

$$\Gamma_2(f, g) = \frac{1}{2}\{L\Gamma(f, g)-\Gamma(Lf, g)-\Gamma(f, Lg)\}. \tag{2.3}$$

We simply denote $\Gamma(f, f)$ by $\Gamma(f)$ and $\Gamma_2(f, f)$ by $\Gamma_2(f)$, respectively and we remark that $\Gamma(f)\geqq 0$ (see e.g., Bakry-Emery [**3**]). We set $E=M\times\boldsymbol{C}^n$, i.e., E is a trivial vector bundle over M with a fiber $\boldsymbol{C}^n$. We denote the set of all sections of E by $\Gamma(E)$. In general, we denote the space of L^p-sections by $L^p(\Gamma(E); m)$. We also denote the set of all sections whose components belong to $\mathcal{A}$ by $\mathcal{A}(\boldsymbol{C}^n)$. Then L can be extended to $\mathcal{A}(\boldsymbol{C}^n)$ componentwise. Also Γ can be extended to $\mathcal{A}(\boldsymbol{C}^n)$ naturally as follows;

$$\begin{aligned}\Gamma(u, v) &= \frac{1}{2}\{L(u\cdot v)-Lu\cdot v-u\cdot Lv\} \\ &= \sum_{i=1}^{n}\Gamma(u^i, v^i), \qquad \text{for} \quad u=(u^1, \cdots, u^n), v=(v^1, \cdots, v^n)\in\mathcal{A}(\boldsymbol{C}^n),\end{aligned}$$

where $\cdot$ stands for the inner product in $\boldsymbol{C}^n$: $z\cdot z'=\sum_{i=1}^n z^i\overline{z'^i}$. We consider an operator of the form $L-U(x)$ where $U(x)$ is an $n\times n$ Hermitian matrix function which is locally bounded and we assume that $U(x)$ is *bounded from below*, i.e., there exists $\beta\geqq 0$ so that

$$U(x) \geqq -\beta I_n \qquad \text{for} \quad x \in M. \tag{A.1}$$

Further $\vec{\Gamma}_2$ associated with $L-U$ is defined by

$$\vec{\Gamma}_2(u, v) = \frac{1}{2}\{L\Gamma(u, v) - \Gamma((L-U)u, v) - \Gamma(u, (L-U)v)\}.$$

We consider an operator $\vec{L}$ of the following form;

$$\vec{L} = L - U - \alpha I_n \tag{2.4}$$

where α is a positive constant. We denote by $\{\vec{P}_t\}$ the semigroup generated by $\vec{L}$.

We assume that $\vec{\Gamma}_2$ is *bounded from below*, i.e., there exist constants $a, b \geqq 0$ such that

(A.2) $\vec{\Gamma}_2(u) \geqq -a\Gamma(u) - b|u|^2$ for $u \in \mathcal{A}(\boldsymbol{C}^n)$ and $u = \vec{P}_t v, v \in \mathcal{A}(\boldsymbol{C}^n)$.

Here we have to assume that $\vec{\Gamma}_2$ is well-defined for not only $u \in \mathcal{A}(\boldsymbol{C}^n)$ but also $\vec{P}_t u$, $u \in \mathcal{A}(\boldsymbol{C}^n)$ because $\vec{P}_t u$, $u \in \mathcal{A}(\boldsymbol{C}^n)$ is not in $\mathcal{A}(\boldsymbol{C}^n)$ generally. A sufficient condition is that for $u \in \mathcal{A}(\boldsymbol{C}^n)$, $|\vec{P}_t u|^2$ belongs to $\mathrm{Dom}(L)$ and further $\Gamma(\vec{P}_t u)$ belongs to $\mathrm{Dom}(L)$.

We now give a probabilistic representation of the semigroup $\{\vec{P}_t\}$ and thereby we show that $\{\vec{P}_t\}$ is a contraction semigroup if we take α large enough. First we define a multiplicative functional $M_t = M_t(X)$ of X as the solution to the following differential equation;

$$\begin{cases} dM_t = -M_t U(X_t)dt \\ M_0 = I_n. \end{cases} \tag{2.5}$$

Define a semigroup $\{\vec{P}_t\}$ on $L^2(\Gamma(E); m)$ by

$$\vec{P}_t u(x) = E_x[e^{-\alpha t} M_t(X) u(X_t)], \qquad \text{for } u \in L^2(\Gamma(E); m). \tag{2.6}$$

The following proposition is a generalization of Feynman-Kac formula.

PROPOSITION 2.1. *$\{\vec{P}_t\}$ is a strongly continuous symmetric semigroup on $L^2(\Gamma(E); m)$ with the generator $\vec{L}$. Moreover it holds that*

$$|\vec{P}_t u(x)| \leqq e^{-(\alpha-\beta)t} P_t |u|(x) \tag{2.7}$$

PROOF. Let $M_t(X)^*$ be the adjoint matrix of $M_t(X)$. Then $M_t(X)^*$ satisfies the following differential equation;

$$\begin{cases} dM_t(X)^* = -U(X_t)M_t(X)^* dt \\ M_0(X)^* = I_n. \end{cases} \tag{2.8}$$

Hence for $\xi \in \boldsymbol{C}^n$,

$$\begin{aligned}\frac{d}{dt}|M_t(X)^*\xi|^2 &= -(U(X_t)M_t(X)^*\xi, M_t(X)^*\xi) - (M_t(X)^*\xi, U(X_t)M_t(X)^*\xi) \\ &\leqq 2\beta |M_t(X)^*\xi|^2.\end{aligned}$$

By the Gronwall inequality, we have

$$|M_t(X)^*\xi|^2 \leqq e^{2\beta t}|\xi|^2.$$

Thus we have

$$\|M_t(X)\|_{\mathcal{L}(C^n)} = \|M_t(X)^*\|_{\mathcal{L}(C^n)} \leqq e^{\beta t}$$

where $\|\cdot\|_{\mathcal{L}(C^n)}$ stands for the operator norm.

Now it is easy to see that $\{\vec{P}_t\}$ is a strongly continuous semigroup satisfying (2.7). Moreover, by using the Itô formula, we can show that $\vec{L}$ is the generator of $\{\vec{P}_t\}$.

Next we show that $\{\vec{P}_t\}$ is symmetric. To show this, let E_m denote the expectation for the process (X_t) with initial distribution m. Take any $T>0$ and fix it. Let us consider the reversed process $Y_t=X_{T-t}$, $0\leqq t\leqq T$. Note that $\{M_t(Y)^{-1}\}$ satisfies

$$\begin{cases} dM_t(Y)^{-1} = U(Y_t)M_t(Y)^{-1}dt \\ M_0(Y)^{-1} = I_n. \end{cases}$$

Hence

$$\begin{cases} \dfrac{d}{dt}M_{T-t}(Y)^{-1}M_T(Y) = -U(X_t)M_{T-t}(Y)^{-1}M_T(Y), \\ M_{T-0}(Y)^{-1}M_T(Y) = I_n. \end{cases}$$

By the uniqueness of the solution to (2.8), we have for $0\leqq t\leqq T$,

$$M_{T-t}(Y)^{-1}M_T(Y) = M_t(X)^*.$$

In particular, it holds that $M_T(Y)=M_T(X)^*$. By the symmetry of (X_t), $(X_t)_{0\leq t\leq T}$ and $(Y_t)_{0\leq t\leq T}$ have the same law under P_m and hence we have

$$\begin{aligned} E_m[(M_T(X)u(X_T), v(X_0))] &= E_m[(u(X_T), M_T(X)^*v(X_0))] \\ &= E_m[(u(Y_0), M_T(Y)v(Y_T))] \\ &= E_m[(u(X_0), M_T(X)v(X_T))] \end{aligned}$$

which implies that $\{\vec{P}_t\}$ is symmetric. $\square$

By the above proposition, $\{\vec{P}_t\}$ is a contraction semigroup if $\alpha\geqq\beta$. Therefore, throughout this paper, we always assume that $\alpha\geqq\beta$. We construct the Cauchy semigroup (or Poisson semigroup) by the following subordination method. For any $t\geqq 0$, let μ_t be the probability measure on $[0, \infty)$ such that

$$\int_0^\infty e^{-\lambda s}\mu_t(ds) = e^{-\sqrt{\lambda}\,t} \qquad \text{for} \quad \lambda > 0.$$

As is well-known, μ_t is of the following form;

(2.9) $$\mu_t(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds .$$

Then the Cauchy semigroup is defined by

(2.10) $$\vec{Q}_t = \int_0^\infty \vec{P}_s \mu_t(ds).$$

The generator of $\{\vec{Q}_t\}$ in $L^2(\Gamma(E); m)$ is $-\sqrt{-\vec{L}}$. We call it the Cauchy generator and denote by $\vec{C}$.

Next we consider $\Gamma(\vec{P}_t u)$ and $\Gamma(\vec{Q}_t u)$ and have the following proposition.

PROPOSITION 2.2. *Assume that* (A.1) *and* (A.2) *hold. Take* $\alpha, \gamma > 0$ *so that* $\alpha \geqq a + \gamma$ *and* $\alpha > \beta + \gamma$. *Then we have*

(2.11) $$\Gamma(\vec{P}_t u) \leqq P_t \Gamma(u) + K P_t^{(2\gamma)} |u|^2$$

where $K = b/(\alpha - \beta - \gamma)$ *and* $P_t^{(2\gamma)} = e^{-2\gamma t} P_t$.

PROOF. Take any $T > 0$ and fix it. Define $g(t)$ for $0 \leqq t \leqq T$ by

$$g(t) = P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u) + K P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 .$$

We first show that $g'(t) \geqq 0$. In fact, by using (A.1), (A.2) and (2.7) we have

$$\begin{aligned}
g'(t) &= P_t^{(2\gamma)} L \Gamma(\vec{P}_{T-t} u) - 2\gamma P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u) \\
&\quad - P_t^{(2\gamma)} \Gamma(\vec{L} \vec{P}_{T-t} u, \vec{P}_{T-t} u) - P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u, \vec{L} \vec{P}_{T-t} u) \\
&\quad + K P_t^{(2\gamma)} L |\vec{P}_{T-t} u|^2 - 2\gamma K P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 \\
&\quad - K P_t^{(2\gamma)} (\vec{L} \vec{P}_{T-t} u, \vec{P}_{T-t} u) - K P_t^{(2\gamma)} (\vec{P}_{T-t} u, \vec{L} \vec{P}_{T-t} u) \\
&= P_t^{(2\gamma)} L \Gamma(\vec{P}_{T-t} u) - 2\gamma P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u) - P_t^{(2\gamma)} \Gamma((L-U) \vec{P}_{T-t} u, \vec{P}_{T-t} u) \\
&\quad - P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u, (L-U) \vec{P}_{T-t} u) + 2\alpha P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u) \\
&\quad + K P_t^{(2\gamma)} L |\vec{P}_{T-t} u|^2 - 2\gamma K P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 - K P_t^{(2\gamma)} (L \vec{P}_{T-t} u, \vec{P}_{T-t} u) \\
&\quad - K P_t^{(2\gamma)} (P_{T-t} u, L \vec{P}_{T-t} u) + 2K\alpha P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 \\
&\quad + K P_t^{(2\gamma)} (U \vec{P}_{T-t} u, \vec{P}_{T-t} u) + K P_t^{(2\gamma)} (P_{T-t} u, U \vec{P}_{T-t} u) \\
&\geqq 2 P_t^{(2\gamma)} \vec{\Gamma}_2(\vec{P}_{T-t} u) + 2(\alpha - \gamma) P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u) \\
&\quad + 2K P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u) + 2(\alpha - \gamma) K P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 \\
&\quad - 2\beta K P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 \\
&\geqq 2(\alpha - \gamma - a + K) P_t^{(2\gamma)} \Gamma(\vec{P}_{T-t} u) - 2b P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 \\
&\quad + 2K(\alpha - \beta - \gamma) P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 \\
&\geqq \{2K(\alpha - \beta - \gamma) - 2b\} P_t^{(2\gamma)} |\vec{P}_{T-t} u|^2 \\
&= 0 .
\end{aligned}$$

Thus we have $g(0)\leqq g(T)$ and hence, we have

$$\Gamma(\vec{P}_T u)+K|\vec{P}_T u|^2 \leqq P_T^{(2\gamma)}\Gamma(u)+KP_T^{(2\gamma)}|u|^2 .$$

Now (2.11) easily follows. □

By the above proposition, we have the following key inequality. We denote the subordination of $\{P_t^{(2\gamma)}\}$ by $\{Q_t^{(2\gamma)}\}$ i.e.,

$$Q_t^{(2\gamma)} = \int_0^\infty P_s^{(2\gamma)} \mu_t(ds).$$

PROPOSITION 2.3. *Under the same assumptions as in Proposition 2.2, we have*

$$\Gamma(\vec{Q}_t u) \leqq Q_t\Gamma(u)+KQ_t^{(2\gamma)}|u|^2 . \tag{2.12}$$

PROOF. We note the Schwarz inequality for Γ, i.e., $|\Gamma(u, v)|\leqq\sqrt{\Gamma(u)}\times\sqrt{\Gamma(v)}$. Then we have,

$$\begin{aligned}
\Gamma(\vec{Q}_t u) &= \Gamma\Big(\int_0^\infty \vec{P}_s u \mu_t(ds)\Big) \\
&= \Gamma\Big(\int_0^\infty \vec{P}_s u \mu_t(ds), \int_0^\infty \vec{P}_\tau u \mu_t(d\tau)\Big) \\
&= \int_0^\infty \mu_t(ds)\int_0^\infty \mu_t(d\tau)\Gamma(\vec{P}_s u, \vec{P}_\tau u) \\
&\leqq \int_0^\infty \mu_t(ds)\int_0^\infty \mu_t(d\tau)\sqrt{\Gamma(\vec{P}_s u)}\sqrt{\Gamma(\vec{P}_\tau u)} \\
&\leqq \Big\{\int_0^\infty \sqrt{\Gamma(\vec{P}_s u)}\,\mu_t(ds)\Big\}^2 \leqq \int_0^\infty \Gamma(\vec{P}_s u)\mu_t(ds) \\
&\leqq \int_0^\infty \{P_s\Gamma(u)+KP_s^{(2\gamma)}|u|^2\}\mu_t(ds) = Q_t\Gamma(u)+KQ_t^{(2\gamma)}|u|^2
\end{aligned}$$

which is the desired result. □

So far, we take $\boldsymbol{C}^n$ as a fiber space. More generally, we can take a Hilbert space $\mathcal{H}$ in place of $\boldsymbol{C}^n$. In this case, we sometimes need to consider an unbounded potential U. It is difficult to handle the general case however and we assume that U is constant: $U(x)=A$, for all $x\in M$. Furthermore, we assume that A is a self-adjoint operator and bounded from below, i.e., there exists a constant β so that

$$A \geqq -\beta I_{\mathcal{H}} \tag{A.1$'$}$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. This condition is similar to (A.1). So we consider an operator of the form $\vec{L}=L-A-\alpha I_{\mathcal{H}}$ on $L^2(\Gamma(E); m)$, where in this case, $E=M\times\mathcal{H}$. We set $\mathcal{A}(\mathcal{H})$ to be the set of all $\mathcal{H}$-valued functions u of the form

$$u = \sum_{i=1}^N f_i h_i, \qquad \text{for} \quad f_i \in \mathcal{A},\ h_i \in C^\infty(A)$$

where $C^\infty(A)=\bigcap_{n=1}^\infty \mathrm{Dom}(A^n)$.

The semigroup $\{\vec{P}_t\}$ generated by $\vec{L}$ is represented by

$$\vec{P}_t u(x) = E_x[e^{-\alpha t} T_t u(X_t)] \tag{2.13}$$

where $T_t=e^{-tA}$. Note that $|\vec{P}_t u(x)|_{\mathcal{H}} \leqq e^{-(\alpha-\beta)t} P_t |u|_{\mathcal{H}}(x)$ where $|\cdot|_{\mathcal{H}}$ is the Hilbert norm in $\mathcal{H}$. In fact,

$$\begin{aligned}|\vec{P}_t u(x)|_{\mathcal{H}} &\leqq E_x[|e^{-\alpha t} T_t u(X_t)|_{\mathcal{H}}] \leqq E_x[e^{-\alpha t} e^{\beta t} |u(X_t)|_{\mathcal{H}}] \\ &\leqq e^{-(\alpha-\beta)t} P_t |u|_{\mathcal{H}}(x).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\vec{\Gamma}_2(u, v) &= \frac{1}{2}\{L\Gamma(u, v) - \Gamma((L-U)u, v) - \Gamma(u, (L-U)v)\} \\ &= \frac{1}{2}\{L\Gamma(u, v) - \Gamma(Lu, v) - \Gamma(u, Lv) + \Gamma(Au, v) + \Gamma(Au, v)\}.\end{aligned}$$

By using (A.1)′, we easily have

$$\Gamma(Au, u) \geqq -\beta\Gamma(u, u).$$

Assuming that Γ_2 associated with L is bounded from below, i.e., there exist constants $a, b \geqq 0$ such that

(A.2)′ $\quad \Gamma_2(f) \geqq -a\Gamma(f) - b|f|^2 \quad$ for $\ f \in \mathcal{A}$ and $f = P_t g,\ g \in \mathcal{A}$,

we have

$$\vec{\Gamma}_2(u) \geqq -(a+\beta)\Gamma(u) - b|u|^2 \qquad \text{for} \quad u \in \mathcal{A}(\mathcal{H}) \text{ and } u = P_t v,\ v \in \mathcal{A}(\mathcal{H}).$$

Hence by a similar proof to that of Proposition 2.2 and Proposition 2.3, we have the same result in infinite dimensional case;

PROPOSITION 2.4. *Assume that* (A.1)′ *and* (A.2)′ *hold. Take* $\alpha, \gamma > 0$ *so that* $\alpha > a+\beta+\gamma$. *Then for* $u \in \mathcal{A}(\mathcal{H})$,

$$\Gamma(\vec{P}_t u) \leqq P_t\Gamma(u) + KP_t^{(2\gamma)}|u|^2 \tag{2.14}$$

and

$$\Gamma(\vec{Q}_t u) \leqq Q_t\Gamma(u) + KQ_t^{(2\gamma)}|u|^2. \tag{2.15}$$

where $K = b/(\alpha-\beta-\gamma)$.

For simplicity, we consider, in the sequel, only the finite dimensional case, the infinite dimensional case being similarly discussed by virtue of Proposition 2.4 under the assumptions (A.1)′ and (A.2)′.

3. Littlewood-Paley G-functions.

Let us introduce the Littlewood-Paley G-functions. For any $u \in \mathcal{A}(\boldsymbol{C}^n)$ (or $\mathcal{A}(\mathcal{H})$), define

$$g^{\rightarrow}(x, t) = \left|\frac{\partial}{\partial t}\vec{Q}_t u(x)\right|^2$$

$$g^{\uparrow}(x, t) = \Gamma(\vec{Q}_t u)(x)$$

$$g(x, t) = g^{\rightarrow}(x, t) + g^{\uparrow}(x, t).$$

Then, Littlewood-Paley's G-functions are defined by

$$G^{\rightarrow}u(x) = \left\{\int_0^\infty t g^{\rightarrow}(x, t)dt\right\}^{1/2} \tag{3.1}$$

$$G^{\uparrow}u(x) = \left\{\int_0^\infty t g^{\uparrow}(x, t)dt\right\}^{1/2} \tag{3.2}$$

$$Gu(x) = \left\{\int_0^\infty t g(x, t)dt\right\}^{1/2}. \tag{3.3}$$

Moreover, we define the H-functions by

$$H^{\rightarrow}u(x) = \left\{\int_0^\infty t Q_t g^{\rightarrow}(x, t)dt\right\}^{1/2} \tag{3.4}$$

$$H^{\uparrow}u(x) = \left\{\int_0^\infty t Q_t g^{\uparrow}(x, t)dt\right\}^{1/2} \tag{3.5}$$

$$Hu(x) = \left\{\int_0^\infty t Q_t g(x, t)dt\right\}^{1/2}. \tag{3.6}$$

The following proposition is easily obtained by the spectral decomposition:

PROPOSITION 3.1. *For $\alpha \geqq \beta$, it holds that*

$$\|G^{\rightarrow}u\|_2 = \frac{1}{2}\|u - E_0 u\|_2, \tag{3.7}$$

where E_0 is the projection to $\mathrm{Ker}(\vec{L})$ *and further*

$$\|G^{\uparrow}u\|_2 \leqq \frac{1}{2}\|u\|_2. \tag{3.8}$$

PROOF. (3.7) is well known. We show (3.8). By the spectral decomposition for $\vec{L}$, we have

$$\vec{L} = -\int_{\alpha-\beta}^\infty \lambda dE_\lambda.$$

Hence

$$\begin{aligned}\|G^{\uparrow}u\|_2^2 &= \int_0^\infty t\,dt\int_M \Gamma(\vec{Q}_t u)(x)m(dx) = -\int_0^\infty t\,dt\int_M (L\vec{Q}_t u, \vec{Q}_t u)(x)m(dx) \\ &= -\int_0^\infty t\,dt\int_M (\vec{L}\vec{Q}_t u, \vec{Q}_t u)(x)m(dx) - \int_0^\infty t\,dt\int_M ((U+\alpha)\vec{Q}_t u, \vec{Q}_t u)(x)m(dx)\end{aligned}$$

$$\leqq \int_0^\infty t\,dt \int_{\alpha-\beta}^\infty \lambda e^{-2t\sqrt{\lambda}} d|E_\lambda u|^2 - (\alpha-\beta)\int_0^\infty t\,dt \int_{\alpha-\beta}^\infty e^{-2t\sqrt{\lambda}} d|E_\lambda u|^2$$

$$\leqq \int_{\alpha-\beta}^\infty \frac{\lambda}{4\lambda} d|E_\lambda u|^2 = \frac{1}{4}\|u\|_2^2.$$

Here in the fifth line we used $\int_0^\infty te^{-2\xi t}dt = 1/4\xi^2$. □

Next we establish the relation between G-functions and H-functions. For notational simplicity, we write $\|u\|_p \lesssim \|v\|_p$ if there exists a positive constant c_p depending only on p so that $\|u\|_p \leqq c_p\|v\|_p$. We use this convention without mentioning.

PROPOSITION 3.2. *For $\alpha \geqq \beta$ it holds that*

$$G^{\rightarrow} u \leqq 2H^{\rightarrow} u. \tag{3.9}$$

Further assuming the same assumptions as in Proposition 2.2, it holds that for $p \geqq 2$,

$$\|G^{\uparrow} u\|_p \lesssim \|H^{\uparrow} u\|_p + \sqrt{K/\gamma}\|u\|_p. \tag{3.10}$$

PROOF. By Proposition 2.1 we have,

$$|\vec{Q}_t u(x)|^2 \leqq \int_0^\infty |\vec{P}_s u(x)|^2 \mu_t(ds) \leqq \int_0^\infty e^{-2(\alpha-\beta)s} P_s|u|^2(x)\mu_t(ds)$$

$$\leqq \int_0^\infty P_s|u|^2(x)\mu_t(ds) = Q_t|u|^2(x).$$

Therefore

$$g^{\rightarrow}(x, 2t) = \left|\frac{\partial}{\partial s}\vec{Q}_s u(x)\right|_{s=2t}^2 = |\vec{C}\vec{Q}_{2t}u(x)|^2$$

$$= |\vec{Q}_t\vec{C}\vec{Q}_t u(x)|^2 \leqq Q_t|\vec{C}\vec{Q}_t u|^2(x) = Q_t g^{\rightarrow}(x, t).$$

Thus we have,

$$G^{\rightarrow}u(x) = \left\{\int_0^\infty tg^{\rightarrow}(x, t)dt\right\}^{1/2} = \left\{4\int_0^\infty tg^{\rightarrow}(x, 2t)dt\right\}^{1/2}$$

$$\leqq \left\{4\int_0^\infty tQ_t g^{\rightarrow}(x, t)dt\right\}^{1/2} = 2H^{\rightarrow}u(x).$$

Next we show (3.10). By using Proposition 2.3 and the Hölder inequality, we have

$$|G^{\uparrow}u(x)|^p = \left\{\int_0^\infty t\Gamma(\vec{Q}_t u)(x)dt\right\}^{p/2}$$

$$\leqq \left\{\int_0^\infty t(Q_t\Gamma(u)(x) + KQ_t^{(2\gamma)}|u|^2(x))dt\right\}^{p/2}$$

$$= \left\{H^{\uparrow}u(x)^2 + K\int_0^\infty tQ_t^{(2\gamma)}|u|^2(x)dt\right\}^{p/2}$$

$$\lesssim H^{\uparrow}u(x)^p+K^{p/2}\left\{\int_0^\infty tQ_t^{(2\gamma)}|u|^2(x)dt\right\}^{p/2}.$$

Let q be a conjugate exponent of $p/2$: $(1/q)+(2/p)=1$. Then we have

$$\left\|\left\{\int_0^\infty tQ_t^{(2\gamma)}|u|^2(x)dt\right\}^{p/2}\right\|_1$$

$$= \left\|\left\{\int_0^\infty te^{-t\sqrt{\gamma/p}}e^{t\sqrt{\gamma/p}}dt\int_0^\infty e^{-2\gamma s}P_s|u|^2(x)\mu_t(ds)\right\}^{p/2}\right\|_1$$

$$\leqq \left\|\left\{\int_0^\infty t^q e^{-qt\sqrt{\gamma/p}}dt\right\}^{p/2q}\int_0^\infty e^{pt\sqrt{\gamma/p}/2}dt\left\{\int_0^\infty e^{-2\gamma s}P_s|u|^2(x)\mu_t(ds)\right\}^{p/2}\right\|_1$$

$$\leqq \left\{\int_0^\infty\left(\frac{u}{q\sqrt{\gamma/p}}\right)^q e^{-u}\frac{du}{q\sqrt{\gamma/p}}\right\}^{p/2q}\int_0^\infty e^{t\sqrt{\gamma p}/2}dt\int_0^\infty e^{-\gamma ps}\|P_s|u|^p(x)\|_1\mu_t(ds)$$

$$\lesssim \sqrt{\gamma}^{-(q+1)p/2q}\|u\|_p^p\int_0^\infty e^{t\sqrt{\gamma p}/2}e^{-t\sqrt{\gamma p}}dt$$

$$= \|u\|_p^p\sqrt{\gamma}^{-(q+1)p/2q}\frac{2}{\sqrt{\gamma p}}\lesssim \|u\|_p^p\sqrt{\gamma}^{-p}.$$

Thus we have

$$\|G^{\uparrow}u\|_p\lesssim\|H^{\uparrow}u\|_p+\sqrt{K/\gamma}\|u\|_p$$

which completes the proof. □

LEMMA 3.3. *For* $u\in\mathcal{A}(\boldsymbol{C}^n)$, *set* $f(x, a)=|\vec{Q}_a u(x)|$ *and for* $\varepsilon>0$, $f_\varepsilon(x, a)=\sqrt{f(x, a)^2+\varepsilon^2}$. *Then for* $p\geqq 2$ *it holds that*

$$\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p\geqq 0 \tag{3.11}$$

and for $1\leqq p\leqq 2$, *it holds that*

$$\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p\geqq 2p(p-1)f^{p-2}g \tag{3.12}$$

where $g=g(x, a)$ *is defined by*

$$g(x, a)=\left|\frac{\partial}{\partial a}\vec{Q}_a u(x)\right|^2+\Gamma(\vec{Q}_a u)(x).$$

PROOF. We first show

$$\left(\frac{\partial^2}{\partial a^2}+L\right)f(x, a)^2\geqq 2g(x, a). \tag{3.13}$$

To show this, we note that $\left(\frac{\partial^2}{\partial a^2}+\vec{L}\right)\vec{Q}_a u(x)=0$. Hence

$$
\begin{aligned}
&\Big(\frac{\partial^2}{\partial a^2}+L\Big)f(x,\,a)^2\\
&=\Big(\frac{\partial^2}{\partial a^2}+L\Big)|\vec{Q}_a u|^2\\
&=2\,\mathrm{Re}\Big(\frac{\partial^2}{\partial a^2}\vec{Q}_a u,\,\vec{Q}_a u\Big)+2\Big(\frac{\partial}{\partial a}\vec{Q}_a u,\,\frac{\partial}{\partial a}\vec{Q}_a u\Big)+2\,\mathrm{Re}(L\vec{Q}_a u,\,\vec{Q}_a u)+2\Gamma(\vec{Q}_a u)\\
&=-2\,\mathrm{Re}((L-U-\alpha)\vec{Q}_a u,\,\vec{Q}_a u)+2\Big|\frac{\partial}{\partial a}\vec{Q}_a u\Big|^2+2\,\mathrm{Re}(L\vec{Q}_a u,\,\vec{Q}_a u)+2\Gamma(\vec{Q}_a u)\\
&\geqq 2(\alpha-\beta)|\vec{Q}_a u|^2+2g(x,\,a)\\
&\geqq 2g(x,\,a).
\end{aligned}
$$

Here Re denotes the real part and we used (A.1) in the fourth line.

Secondly we show (3.11). To show this we recall the following fundamental relation of L and Γ: for $F(\xi^1, \xi^2, \cdots, \xi^n)\in C^\infty(\boldsymbol{R}^n)$ and $f^1, f^2, \cdots, f^n\in\mathcal{A}$

$$
LF(f^1,\,f^2,\,\cdots,\,f^n)=\sum_{i=1}^n\frac{\partial F}{\partial\xi^i}Lf^i+\sum_{i,j=1}^n\frac{\partial^2 F}{\partial\xi^i\partial\xi^j}\Gamma(f^i,\,f^j)
$$

(see [3] Lemme 1). Hence we have,

$$
\begin{aligned}
\Big(\frac{\partial^2}{\partial a^2}+L\Big)f_\varepsilon^p&=\Big(\frac{\partial^2}{\partial a^2}+L\Big)(f_\varepsilon^2)^{p/2}\\
&=\frac{p}{2}(f_\varepsilon^2)^{p/2-1}\Big(\frac{\partial^2}{\partial a^2}+L\Big)f_\varepsilon^2+\frac{p}{2}\Big(\frac{p}{2}-1\Big)(f_\varepsilon^2)^{p/2-2}\Big\{\Big(\frac{\partial}{\partial a}f_\varepsilon^2\Big)^2+\Gamma(f_\varepsilon^2)\Big\}\\
&=\frac{p}{2}f_\varepsilon^{p-2}\Big(\frac{\partial^2}{\partial a^2}+L\Big)f_\varepsilon^2+\frac{p}{4}(p-2)f_\varepsilon^{p-4}\Big\{\Big(\frac{\partial}{\partial a}f_\varepsilon^2\Big)^2+\Gamma(f_\varepsilon^2)\Big\}.
\end{aligned}
$$

Hence, by using (3.13) for $p\geqq 2$,

$$
\Big(\frac{\partial^2}{\partial a^2}+L\Big)f_\varepsilon^p\geqq pf_\varepsilon^{p-2}g(x,\,a)\geqq 0
$$

which proves (3.11).

Lastly we show (3.12) for $1<p\leqq 2$. Let us recall the derivation property of Γ (see [3]);

$$
\Gamma(fg,\,h)=f\Gamma(g,\,h)+g\Gamma(f,\,h).
$$

Then, writing $\vec{Q}_a u=v=(v^1, v^2, \cdots, v^n)$, we have

$$
\begin{aligned}
\Gamma(f_\varepsilon^2)&=\Gamma(f^2)=\Gamma(f^2,\,f^2)\\
&=\Gamma\Big(\sum_{i=1}^n v^i\overline{v^i},\,\sum_{j=1}^n v^j\overline{v^j}\Big)\\
&=\sum_{i,j=1}^n\{v^i\Gamma(\overline{v^i},\,v^j\overline{v^j})+\overline{v^i}\Gamma(v^i,\,v^j\overline{v^j})\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{n} \{v^i\overline{v^j}\Gamma(\overline{v^i}, \overline{v^j})+v^i\overline{v^j}\Gamma(\overline{v^i}, v^j)+\overline{v^i v^j}\Gamma(v^i, \overline{v^j})+\overline{v^i}v^j\Gamma(v^i, v^j)\} \\
&\leqq \sqrt{\sum_{i,j=1}^{n} |v^i v^j|^2}\Big\{\sqrt{\sum_{i,j=1}^{n} |\Gamma(\overline{v^i}, \overline{v^j})|^2}+\sqrt{\sum_{i,j=1}^{n} |\Gamma(\overline{v^i}, v^j)|^2} \\
&\qquad\qquad +\sqrt{\sum_{i,j=1}^{n} |\Gamma(v^i, \overline{v^j})|^2}+\sqrt{\sum_{i,j=1}^{n} |\Gamma(v^i, v^j)|^2}\Big\} \\
&\leqq 4|v|^2\sqrt{\sum_{i,j=1}^{n} \Gamma(v^i)\Gamma(v^j)} \\
&\leqq 4|v|^2\sum_{i=1}^{n} \Gamma(v^i) \\
&\leqq 4|\vec{Q}_a u|^2\Gamma(\vec{Q}_a u).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big(\frac{\partial^2}{\partial a^2}+L\Big)f_\varepsilon^p \\
&\geqq pf_\varepsilon^{p-2}g(x, a)+\frac{p}{4}(p-2)f_\varepsilon^{p-4}\Big\{4\,\mathrm{Re}\Big(\frac{\partial}{\partial a}\vec{Q}_a u, \vec{Q}_a u\Big)^2+4|\vec{Q}_a u|^2\Gamma(\vec{Q}_a u)\Big\} \\
&\geqq pf_\varepsilon^{p-2}g(x, a)+\frac{p}{4}(p-2)f_\varepsilon^{p-4}\Big\{4\Big|\frac{\partial}{\partial a}\vec{Q}_a u\Big|^2|\vec{Q}_a u|^2+4|\vec{Q}_a u|^2\Gamma(\vec{Q}_a u)\Big\} \\
&\geqq pf_\varepsilon^{p-2}g(x, a)+p(p-2)f_\varepsilon^{p-2}\Big\{\Big|\frac{\partial}{\partial a}\vec{Q}_a u\Big|^2+\Gamma(\vec{Q}_a u)\Big\} \\
&\geqq p(p-1)f^{p-2}g(x, a)
\end{aligned}
$$

which completes the proof. □

4. The proof of Littlewood-Paley-Stein inequalities by martingale approach.

In this section, we give estimates of G and H by a probabilistic method. The original idea is due to P. A. Meyer [**18**] but we mainly follow Bakry [**6**]. So many parts are merely repetition of Bakry [**4, 6**] or Meyer [**18, 19**] with slight modification, but we give proofs for the completeness. Let (X_t, P_x) be the diffusion process on M as before. We need an additional 1-dimensional Brownian motion $(B_t)_{t\geq 0}$ and we regard M as a vertical space. So, from now on, we write $P_x^{\uparrow}$ in place of P_x. Let $(B_t, P_a^{\to})$ be a 1-dimensional Brownian motion starting at $a\in\boldsymbol{R}$ with the generator d^2/da^2. Note that the time scale of this Brownian motion is different from the standard one up to constant, but we use this for notational simplicity. Let τ be the hitting time of (B_t) to 0, i.e.,

$$\tau=\inf\{t\,;\ B_t=0\}.$$

We consider the following stopped diffusion $(Y_t, P_{(x,a)})$ on the state space $M\times \boldsymbol{R}_+$ where $\boldsymbol{R}_+=[0, \infty)$;

$$Y_t := (X_{t\wedge\tau}, B_{t\wedge\tau}), \qquad P_{(x,a)} := P_x^{\uparrow}\otimes P_a^{\rightarrow}. \tag{4.1}$$

So the generator of (Y_t) is $(\partial^2/\partial a^2)+L$. We denote the integration with respect to $P_{(x,a)}$ and $\int_M P_{(x,a)}m(dx)$ by $E_{(x,a)}$ and $\boldsymbol{E}_a$, respectively.

The following relation is fundamental.

LEMMA 4.1. *Let $\eta: M\times\boldsymbol{R}_+\to[0, \infty)$ be measurable. Then*

$$\boldsymbol{E}_a\left[\int_0^\tau \eta(Y_t)dt\right] = \int_M\int_0^\infty \eta(x, t)(t\wedge a)dt \tag{4.2}$$

and

$$\boldsymbol{E}_a\left[\int_0^\tau \eta(Y_t)dt\,|\,X_\tau=x\right] = \int_0^\infty Q_t\eta(\cdot, t)(x)(t\wedge a)dt. \tag{4.3}$$

PROOF. See e.g., Meyer [**18**]. □

Set $N_t=\vec{Q}_{B_{t\wedge\tau}}u(X_{t\wedge\tau})$ for $u\in\mathcal{A}(\boldsymbol{C}^n)$. Then, by noting $((\partial^2/\partial a^2)+L)\vec{Q}_a u(x)=0$, (N_t) is a $\boldsymbol{C}^n$-valued martingale. Hence $(|N_t|)$ is a non-negative submartingale and by the Doob inequality, it holds that for $p>1$

$$\begin{aligned} E_{(x,a)}[\sup_{t\geq 0}|N_t|^p] &\leqq (p/(p-1))^p E_{(x,a)}[|N_\tau|^p] \\ &= (p/(p-1))^p E_{(x,a)}[|u(X_\tau)|^p]. \end{aligned} \tag{4.4}$$

We need another inequality for submartingales. Let (Z_t) be a continuous submartingale with the following Doob-Meyer decomposition;

$$Z_t = M_t + A_t$$

where (M_t) is a continuous martingale and (A_t) is a continuous increasing process with $A_0=0$. Then, for $p>0$, it holds that

$$E[A_\infty^p] \leqq (2p)^p E[\sup_{t\geq 0}|Z_t|^p]. \tag{4.5}$$

For the proof, see Lenglart-Lépingle-Pratelli [**15**].

Now we have the following proposition.

PROPOSITION 4.2. *For $p\geqq 2$, it holds that*

$$\|Hu\|_p \lesssim \|u\|_p \qquad \textit{for} \quad u\in\mathcal{A}(\boldsymbol{C}^n). \tag{4.6}$$

PROOF. For $u\in\mathcal{A}(\boldsymbol{C}^n)$, set $f(x, a)=|\vec{Q}_a u(x)|$ as in Lemma 3.3. Define $(Z_t)_{t\geq 0}$ by

$$Z_t = f(Y_t)^2.$$

Then (Z_t) is a submartingale under $P_{(x,a)}$. In fact, set

$$M_t = f(Y_t)^2 - \int_0^{\tau\wedge t}\left(\frac{\partial^2}{\partial a^2}+L\right)f^2(Y_s)ds$$

and

$$A_t = \int_0^{\tau\wedge t}\left(\frac{\partial^2}{\partial a^2}+L\right)f^2(Y_s)ds\,.$$

Then (M_t) is a martingale and (A_t) is an increasing process because of (3.11). Thus $Z_t = M_t + A_t$ is a submartingale. Hence, by (4.5) and (4.4), we have

$$\begin{aligned}E_{(x,a)}\left[\left\{\int_0^{\tau\wedge t}\left(\frac{\partial^2}{\partial a^2}+L\right)f^2(Y_s)ds\right\}^{p/2}\right] &\lesssim E_{(x,a)}[\sup_{t\geqq 0}|Z_t|^{p/2}] \lesssim E_{(x,a)}[|Z_\infty|^{p/2}] \\ &= E_{(x,a)}[f(Y_\tau)^p] = E_{(x,a)}[|\vec{Q}_0 u(X_\tau)|^p] = E_{(x,a)}[|u(X_\tau)|^p].\end{aligned} \tag{4.7}$$

On the other hand, using (3.13) and (4.3) of Lemma 4.1, we have

$$\begin{aligned}Hu(x) &= \left\|\left\{\int_0^\infty tQ_t g(x,t)dt\right\}^{p/2}\right\|_1 \\ &\leqq \left\|\left\{\int_0^\infty tQ_t\left(\frac{\partial^2}{\partial t^2}+L\right)f^2(x,t)dt\right\}^{p/2}\right\|_1 \\ &\leqq \varliminf_{a\to\infty}\int_M m(dx)\left\{\int_0^\infty Q_t\left(\frac{\partial^2}{\partial t^2}+L\right)f^2(x,t)(t\wedge a)dt\right\}^{p/2} \\ &= \varliminf_{a\to\infty}\int_M m(dx)\boldsymbol{E}_a\left[\int_0^\tau\left(\frac{\partial^2}{\partial t^2}+L\right)f^2(Y_s)ds\,|\,X(\tau)=x\right]^{p/2} \\ &\leqq \varliminf_{a\to\infty}\boldsymbol{E}_a\left[\left\{\int_0^\tau\left(\frac{\partial^2}{\partial t^2}+L\right)f^2(Y_s)ds\right\}^{p/2}\right].\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we have

$$Hu(x) \lesssim \varliminf_{a\to\infty}\int_M m(dx)E_{(x,a)}[|u(X_\tau)|^p] = \varliminf_{a\to\infty}\int_M |u(x)|^p m(dx) = \|u\|_p^p$$

which completes the proof. □

PROPOSITION 4.3. *For $1<p\leqq 2$, it holds that*

$$\|Gu\|_p \lesssim \|u\|_p \qquad for \quad u\in\mathcal{A}(\boldsymbol{C}^n). \tag{4.9}$$

PROOF. Let f and f_ε be as in Lemma 3.3. Then, by Lemma 3.3, we have

$$g(x,a) \leqq \frac{1}{p(p-1)}\varliminf_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p f^{2-p}.$$

On the other hand,

$$f(x,a) = |\vec{Q}_a u(x)| \leqq \int_0^\infty P_s|u|(x)\mu_a(ds) \leqq |u|^*(x)$$

where

$$|u|^*(x)=\sup_{t\geq 0}P_t|u|(x).$$

Hence we have,

$$\begin{aligned}\|Gu\|_p^p &= \left\|\left\{\int_0^\infty a g(x,a)da\right\}^{p/2}\right\|_1\\ &\lesssim \left\||u|^{*p(2-p)/2}\left\{\int_0^\infty a\lim_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p da\right\}^{p/2}\right\|_1\\ &\lesssim \||u|^{*p}\|_1^{(2-p)/2}\left\|\int_0^\infty a\lim_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p da\right\|_1^{p/2}\end{aligned}$$

by the Hölder inequality for $2/(2-p)$ and $2/p$. The following maximal inequality is well-known: $\|u^*\|_p\lesssim\|u\|_p$ (see e.g., [21]). Hence it is easy to see that $\||u|^{*p}\|_1^{(2-p)/2}=\||u|^*\|_p^{p(2-p)/2}\lesssim\|u\|_p^{p(2-p)/2}$. Moreover, by (4.2) of Lemma 4.1, we have

$$\begin{aligned}&\left\|\int_0^\infty a\lim_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p da\right\|_1\\ &=\lim_{a\to\infty}\int_M m(dx)\int_0^\infty\lim_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)f_\varepsilon^p(x,t)(t\wedge a)dt\\ &=\lim_{a\to\infty}\boldsymbol{E}_a\left[\int_0^\tau\lim_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)f_\varepsilon^p(Y_s)ds\right].\end{aligned}$$

Now we set $Z_t=f_\varepsilon(Y_t)^p$. Then (Z_t) is a submartingale such that

$$Z_t=M_t+A_t$$

where (M_t) is a martingale defined by

$$M_t=f_\varepsilon(Y_t)^p-\int_0^{\tau\wedge t}\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p(Y_s)ds$$

and (A_t) is an increasing process (recall (3.11)) defined by

$$A_t=\int_0^{\tau\wedge t}\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p(Y_s)ds.$$

Thus by (4.5) and (4.4), we have

$$\begin{aligned}E_{(x,a)}\left[\int_0^\tau\left(\frac{\partial^2}{\partial a^2}+L\right)f_\varepsilon^p(Y_s)ds\right] &\lesssim E_{(x,a)}[\sup_{t\geq 0}f_\varepsilon(Y_t)^p]\\ &\leqq E_{(x,a)}[\sup_{t\geq 0}f(Y_t)^p+\varepsilon^p]\\ &\lesssim E_{(x,a)}[f(Y_\tau)^p]+\varepsilon^p\\ &= E_{(x,a)}[|u(X_\tau)|^p]+\varepsilon^p.\end{aligned}$$

By the Fatou lemma, we have

$$E_{(x,a)}\Big[\int_0^\tau \lim_{\varepsilon\to 0}\Big(\frac{\partial^2}{\partial a^2}+L\Big)f_\varepsilon^p(Y_s)ds\Big] \lesssim E_{(x,a)}[|u(X_\tau)|^p]$$

and hence

$$\boldsymbol{E}_a\Big[\int_0^\tau \lim_{\varepsilon\to 0}\Big(\frac{\partial^2}{\partial a^2}+L\Big)f_\varepsilon^p(Y_s)ds\Big] \lesssim \|u\|_p^p.$$

Therefore we have,

$$\|Gu\|_p^p \lesssim \|u\|_p^{p(2-p)/2}(\|u\|_p^p)^{p/2} = \|u\|_p^p$$

as desired. □

Now the following main theorem is easily obtained.

THEOREM 4.4. *If* $\alpha \geqq \beta$, *then for* $1<p<\infty$, *it holds that*

$$\|u-E_0u\|_p \lesssim \|G^{\rightarrow}u\|_p \lesssim \|u-E_0u\|_p \qquad \textit{for} \quad u\in\mathcal{A}(\boldsymbol{C}^n). \tag{4.10}$$

where E_0 *is the projection to* $\operatorname{Ker}(\vec{L})$. *Moreover, we suppose* (A.1), (A.2) *and* $\alpha \geqq a+\gamma$, $\alpha>\beta+\gamma$. *Then it holds that*

$$\|G^{\uparrow}u\|_p \lesssim (1+\sqrt{K/\gamma})\|u\|_p \qquad \textit{for} \quad u\in\mathcal{A}(\boldsymbol{C}^n) \tag{4.11}$$

where $K=b/(\alpha-\beta-\gamma)$.

PROOF. For $1<p\leqq 2$, we have, by Proposition 4.3,

$$\|G^{\rightarrow}u\|_p \lesssim \|u-E_0u\|_p, \qquad \|G^{\uparrow}u\|_p \lesssim \|u\|_p.$$

Here we used $G^{\rightarrow}u=G^{\rightarrow}(u-E_0u)$. Similarly, for $p\geqq 2$ by Proposition 3.2 and Proposition 4.2,

$$\|G^{\rightarrow}u\|_p \leqq 2\|H^{\rightarrow}u\|_p \lesssim \|u-E_0u\|_p.$$

By using $\|G^{\rightarrow}u\|_p \lesssim \|u\|_p$, we can show $\|u-E_0u\|_q \lesssim \|G^{\rightarrow}u\|_q$ by the duality where q is the conjugate exponent of p: $(1/p)+(1/q)=1$. In fact, by using (3.7) and Proposition 3.1 and the polarization,

$$\begin{aligned}&(u-E_0u,\ v-E_0v)_{L^2(\Gamma(E);\,m)}\\ &\quad = 4\int_M m(dx)\int_0^\infty t\Big(\frac{\partial}{\partial t}\vec{Q}_t u(x),\ \frac{\partial}{\partial t}\vec{Q}_t v(x)\Big)dt, \qquad u,v\in\mathcal{A}(\boldsymbol{C}^n).\end{aligned}$$

Hence

$$\begin{aligned}&\Big|\int_M (u(x),\ v(x)-E_0v(x))m(dx)\Big|\\ &\quad = \Big|\int_M (u(x)-E_0u,\ v(x)-E_0v(x))m(dx)\Big|\end{aligned}$$

$$\leqq 4\int_M m(dx)\int_0^\infty t\left|\frac{\partial}{\partial t}\vec{Q}_t u(x)\right|\left|\frac{\partial}{\partial t}\vec{Q}_t v(x)\right|dt$$

$$\leqq 4\int_M m(dx)\left\{\int_0^\infty t\left|\frac{\partial}{\partial t}\vec{Q}_t u(x)\right|^2 dt\right\}^{1/2}\left\{\int_0^\infty t\left|\frac{\partial}{\partial t}\vec{Q}_t v(x)\right|^2 dt\right\}^{1/2}$$

$$= 4\int_M G^{\rightarrow}u(x)G^{\rightarrow}v(x)m(dx)$$

$$\leqq 4\|G^{\rightarrow}u\|_p\|G^{\rightarrow}v\|_q \lesssim 4\|u\|_p\|G^{\rightarrow}v\|_q.$$

Thus we have $\|v-E_0v\|_q \lesssim \|G^{\rightarrow}v\|_q$.

To show (4.11) for $p\geqq 2$, we assume (A.1), (A.2) and $\alpha\geqq a+\gamma$, $\alpha>\beta+\gamma$. Then by (3.10) of Proposition 3.2 and Proposition 4.2, we have

$$\|G^{\uparrow}u\|_p \lesssim \|H^{\uparrow}u\|_p+\sqrt{K/\gamma}\|u\|_p \lesssim \|u\|_p+\sqrt{K/\gamma}\|u\|_p$$

which completes the proof. □

5. Examples.

We shall give two examples in this section.

EXAMPLE 5.1. Let (B, H, μ) be an abstract Wiener space: B is a separable real Banach space, H is a separable real Hilbert space which is imbedded densely and continuously in B, and μ is the Gaussian measure satisfying

$$\hat{\mu}(l)=\int_B \exp\{\sqrt{-1}\,{}_B\langle x, l\rangle_{B^*}\}\mu(dx)=\exp\left\{-\frac{1}{2}|l|_{H^*}^2\right\}, \quad l\in B^*\hookrightarrow H^*.$$

We consider the following Ornstein-Uhlenbeck semigroup;

$$P_tf(x)=\int_B f(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy) \qquad \text{for} \quad f\in L^2(\mu). \tag{5.1}$$

Here A is a non-negative definite self-adjoint operator in H. The above expression (5.1) is well-defined if the semigroup $\{e^{-tA}\}$ generated by A can be extended to a strongly continuous contraction semigroup in B so that

$$\|e^{-tA}\|_{\mathcal{L}(B)}<1, \tag{5.2}$$

where $\|\cdot\|_{\mathcal{L}(B)}$ denotes the operator norm. In this case, $\{P_t\}$ is a Feller semigroup with the probability kernel given by

$$p(t, x, C)=\int_B 1_C(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy),$$

and it defines a symmetric diffusion process on B. We give the corresponding Dirichlet form. Set $\mathcal{A}$ to be the set of all functions of the form

$$f(x)=p({}_B\langle x, l_1\rangle_{B^*}, \cdots, {}_B\langle x, l_n\rangle_{B^*}), \qquad n\in \boldsymbol{N}, \tag{5.3}$$

where p is a polynomial on $\boldsymbol{R}^n$ and $l_1, \cdots, l_n \in C^\infty(A^*) \cap B^*$, A^* being the dual operator of A in the dual space H^* (we do not identify H and H^*) and $C^\infty(A^*) = \bigcap_{n=1}^\infty \mathrm{Dom}(A^{*n})$. Then the Dirichlet form is given by

$$\mathcal{E}(f, g) = \int_B (\sqrt{A^*} Df(x), \sqrt{A^*} Dg(x))_{H^*} \mu(dx). \tag{5.4}$$

Here $Df(x) \in H^*$ is a H-derivative of f at x;

$$_H\langle h, Df(x)\rangle_{H^*} = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}. \tag{5.5}$$

In place of the assumption (5.2) for A, it is enough to assume that $C^\infty(A^*) \cap B^*$ is *dense* in H^* to ensure the existence of a diffusion process with the Dirichlet form (5.4) (see e.g., [**14**, **1**, **22**] for the construction of diffusion processes).

We denote $\sqrt{A^*} D$ by D_A and the generator by L_A to specify A. The generator L_A is given as follows; for $f(x) = p(_B\langle x, l_1\rangle_{B^*}, \cdots, {}_B\langle x, l_n\rangle_{B^*})$

$$\begin{aligned} L_A f(x) = & \sum_{i,j}^n (A^* l_i, l_j)_{H^*} \frac{\partial^2 p}{\partial \xi^i \partial \xi^j} (_B\langle x, l_1\rangle_{B^*}, \cdots, {}_B\langle x, l_n\rangle_{B^*}) \\ & - \sum_i^n \langle x, A^* l_i\rangle \frac{\partial p}{\partial \xi^i} (_B\langle x, l_1\rangle_{B^*}, \cdots, {}_B\langle x, l_n\rangle_{B^*}). \end{aligned} \tag{5.6}$$

Here $\langle x, A^* l_i\rangle$ stands for the Wiener integral for $A^* l_i \in H^*$ (so it is defined μ-almost everywhere). Moreover, by using the Wiener integral, the semigroup (5.1) is well-defined for $f \in \mathcal{A}$. By H-differentiating both hands in (5.1), we have,

$$D(P_t f)(x) = \int_B e^{-tA^*} Df(e^{-tA} x + \sqrt{1 - e^{-2tA}} y) \mu(dy) = e^{-tA^*} P_t Df(x).$$

Hence we have the following commutation relation;

$$D_A P_t = e^{-tA^*} P_t D_A. \tag{5.7}$$

By differentiating in t, we have

$$D_A L_A = (L_A - A^*) D_A. \tag{5.8}$$

Now we can compute Γ_2. First note that Γ is given by

$$\Gamma(f, g) = (\sqrt{A^*} Df(x), \sqrt{A^*} Dg(x))_{H^*}. \tag{5.9}$$

Then,

$$\begin{aligned} & 2\Gamma_2(f, g)(x) \\ & \quad = L_A \Gamma(f, g)(x) - \Gamma(L_A f, g)(x) - \Gamma(f, L_A g)(x) \\ & \quad = L_A(D_A f(x), D_A g(x)) - (D_A L_A f(x), D_A g(x)) - (D_A f(x), D_A L_A g(x)) \\ & \quad = L_A(D_A f(x), D_A g(x)) - (L_A D_A f(x), D_A g(x)) - (D_A f(x), L_A D_A g(x)) \end{aligned}$$

$$+(A^*D_Af(x),\ D_Ag(x))_{H^*}+(D_Af(x),\ A^*D_Ag(x))_{H^*}$$
$$=2(D_A^2f(x),\ D_A^2g(x))_{H^*\otimes H^*}+2(A^*D_Af(x),\ D_Ag(x))_{H^*}.$$

Hence we have

$$\Gamma_2(f)(x)=|D_A^2f(x)|^2_{H^*\otimes H^*}+(A^*D_Af(x),\ D_Af(x))_{H^*}\geqq 0 \tag{5.10}$$

because A^* is non-negative definite. Thus Γ_2 is non-negative in this case.

Further let $\mathcal{H}$ be a separable real Hilbert space and C be a non-negative self-adjoint operator in $\mathcal{H}$. We consider the following operator $\vec{L}$ in $L^2(\mu)\otimes\mathcal{H}$;

$$\vec{L}:=L_A-C. \tag{5.11}$$

Then the assumptions of Theorem 4.4 are all satisfied. Hence we have for $1<p<\infty$,

$$\|u\|_p\lesssim\|G^{\rightarrow}u\|_p\lesssim\|u\|_p,\qquad u\in\mathcal{A}(\mathcal{H})$$

and

$$\|G^{\uparrow}u\|_p\lesssim\|u\|_p,\qquad u\in\mathcal{A}(\mathcal{H}).$$

EXAMPLE 5.2. Let M be a d-dimensional complete Riemannian manifold. We shall consider a diffusion process on M with the Dirichlet form on $L^2(e^{-2\rho}dx)$ of the following form.

$$\mathcal{E}(f,\ g)=\frac{1}{2}\int_M(\nabla f(x),\ \nabla g(x))_{T_x^*M}e^{-2\rho(x)}dx \tag{5.12}$$

where ρ is a C^∞ function on M and dx is the Riemannian volume. We set $m=e^{-2\rho}dx$ for simplicity. We denote the generator by L. Then it is easy to see that

$$L=\frac{1}{2}\Delta+b \tag{5.13}$$

where b is a vector field defined by $b=-\mathrm{grad}\rho$. We assume that the diffusion process generated by L is *conservative*. A sufficient condition is given in Bakry [**5**] for example.

Moreover we consider a complex vector bundle E with fiber dimension n equipped with a Hermitian fiber metric. We assume that a unitary connection $\nabla:\Gamma(E)\to\Gamma(E\otimes T^*M)$ is given where $\Gamma(E)$ and $\Gamma(E\otimes T^*M)$ denote C^∞ sections. We consider a sesquilinear form q on $\Gamma(E)$ of the form

$$\begin{aligned}q(u,\ v)&=\frac{1}{2}\int_M(\nabla u(x),\ \nabla v(x))_{E_x\otimes T_x^*M}e^{-2\rho(x)}dx\\&\quad+\int_M((Ux)u(x),\ v(x))_{E_x}e^{-2\rho(x)}dx\qquad\text{for}\quad u,\ v\in\Gamma(E)\end{aligned} \tag{5.14}$$

where $U\in\Gamma(\mathrm{Hom}(E;\ E))$ is a potential. First we assume that there exists a constant $\beta\geqq 0$ such that

(M.1) $$U(x) \geqq -\beta I_E \quad \text{for} \quad x \in M.$$

Let $\vec{L}$ is the associated symmetric operator in $L^2(\Gamma(E); m)$ where $L^2(\Gamma(E); m)$ is a Hilbert space of all square integrable sections of E with respect to the measure m. We can write $\vec{L}$ as

(5.15) $$\vec{L} = \frac{1}{2}\Delta_E + \nabla_b + U$$

where Δ_E is the covariant Laplacian: $\Delta_E = \sum_{i=1}^d \nabla^i \nabla_i$. In this case, the vector bundle E is not trivial and so our results are not applicable. Hence we have to introduce horizontal lifts.

Let $O(M)$ be the orthonormal frame bundle and P be the principal fiber bundle associated with E. The structure group of P is $U(n)$, the set of all unitary matrices of order n. Since M is a Riemannian manifold, we can introduce the Levi-Civita connection on M which defines a connection form ω' on $O(M)$. Similarly, covariant derivative ∇ on E defines a connection form ω'' on P. Let $O(M)+P$ be the product bundle, i.e., the set of all $(r, s) \in O(M) \times P$ such that $\pi(r) = \pi(s)$. Let ω be the connection form on $O(M)+P$ defined by $\omega = \omega' + \omega''$. So ω is a differential form with values in $\mathfrak{o}(d) + \mathfrak{u}(n)$ where $\mathfrak{o}(d)$ and $\mathfrak{u}(n)$ are Lie algebras of $O(d)$ and $U(n)$, respectively. We can regard $r \in O(M)$ and $s \in P$ as isometric linear mappings in the following way;

$$r: \boldsymbol{R}^d \longrightarrow T_{\pi(r)}M, \qquad s: \boldsymbol{C}^n \longrightarrow E_{\pi(s)}.$$

Let $(X_t, P_x)_{x \in M}$ be the diffusion process generated by L. Then the horizontal lift of (X_t) is realized as follows. Let $L_1, \cdots, L_d$ be the system of basic horizontal vector fields, i.e.,

$$\pi_*(L_i(r, s)) = r(\delta_i) \in T_{\pi(r)}M \qquad \text{for} \quad i = 1, \cdots, d$$

where $\delta_1, \cdots, \delta_d$ is the canonical basis in $\boldsymbol{R}^d$. Moreover let L_0 be a horizontal lift of b.

Let us consider the following stochastic differential equation on $O(M)+P$;

(5.16) $$\begin{cases} dV_t = \sum_{i=1}^d L_i(V_t) \circ dw_t^i + L_0(V_t)dt \\ V_0 = (r, s) \in O(M)+P. \end{cases}$$

Here $(w_t^1, \cdots, w_t^d)$ is a d-dimensional Brownian motion starting at 0 and $\circ$ stands for the Stratonovich symmetric integral. We denote a solution to (5.16) by $(V_t(r, s))$. The generator of $(V_t(r, s))$ is $\hat{L} = (1/2)\sum_{i=1}^d L_i^2 + L_0$. Moreover it is well-known that $(\pi(V_t(r, s))$ is a diffusion process on M generated by L.

We introduce a symmetrizing measure $\hat{m}$ for $(V_t(r, s))$ on $O(M)+P$. Let ν be a Haar measure on $O(d) \times U(n)$ with total mass 1. Then $\hat{m}$ is given locally as

$$\hat{m} = m \times \nu \quad \text{on} \quad \pi^{-1}(O) \cong O \times O(d) \times U(n)$$

where O is a neighborhood in M. Then $\hat{m}$ is well-defined since ν is invariant under the action of $O(d)\times U(n)$. Further $\hat{m}$ is invariant under the action of $O(d)\times U(n)$ on $O(M)+P$ on the right and $\pi_*\hat{m}=m$.

For any $u\in\Gamma(T^p_q(M)\otimes E)$, we can define a scalarization $\bar{u}$: $O(M)+P\to (\boldsymbol{R}^d)^{p\otimes}\otimes(\boldsymbol{R}^d)^{*q\otimes}\otimes\boldsymbol{C}^n$ as follows

$$\bar{u}(r, s) = (r^{-1}\otimes s^{-1})u(\pi(r, s)).$$

We use $^-$ to denote the scalarization. Fortunately, we do not need to use complex conjugate in the sequel, so there is no fear of confusion. We note that $\bar{u}$ is equivariant, i.e., for $g\in O(d)\times U(n)$,

$$\bar{u}((r, s)g) = g^{-1}\bar{u}(r, s).$$

Here the action of $O(d)\times U(n)$ is extended to $(\boldsymbol{R}^d)^{p\otimes}\otimes(\boldsymbol{R}^d)^{*q\otimes}\otimes\boldsymbol{C}^n$ in natural way.

We note the following fact; for $u\in\Gamma(E)$

$$\overline{\nabla u_{;i}} = L_i\bar{u}$$

where $;i$ denotes the i-th component of covariant derivative. Moreover by noting that $L_0=\sum_{i=1}^d \bar{b}^i L_i$, we have for $u\in\Gamma(E)$,

$$\hat{L}\bar{u} = \left(\frac{1}{2}\sum_{i=1}^d L_i^2+L_0\right)\bar{u} = \sum_{i=1}^d\left\{\frac{1}{2}\overline{\nabla^2 u_{;i;i}}+\bar{b}^i\overline{\nabla u_{;i}}\right\} = \frac{1}{2}\overline{\Delta_E u}+\overline{\nabla_b u}. \tag{5.17}$$

We shall give the Dirichlet form on $L^2(\hat{m})$ for the diffusion process $(V_t(r, s))$. To do this, we give another expression of $\hat{m}$. Let $\{A'_\alpha\}$ and $\{A''_I\}$ be bases of $\mathfrak{o}(d)$ and $\mathfrak{u}(n)$, respectively. Then we can write

$$\omega = \sum_\alpha \omega'^\alpha A_\alpha+\sum_I \omega''^I A_I .$$

Moreover let $\theta=(\theta^1, \cdots, \theta^d)$ be a canonical 1-form on $O(M)+P$ defined by

$$\theta_{(r,s)}(X) = r^{-1}\pi_* X \qquad \text{for} \quad X\in T_{(r,s)}(O(M)+P).$$

Define a volume form η by

$$\eta = C\theta^1\wedge\cdots\wedge\theta^d\wedge\omega'^1\wedge\cdots\wedge\omega'^{d(d-1)/2}\wedge\omega''^1\wedge\cdots\wedge\omega''^{n(n+1)/2}$$

where C is a normalizing constant. It is easy to see that $e^{-2\bar{\rho}}\eta$ defines a measure $\hat{m}$. For any $X\in\Gamma(T(O(M)+P))$, we denote the Lie derivative by L_X. Then by the structure equation (see [**13**] Theorem III. 2.4), we can see

$$L_{L_i}\eta = 0 \qquad \text{for} \quad i = 1, \cdots, d .$$

Then the Dirichlet from of $(V_t(r, s))$ is given by

$$\hat{\mathcal{E}}(f, g)=\frac{1}{2}\int_{O(M)+P}\sum_{i=1}^{d} L_i f L_i g d\hat{m} \qquad \text{for} \quad f, g \in C_c^\infty(O(M)+P)$$

where $C_c^\infty(O(M)+P)$ is the set of all C^∞ functions on $O(M)+P$ with compact support. To see this, we note that for $f\in C_c^\infty(O(M)+P)$,

$$\int_{O(M)+P} L_{L_i}(f\eta)=0.$$

Hence

$$\begin{aligned} 0 &= \int_{O(M)+P} L_{L_i}(fg\eta) \\ &= \int_{O(M)+P}(L_{L_i}f)g\eta+\int_{O(M)+P}f(L_{L_i}g\eta)+\int_{O(M)+P}fgL_{L_i}\eta \end{aligned}$$

which implies

$$\int_{O(M)+P}(L_{L_i}f)g\eta=-\int_{O(M)+P}f(L_{L_i}g)\eta.$$

Thus, by using $L_0=\sum_{i=1}^d \bar{b}^i L_i=-\sum_{i=1}^d \overline{\nabla\rho_{;i}}L_i$, we have

$$\begin{aligned} \hat{\mathcal{E}}(f, g) &= \frac{1}{2}\int_{O(M)+P}\sum_{i=1}^{d} L_i f L_i g e^{-2\bar{\rho}}\eta \\ &= -\frac{1}{2}\int_{O(M)+P}\sum_{i=1}^{d} L_i(L_i f e^{-2\bar{\rho}})g\eta = -\frac{1}{2}\int_{O(M)+P}\sum_{i=1}^{d}(L_i^2 f-2L_i\bar{\rho}L_i f)ge^{-2\bar{\rho}}\eta \\ &= -\int_{O(M)+P}\Big(\frac{1}{2}\sum_{i=1}^{d} L_i^2 f-\overline{\nabla\rho_{;i}}L_i f\Big)ge^{-2\bar{\rho}}\eta = -\int_{O(M)+P}(\hat{L}f)gd\hat{m} \end{aligned}$$

which means that $\hat{L}$ is an associated generator. We take $C_c^\infty(O(M)+P)$ as an algebra $\mathcal{A}$.

Define $\hat{\vec{L}}$ by

$$\hat{\vec{L}}=\frac{1}{2}\sum_{i=1}^{d} L_i^2+L_0+\bar{U}, \tag{5.18}$$

then, by using (5.17), it is easy to see that for $u\in\Gamma(E)$,

$$\hat{\vec{L}}\bar{u}=\overline{\vec{L}u}. \tag{5.19}$$

Let $\{\vec{P}_t\}$ and $\{\hat{\vec{P}}_t\}$ be semigroups generated by $\hat{L}$ and $\hat{\vec{L}}$, respectively. Then we have

$$\hat{\vec{P}}\bar{u}=\overline{\vec{P}u}. \tag{5.20}$$

Moreover, by the definition of $\hat{m}$, the scalarization

$$\Gamma(E)\ni u\longmapsto \bar{u}\in\Gamma((O(M)+P)\times \boldsymbol{C}^n)$$

is an isometric linear mapping from $L^p(\Gamma(E); m)$ into $L^p(\Gamma((O(M)+P)\times\boldsymbol{C}^n); \hat{m})$. Now we can discuss everything on $O(M)+P$. But we remark here that we treat only equivariant functions on $O(M)+P$ since our interest is in $\Gamma(E)$. So to be precise, we consider the set of all *equivariant* $\boldsymbol{C}^n$-valued functions in place of $\mathcal{A}(\boldsymbol{C}^n)$.

Let us check assumptions in Theorem 4.4. First of all, let us compute Γ for $(V_t(r, s))$. For $\bar{u}, \bar{v} \in \Gamma((O(M)+P)\times \boldsymbol{C}^n)$,

$$\begin{aligned}\Gamma(\bar{u}, \bar{v}) &= \frac{1}{2}\{\hat{L}(\bar{u}\cdot\bar{v})-(\hat{L}\bar{u})\cdot\bar{v}-\bar{u}\cdot(\hat{L}\bar{v})\} \\ &= \frac{1}{4}\sum_{i=1}^{d}\{L_i^2(\bar{u}\cdot\bar{v})-(L_i^2\bar{u})\cdot\bar{v}-\bar{u}\cdot(L_i^2\bar{v})\} \\ &= \frac{1}{4}\sum_{i=1}^{d}\{(L_i^2\bar{u})\cdot\bar{v}+2L_i\bar{u}\cdot L_i\bar{v}+\bar{u}\cdot(L_i^2\bar{v})-(L_i^2\bar{u})\cdot\bar{v}-\bar{u}\cdot(L_i^2\bar{v})\} \\ &= \frac{1}{2}\sum_{i=1}^{d}L_i\bar{u}\cdot L_i\bar{v}\end{aligned}$$

which is a well-known result. Here $\cdot$ stands for the Hermitian inner product in $\boldsymbol{C}^n$.

To compute $\vec{\Gamma}_2$, the commutation relation is fundamental. So we shall obtain the explicit form of $[L_i, L_j]$. We note that $[L_i, L_j]$ is vertical since the torsion vanishes (see [**13**] Proposition III.5.4) and $\omega([L_i, L_j])=-2\Omega(L_i, L_j)$ (see [**13**] Corollary II.5.3) where Ω is the curvature form on $O(M)+P$. For any $A\in\mathfrak{o}(d)+\mathfrak{u}(n)$, a 1-parameter subgroup $\{\exp tA\}$ induces a vector field on $O(M)+P$ since $O(d)\times U(n)$ acts on $O(M)+P$ on the right. We denote it by A^*. Then it holds that $[A'^*, L_i]=\sum_j A'^j{}_i L_j$ for $A'\in\mathfrak{o}(d)$, and $[A''^*, L_i]=0$ for $A''\in\mathfrak{u}(n)$ (see [**13**] Proposition III.2.3) where $A'^j{}_i$ are components of A'. Hence, writing a basis of $\mathfrak{o}(d)$ and $\mathfrak{u}(n)$ by $\{A'_\alpha\}$ and $\{A''_I\}$ respectively, we have

$$[L_i, L_j] = -2\sum_\alpha \Omega'^\alpha(L_i, L_j)A'^*_\alpha-2\sum_I \Omega''^I(L_i, L_j)A''^*_I$$

where Ω'^α, Ω''^I are components of curvature forms Ω', Ω''. Hence, by noting that $L_0=\sum_{i=1}^d \bar{b}^i L_i$, we have

$$\begin{aligned}[L_0, L_j] &= \left[\sum_{i=1}^{d}\bar{b}^i L_i, L_j\right] = \sum_{i=1}^{d}\{\bar{b}^i[L_i, L_j]-(L_j\bar{b}^i)L_i\} \\ &= \sum_{i=1}^{d}\{-\overline{\nabla\rho_{;i}}[L_i, L_j]+\overline{\nabla^2\rho_{;i;j}}L_i\} \\ &= \sum_{i=1}^{d}\{2\sum_\alpha\overline{\nabla\rho_{;i}}\Omega'^\alpha(L_i, L_j)A'^*_\alpha+2\sum_I\overline{\nabla\rho_{;i}}\Omega''^I(L_i, L_j)A''^*_I+\overline{\nabla^2\rho_{;i;j}}L_i\}\end{aligned}$$

and further

$$
\begin{aligned}
[L_i^2, L_j] &= L_i^2 L_j - L_j L_i^2 = L_i[L_i, L_j] + [L_i, L_j]L_i \\
&= -2L_i(\sum_\alpha \Omega'^\alpha(L_i, L_j)A'^*_\alpha) - 2\sum_\alpha \Omega'^\alpha(L_i, L_j)A'^*_\alpha L_i \\
&\quad -2L_i(\sum_I \Omega''^I(L_i, L_j)A''^*_I) - 2\sum_I \Omega''^I(L_i, L_j)A''^*_I L_i \\
&= -2\sum_\alpha (L_i \Omega'^\alpha(L_i, L_j)A'^*_\alpha - 2\sum_\alpha \Omega'^\alpha(L_i, L_j)L_i A'^*_\alpha \\
&\quad -2\sum_\alpha \Omega'^\alpha(L_i, L_j)L_i A'^*_\alpha - 2\sum_\alpha \Omega'^\alpha(L_i, L_j)[A'^*_\alpha, L_i] \\
&\quad -2\sum_I (L_i \Omega''^I(L_i, L_j)A''^*_I - 2\sum_I \Omega''^I(L_i, L_j)L_i A''^*_I \\
&\quad -2\sum_I \Omega''^I(L_i, L_j)L_i A''^*_I - 2\sum_I \Omega''^I(L_i, L_j)[A''^*_I, L_i] \\
&= -2\sum_\alpha (L_i \Omega'^\alpha(L_i, L_j))A'^*_\alpha - 4\sum_\alpha \Omega'^\alpha(L_i, L_j)L_i A'^*_\alpha \\
&\quad -2\sum_\alpha \sum_{k=1}^d \Omega'^\alpha(L_i, L_j)A'^k_{\alpha i} L_k - 2\sum_I (L_i \Omega''^I(L_i, L_j))A''^*_I \\
&\quad -4\sum_I \Omega''^I(L_i, L_j)L_i A''^*_I \\
&= -2\sum_\alpha (L_i \Omega'^\alpha(L_i, L_j))A'^*_\alpha - 4\sum_\alpha \Omega'^\alpha(L_i, L_j)L_i A'^*_\alpha \\
&\quad -2\sum_{k=1}^d \Omega'^k{}_i(L_i, L_j)L_k - 2\sum_I (L_i \Omega''^I(L_i, L_j))A''^*_I \\
&\quad -4\sum_I \Omega''^I(L_i, L_j)L_i A''^*_I.
\end{aligned}
$$

Note that $A'^*\bar{u}=0$ and $A''^*\bar{u}=-A''\bar{u}$ for a scalarization $\bar{u}$ of $u\in\Gamma(E)$ and $A'\in \mathfrak{o}(d)$, $A''\in\mathfrak{u}(n)$. Moreover

$$
\begin{aligned}
2\Omega'(L_i, L_j) &= \overline{R(TM)}_{ij}, \\
2\Omega''(L_i, L_j) &= \overline{R(E)}_{ij}, \\
2L_i\Omega''(L_i, L_j) &= \overline{\nabla R(E)}_{ij;i},
\end{aligned}
$$

(see [13] Theorem III.5.1 and Proposition III.5.2). Here $R(TM)$ and $R(E)$ are curvature tensor of TM and E, respectively. Hence we have

$$
\begin{aligned}
[L_0, L_j]\bar{u} &= \sum_{i=1}^d \{2\sum_I \overline{\nabla\rho_{;i}}\Omega''^I(L_i, L_j)A''^*_I\bar{u} + \overline{\nabla^2\rho_{;i;j}}L_i\bar{u}\} \\
&= \sum_{i=1}^d \{2\overline{\nabla\rho_{;i}}\Omega''(L_i, L_j)\bar{u} + \overline{\nabla^2\rho_{;i;j}}L_i\bar{u}\} \\
&= \sum_{i=1}^d \{\overline{\nabla\rho_{;i}}R(E)_{ij}\bar{u} + \overline{\nabla^2\rho_{;i;j}}L_i\bar{u}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^d [L_i^2, L_j]\bar{u} &= \sum_{i=1}^d \{-2\sum_I (L_i\Omega''^I(L_i, L_j))A''_I\bar{u} - 4\sum_I \Omega''^I(L_i, L_j)L_i A''_I\bar{u}\} \\
&\quad -2\sum_{i=1}^d \sum_{k=1}^d \Omega'^k{}_i(L_i, L_j)L_k\bar{u}
\end{aligned}
$$

$$= \sum_{i=1}^{d} \{-2L_i\Omega''(L_i, L_j)\bar{u} - 4\Omega''(L_i, L_j)L_i\bar{u}\} - 2\sum_{i=1}^{d}\sum_{k=1}^{d} \Omega'^k{}_i(L_i, L_j)L_k\bar{u}$$

$$= \sum_{i=1}^{d} \{-\overline{\nabla R(E)}_{ij;i}\bar{u} - 2\overline{R(E)}_{ij}L_i\bar{u}\} + 2\sum_{k=1}^{d} \bar{S}_{kj}L_k\bar{u}$$

$$= \sum_{i=1}^{d} \{-\overline{\nabla R(E)}_{ij;i}\bar{u} - 2\overline{R(E)}_{ij}L_i\bar{u} + 2\bar{S}_{ij}L_i\bar{u}\}$$

where S is the Ricci tensor;

$$\bar{S}_{ij} = \sum_{k=1}^{d} \overline{R(TM)}^k{}_{ikj} = -\sum_{k=1}^{d} \overline{R(TM)}^i{}_{kkj}.$$

Now we can compute $\vec{\Gamma}_2$;

$$\vec{\Gamma}_2(\bar{u}, \bar{v}) = \frac{1}{2}\{\hat{L}\Gamma(\bar{u}, \bar{v}) - \Gamma((\hat{\hat{L}} - \bar{U})\bar{u}, \bar{v}) - \Gamma(\bar{u}, (\hat{\hat{L}} - \bar{U})\bar{v})\}$$

$$= \frac{1}{2}\sum_{j=1}^{d}\Big\{\Big(\frac{1}{2}\sum_{i=1}^{d} L_i^2 + L_0\Big)(L_j\bar{u}\cdot L_j\bar{v})$$

$$-L_j\Big(\Big(\frac{1}{2}\sum_{i=1}^{d} L_i^2 + L_0 - \bar{U}\Big)\bar{u}\Big)\cdot L_j\bar{v}$$

$$-L_j\bar{u}\cdot L_j\Big(\Big(\frac{1}{2}\sum_{i=1}^{d} L_i^2 + L_0 - \bar{U}\Big)\bar{v}\Big)\Big\}$$

$$= \frac{1}{2}\sum_{j=1}^{d}\Big\{\frac{1}{2}\sum_{i=1}^{d} L_i^2(L_j\bar{u}\cdot L_j\bar{v}) + L_0L_j\bar{u}\cdot L_j\bar{v} + L_j\bar{u}\cdot L_0L_j\bar{v}$$

$$-L_j\Big(\frac{1}{2}\sum_{i=1}^{d} L_i^2\bar{u}\Big)\cdot L_j\bar{v} - L_jL_0\bar{u}\cdot L_j\bar{v}$$

$$+(L_j\bar{U})\bar{u}\cdot L_j\bar{v} + (\bar{U}L_j\bar{u})\cdot L_j\bar{v} - L_j\bar{u}\cdot L_j\Big(\frac{1}{2}\sum_{i=1}^{d} L_i^2\bar{v}\Big)$$

$$-L_j\bar{u}\cdot L_jL_0\bar{v} + L_j\bar{u}\cdot(L_j\bar{U})\bar{v} + L_j\bar{u}\cdot(\bar{U}L_j\bar{v})\Big\}$$

$$= \frac{1}{4}\sum_{i,j=1}^{d}\{L_i^2(L_j\bar{u}\cdot L_j\bar{v}) - (L_jL_i^2\bar{u})\cdot L_j\bar{v} - L_j\bar{u}\cdot L_jL_i^2\bar{v}\}$$

$$+\frac{1}{2}\sum_{j=1}^{d}\{[L_0, L_j]\bar{u}\cdot L_j\bar{v} + L_j\bar{u}\cdot[L_0, L_j]\bar{v} + 2(\bar{U}L_j\bar{u})\cdot L_j\bar{v}$$

$$+(L_j\bar{U})\bar{u}\cdot L_j\bar{v} + L_j\bar{u}\cdot(L_j\bar{U})\bar{v}\}$$

$$= \frac{1}{4}\sum_{i,j=1}^{d}\{[L_i^2, L_j]\bar{u}\cdot L_j\bar{v} + L_j\bar{u}\cdot[L_i^2, L_j]\bar{v} + 2L_iL_j\bar{u}L_iL_j\bar{v}\}$$

$$+\frac{1}{2}\sum_{j=1}^{d}\{[L_0, L_j]\bar{u}\cdot L_j\bar{v} + L_j\bar{u}\cdot[L_0, L_j]\bar{v} + 2(\bar{U}L_j\bar{u})\cdot L_j\bar{v}$$

$$+(L_j\bar{U})\bar{u}\cdot L_j\bar{v} + L_j\bar{u}\cdot(L_j\bar{U})\bar{v}\}$$

$$= \frac{1}{4}\sum_{i,j=1}^{d}\{-\overline{\nabla R(E)}_{ij;i}\bar{u}\cdot L_j\bar{v} - 2(\overline{R(E)}_{ij}L_i\bar{u})\cdot L_i\bar{v}$$

$$
\begin{aligned}
&\qquad +2\bar{S}_{ij}L_i\bar{u}\cdot L_j\bar{v}-L_j\bar{u}\cdot\overline{\nabla R(E)}_{ij;i}\bar{v}-2L_i\bar{u}\cdot(\overline{R(E)}_{ij}L_i\bar{v})\\
&\qquad +2L_j\bar{u}\cdot\bar{S}_{ij}L_i\bar{v}+2L_iL_j\bar{u}L_iL_j\bar{v}\\
&\qquad +2\overline{\nabla\rho_{;i}}\overline{R(E)}_{ij}\bar{u}\cdot L_j\bar{v}+2\overline{\nabla^2\rho_{;i;j}}L_i\bar{u}\cdot L_j\bar{v}\\
&\qquad +2L_j\bar{u}\cdot\overline{\nabla\rho_{;i}}\overline{R(E)}_{ij}\bar{v}+2L_j\bar{u}\cdot\overline{\nabla^2\rho_{;i;j}}L_i\bar{v}\}\\
&+\frac{1}{2}\sum_{j=1}^{d}\{2(\bar{U}L_j\bar{u})\cdot L_j\bar{v}+(L_j\bar{U})\bar{u}\cdot L_j\bar{v}+L_j\bar{u}\cdot(L_j\bar{U})\bar{v}\}\\
=&\ \frac{1}{4}\sum_{i,j=1}^{d}\{-(\overline{\nabla R(E)}_{ij;i}\bar{u})\cdot L_j\bar{v}-L_j\bar{u}\cdot(\overline{\nabla R(E)}_{ij;i}\bar{v})\\
&\qquad -4(\overline{R(E)}_{ij}L_i\bar{u})\cdot L_j\bar{v}+4\bar{S}_{ij}L_i\bar{u}\cdot L_j\bar{v}+2L_iL_j\bar{u}L_iL_j\bar{v}\\
&\qquad +2\overline{\nabla\rho_{;i}}\overline{R(E)}_{ij}\bar{u}\cdot L_j\bar{v}+2L_j\bar{u}\cdot\overline{\nabla\rho_{;i}}\overline{R(E)}_{ij}\bar{v}\\
&\qquad +2\overline{\nabla^2\rho_{;i;j}}L_i\bar{u}\cdot L_j\bar{v}\}\\
&\qquad +\frac{1}{2}\sum_{j=1}^{d}\{2(\bar{U}L_j\bar{u})\cdot L_j\bar{v}+(L_j\bar{U})\bar{u}\cdot L_j\bar{v}+L_j\bar{u}\cdot(L_j\bar{U})\bar{v}\}\\
=&\ \frac{1}{2}\sum_{i,j=1}^{d}\{2(\bar{S}_{ij}+\overline{\nabla^2\rho_{;i;j}})L_i\bar{u}\cdot L_j\bar{v}-2(\overline{R(E)}_{ij}L_i\bar{u})\cdot L_j\bar{v}\\
&\qquad +L_iL_j\bar{u}L_iL_j\bar{v}\}+\sum_{j=1}^{d}(\bar{U}L_j\bar{u})\cdot L_j\bar{v}\\
&+\frac{1}{2}\sum_{i,j=1}^{d}\{-2(\overline{\nabla R(E)}_{ij;i}\bar{u})\cdot L_j\bar{v}-2L_j\bar{u}\cdot(\overline{\nabla R(E)}_{ij;i}\bar{v})\\
&\qquad\quad +2\overline{\nabla\rho_{;i}}\overline{R(E)}_{ij}\bar{u}\cdot L_j\bar{v}+2L_j\bar{u}\cdot\overline{\nabla\rho_{;i}}\overline{R(E)}_{ij}\bar{v}\}\\
&+\frac{1}{2}\sum_{j=1}^{d}\{(L_j\bar{U})\bar{u}\cdot L_j\bar{v}+L_j\bar{u}\cdot(L_j\bar{U})\bar{v}\}\,.
\end{aligned}
$$

For $F\in\Gamma(T_2M)=\Gamma(T^*M\otimes T^*M)$, we define $F^\#\in\Gamma(TM\otimes T^*M)\cong\Gamma(\mathrm{Hom}(TM))$ by

$$g(X,F^\#Y)=F(X,Y)\qquad\text{for}\quad X,Y\in\Gamma(TM)$$

where g is the Riemannian metric on M. Hence $S^\#$, $(\nabla^2\rho)^\#\in\Gamma(\mathrm{Hom}(TM))$. Similarly, we can define $R(E)^\#\in\Gamma(\mathrm{Hom}(TM\otimes E))$. We assume the following conditions: there exists a constant $c\geqq 0$ such that

(M.2) $$S^\#\otimes I_E+(\nabla^2\rho)^\#\otimes I_E-R(E)^\#+I_{TM}\otimes U\geqq -cI_{TM}\otimes I_E$$

and

(M.3) $\sum_i\overline{\nabla R(E)}_{ij;i}$, $\nabla\rho\otimes R(E)$ and ∇U are bounded.

Then under the conditions (M.1), (M.2) and (M.3) we have

$$
\begin{aligned}
\vec{\Gamma}_2(\bar{u}) \geqq & -c|\overline{\nabla u}|^2-\Big(2\Big\|\Big\{\sum_{j=1}^{d}(\sum_{i=1}^{d}\overline{\nabla R(E)}_{ij;i})^2\Big\}^{1/2}\Big\|_\infty \\
& +2\|\nabla\rho\otimes R(E)\|_\infty+\|\nabla U\|_\infty\Big)|\overline{\nabla u}|\,|\bar{u}| \\
\geqq & -c|\overline{\nabla u}|^2-\Big(\Big\|\Big\{\sum_{j=1}^{d}\Big(\sum_{i=1}^{d}\overline{\nabla R(E)}_{ij;\,i}\Big)^2\Big\}^{1/2}\Big\|_\infty \\
& +\|\nabla\rho\otimes R(E)\|_\infty+\frac{1}{2}\|\nabla U\|_\infty\Big)(|\overline{\nabla u}|^2+|\bar{u}|^2) \\
\geqq & -\Big(c+\Big\|\Big\{\sum_{j=1}^{d}\Big(\sum_{i=1}^{d}\overline{\nabla R(E)}_{ij;i}\Big)^2\Big\}^{1/2}\Big\|_\infty \\
& +\|\nabla\rho\otimes R(E)\|_\infty+\frac{1}{2}\|\nabla U\|_\infty\Big)\Gamma(\bar{u}) \\
& -\Big(\Big\|\Big\{\sum_{j=1}^{d}\Big(\sum_{i=1}^{d}\overline{\nabla R(E)}_{ij;i}\Big)^2\Big\}^{1/2}\Big\|_\infty+\|\nabla\rho\otimes R(E)\|_\infty+\frac{1}{2}\|\nabla U\|_\infty\Big)|\bar{u}|^2.
\end{aligned}
$$

The above inequality is valid for equivariant C^∞ sections. By noting the hypoellipticity of $\vec{L}$, we have that $\vec{P}_t\bar{u}$ is equivariant and C^∞ and hence the assumption (A.2) is satisfied. Thus the assumptions of Theorem 4.4 are all satisfied. Hence we have estimates (4.9) (4.10) of G-functions for $\hat{\vec{L}}=\hat{L}-\bar{U}-\alpha$.

By projecting this result to the base manifold M, we have similar estimate of G-functions for $\vec{L}=L-U-\alpha$. We sum up in a theorem.

THEOREM 5.1. *Assume that* (M.1), (M.2) *and* (M.3) *hold. Then for* $\alpha, \gamma>0$ *such that* $\alpha\geqq c+\gamma+\|\{\sum_j(\sum_i\overline{\nabla R(E)}_{ij;i})^2\}^{1/2}\|_\infty+\|\nabla\rho\otimes R(E)\|_\infty+(1/2)\|\nabla U\|_\infty$, $\alpha>\beta+\gamma$, *we have for* $1<p<\infty$,

$$\|u\|_p \lesssim \|G^{\rightarrow}u\|_p \lesssim \|u\|_p \qquad \text{for} \quad u\in\Gamma_c(E)$$

and

$$\|G^{\uparrow}u\|_p \lesssim (1+\sqrt{K/\gamma})\|u\|_p \qquad \text{for} \quad u\in\Gamma_c(E)$$

where $K=(\|\{\sum_j(\sum_i\overline{\nabla R(E)}_{ij;i})^2\}^{1/2}\|_\infty+\|\nabla\rho\otimes R(E)\|_\infty+(1/2)\|\nabla U\|_\infty)/(\alpha-\beta-\gamma)$ *and* $\Gamma_c(E)$ *is the set of all* C^∞ *sections with compact support.*

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